ON A STOCHASTIC CONTROL PROBLEM WITH TERMINAL STATE CONTRAINTS

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Abstract

In this thesis, we consider a stochastic control problem in both finite and infinite time interval, with a terminal state constraint. We consider two approaches in solving the problem i.e, the Maximum Principle and the Dynamic programming approach.

Declaration

No portion of the work in this thesis has been submitted for another degree or qualification of this or any other university or another institution of learning.

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Not forgetting all my colleagues and friends for their encouragement and support throughout the writing of this thesis. God Bless!.

Chapter 1

Some Mathematical Preliminaries

1.0.1 Introduction

We begin with a review of some basic definitions and theory closely related to classical stochastic control. Both the dynamic programming method and the maximum principle method are discussed as well as the relation between them. Also formulated are the corresponding verification theorems involving the Hamilton-Jacobi Bellman(HJB) equation.

Definition 1 *A stochastic process [4] is a family* $\{X_t = X(t,w), t \geq 0, w \in \Omega\}$ *of random variables defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *where in this case* $\mathcal{F}$ *denotes the subsets of* Ω *that are events and are called* $\mathcal{F}$*-measurable and with* P *a probability measure assigning to any event in* $\mathcal{F}$ *its probability i.e.,* $P : \mathcal{F} \rightarrow [0,1]$ *such that*

1. $P(\emptyset) = 0, P(\omega) = 1$

2. If $B_1, B_2, ... \in \mathcal{F}$ and $\{B_i\}_{i=1}^{\infty}$ is disjoint (i.e., $B_i \cap B_j = \emptyset, \quad i \neq j$) then

$$P(\cup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} P(B_i)$$

The set of events corresponding to the information available at time t is denoted by $\mathcal{F}_t \subset \mathcal{F}$ i.e., if an event B is in $\mathcal{F}_t$, then at time t, this event is known to be true or false. A good example of a stochastic process is the Brownian motion.

Definition 2 *A one-dimensional Brownian motion is a stochastic process $B_t(w)$ such that*

1. $P[B_o(w) = x] = 1$ i.e., the process starts at a point x at $t = 0$.

2. It has independent increments i.e., if $0 < t_1 < t_2 < ... < t_m$, then $B_{t_m} - B_{t_{m-1}}, B_{t_{m-1}} - B_{t_{m-2}}, ..., B_{t_1}, B_{t_0}$ are independent.

3. For $s < t$, $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$.

4. B_t is continuous in t.

5. It has stationary increments i.e., if $s \leq t$, $B_t - B_s$ and $B_{t-s} - B_0$ have the same probability law.

Definition 3 *Let U be the collection of all open subsets of Ω and let G_u be the smallest σ-algebra generated by U. If $\Omega = \mathbb{R}^n$, then the family $\mathcal{B} = G_u$ is called the Borel σ-algebra on Ω and the elements $B = \mathcal{B}$ are called the Borel sets. More on this reader is refered to [2]*

Definition 4 *Stochastic Differential Equations*

A stochastic differential equation(S.D.E) is of the form

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t)W_t, \quad b(t, x) \in \mathbb{R}, \quad \sigma(t, x) \in \mathbb{R} \tag{1.1}$$

where W_t is a 1-dimensional white noise.

The Ito interpretation of (1.1) is that X_t satisfies the stochastic integral equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s \quad \text{or in the differential form} \tag{1.2}$$

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \tag{1.3}$$

Thus a(one-dimensional) Ito process(or stochastic integral) is a stochastic process X_t on $(\Omega, \mathcal{F}, \mathbb{P})$ of the form (1.2).

Theorem 1 *Consider the following SDE in $\mathbb{R}^n : X(0) = x \in \mathbb{R}^n$ and*

$$dX(t) = \alpha(t, X(t))dt + \sigma(t, X(t))dB(t)$$

where

$$\alpha : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n,$$

$$\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$$

satisfy the following conditions:

(At most linear growth) There exists a constant $C < \infty$ such that

$$\| \sigma(t, x) \|^2 + | \alpha(t, x) |^2 \leq C(1 + | x |^2) \quad \forall \ \ x \in \mathbb{R}^n.$$

(Lipschitz continuity) There exists a constant $D < \infty$ such that

$$\| \sigma(t, x) - \sigma(t, y) \|^2 + | \alpha(t, x) - \alpha(t, y) |^2$$

$$\leq D \, | x - y |^2 \quad \forall \ \ x, y \in \mathbb{R}^n.$$

Then, there exists a unique cadlag adapted solution $X(t)$ such that
$\mathbb{E}[| X(t) |^2] < \infty \quad \forall \ \ t$

Proof

See [1]

Martingales

An n-dimensional stochastic process $\{M_t\}_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a martingale with respect to a filtration $\{\mathcal{M}_t\}_{t \geq 0}$ if

(i) M_t is $\mathcal{M}_t$-measurable for all t.

(ii) $\mathbb{E}[| M_t |] < \infty$ for all t and,

(iii) $\mathbb{E}[M_s \mid \mathcal{M}_t] = M_t$ for all $s \geq t$.

Properties of Conditional Expectation

Let X be a stochastic process with $\mathbb{E}[X_t] < \infty$

(a) $\mathbb{E}[\mathbb{E}[X \mid \mathcal{M}]] = \mathbb{E}[X]$, $\mathcal{M} - \sigma$-algebra.

(b) $\mathbb{E}[X \mid \mathcal{M}] = X$ if X is $\mathcal{M}$-measurable.

(c) $\mathbb{E}[X \mid \mathcal{M}] = \mathbb{E}[X]$ if X is independent of $\mathcal{M}$.

(d) $\mathbb{E}[YX \mid \mathcal{M}] = Y\mathbb{E}[X \mid \mathcal{M}]$ if Y is $\mathcal{M}$-measurable.

Example

$$\begin{aligned}
\mathbb{E}[B_s e^{2B_t} \mid \mathcal{F}_s] &= B_s \mathbb{E}[e^{2B_t} \mid \mathcal{F}_s] \text{ since } B_s \text{ is } \mathcal{F}_s - \text{ measurable} \\
&= B_s \mathbb{E}[e^{2B_t - 2B_s + 2B_s} \mid \mathcal{F}_s] \\
&= B_s e^{2B_s} \mathbb{E}[e^{2(B_t - B_s)} \mid \mathcal{F}_s] \\
&= B_s e^{2B_s} \mathbb{E}[e^{2(B_t - B_s)}] \text{ since } B_t - B_s \text{ is } \mathcal{F}_s \text{ independent} \\
&= B_s e^{2B_s} \mathbb{E}[e^{2B_{t-s}}] \\
&= B_s e^{2B_s} e^{2(t-s)}.
\end{aligned}$$

NOTE

When $\alpha(t,x) = \alpha(x)$, and $\sigma(t,x) = \sigma(x)$ then we have the time homogeneous case. In the above theorem, by the term " unique", we mean that any other Ito process with the same properties is equal to X almost everywhere. A unique solution in this sense is sometimes called a strong solution or strong uniqueness. We also have weak uniqueness where any two solutions are identical in law i.e., they have the same finite-dimensional distributions.

1.0.2 The Ito Formula

Let $X(t) \in \mathbb{R}^n$ be an Ito process of the form

$$dX(t) = \alpha(t, w)dt + \sigma(t, w)dB(t)$$

where $\alpha : [0, T] \times \Omega \to \mathbb{R}^n$; $\sigma : [0, T] \times \Omega \to \mathbb{R}^{n\times m}$ are adapted processes such that the integrals exist. Also in this case $B(t)$ is an m-dimensional Brownian motion.

Let $f \in C^{1,2}([0, T] \times \mathbb{R}^n) \to \mathbb{R}$.

Then $Y(t) = f(t, X(t))$ is again an Ito process, and

$$dY(t) = \frac{\partial f}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\alpha_i dt + \sigma_i dB(t)) + \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}dt. \tag{1.4}$$

The one dimensional version of this Ito-formula is

$$dY(t) = \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))(\alpha dt + \sigma dB(t)) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X(t))\sigma^2(t)dt \tag{1.5}$$

If we choose to describe the motion of a small particle suspended in a moving liquid, subject to random molecular bombardments, then a reasonable mathematical model for the position X_t of the particle at time t would be a stochastic differential equation of the form (1.1) where $b(t, x) \in \mathbb{R}^3$ is the velocity of the fluid at the point x at time t and $W_t \in \mathbb{R}^3$ denotes 'white noise' and $\sigma(t, x) \in \mathbb{R}^{3\times 3}$. In a stochastic differential equation of the form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$$

where $X_t \in \mathbb{R}^n$, $b(t, x) \in \mathbb{R}^n$, $\sigma(t, x) \in \mathbb{R}^{n\times m}$ and B_t is m-dimensional Brownian motion, we will call b the drift velocity and σ the diffusion coefficient.

We note that the solution of a stochastic differential equation may be thought of as the mathematical description of the motion of a small particle in a moving fluid and such stochastic processes are called Ito diffusions. We thus develop some of the most basic properties and results about Ito diffusions.

The next section introduces a class of stochastic processes that share what is called the 'Markov property': the future is independent of the past, given the present values of the process. Markov models are important models of security prices, because they are often realistic representations of the true prices and yet the Markov property leads to simplified computations.

1.0.3 The Markov Property

Definition 5 *A (time homogeneous) Ito diffusion is a stochastic process $X_t(w) = X(t,w) : [0,\infty) \times \Omega \to \mathbb{R}^n$ satisfying a stochastic differential equation of the form*

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad t \geq s, \quad X_s = x..............(a)$$

where B_t is m-dimensional Brownian motion.
The unique solution of (a) is denoted by $X_t = X_t^{s,x},\ t \geq s$. If $s = 0$, we write X_t^x for $X_t^{0,x}$ where in this case we have assumed that b and σ do not depend on t but only on x.

Consider

$$\begin{aligned} X_{s+h}^{s,x} &= x + \int_s^{s+h} b(X_u^{s,x})du + \int_s^{s+h} \sigma(X_u^{s,x})dB_u \\ &= x + \int_0^h b(X_{s+v}^{s,x})dv + \int_0^h \sigma(X_{s+v}^{s,x})d\tilde{B}_v, \quad u = s+v \end{aligned}$$

where $\tilde{B}_v = B_{s+v} - B_s, \quad v \geq 0$.
On the other hand,

$$X_h^{0,x} = x + \int_0^h b(X_v^{0,x})dv + \int_0^h \sigma(X_v^{0,x})dB_v$$

$\{\tilde{B}_v\}_{v\geq 0}$ and $\{B_v\}_{v\geq 0}$ have the same Q-distributions and thus by the weak uniqueness of the solution of the stochastic differential equation (a),

$$\{X_{s+h}^{s,x}\}_{h\geq 0} \quad \text{and} \quad \{X_h^{0,x}\}_{h\geq 0}$$

have the same Q-distributions i.e., $\{X_t\}_{t\geq 0}$ is time homogeneous.

Definition 6 *Let P and Q be measures on a σ-algebra $\mathcal{U}$. The measure Q is absolutely continuous w.r.t P if for each $A \in \mathcal{U}, P(A) = 0 \Rightarrow Q(A) = 0$. The relation indicated by $Q << P$.*
If $Q << P$ and $P << Q$, the measures are said to be equivalent and are related as follows, $Q \sim P$.

Markov property: The future behavior of the process given what has happened up to time t is the same as the behavior obtained when starting the process at X_t.
We now want to prove that X_t satisfies this property.
NB: $\mathcal{F}_t$ is the σ-algebra generated by $\{B_r, r \leq t\}$. Similarly, we let $\mathcal{M}_t$ be the σ-algebra generated by $\{X_r, r \leq t\}$. Since X_t is measurable with respect to $\mathcal{F}_t$, then $\mathcal{M}_t \subseteq \mathcal{F}_t$.

Theorem 2 *(The Markov property for Ito diffusions)*
Let f be a bounded Borel function from $\mathbb{R}^n \to \mathbb{R}$. Then for $t, h \geq 0$,

$$\mathbb{E}^x[f(X_{t+h}) \mid \mathcal{F}_t] = \mathbb{E}^{X_t(w)}[f(X_h)]$$

Proof

From the theorem, X_t is a markov process with respect to the family of σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$. Since $\mathcal{M}_t \subseteq \mathcal{F}_t$ this implies that X_t is also a Markov process with respect to the σ-algebras $\{\mathcal{M}_t\}_{t\geq 0}$.

Now by using the properties of conditional expectation

$$\begin{aligned}
\mathbb{E}^x[f(X_{t+h}) \mid \mathcal{F}_t] &= \mathbb{E}^x[E^x[f(X_{t+h}) \mid \mathcal{F}_t] \mid \mathcal{M}_t] \\
&= \mathbb{E}^x[\mathbb{E}^{X_t}[f(X_h) \mid \mathcal{F}_t] \mid \mathcal{M}_t] \\
&= \mathbb{E}^x[\mathbb{E}^{X_t}[f(X_h)] \mid \mathcal{M}_t] \\
&= \mathbb{E}^{X_t}[f(X_h)]
\end{aligned}$$

Since $\mathbb{E}^{X_t}[f(X_h)]$ is $\mathcal{M}_t$-measurable

The Strong Markov Property

If the time t is replaced by a random time $\tau(w)$ of a more general type called stopping time, then the strong Markov property states that a relation of the form (a) continues to hold.

Definition 7 *Let $\{\mathcal{N}_t\}$ be an increasing family of σ-algebras, $\mathcal{N}_t \subseteq \Omega$. A function $\tau : \Omega \to [0, \infty)$ is called a strict stopping time w.r.t $\mathcal{N}_t$ if*

$$\{w; \tau(w) \leq t\} \in \mathcal{N}_t, \quad \textit{for all} \ \ t \geq 0$$

In other words, on the basis of the knowledge of $\mathcal{N}_t$, it should be possible to decide whether or not $\tau \leq t$ has occurred since

$$\{w; \tau(w) \leq t\} \ \text{ is } \ \mathcal{N}_t \ \text{ measurable.}$$

Definition 8 *Let τ be a stopping time w.r.t $\{\mathcal{N}_t\}$ and let $\mathcal{N}_\infty$ be the smallest σ-algebra containing $\mathcal{N}_t$ for all $t \geq 0$. Then the σ-algebra $\mathcal{N}_\tau$ consists of all sets $N \in \mathcal{N}_\infty$ such that*

$$N \cap \{\tau \leq t\} \in \mathcal{N}_t \quad \textit{for all} \quad t \geq 0$$

Theorem 3 *The Strong Markov property for Ito diffusions.*
Let f be a bounded Borel function on $\mathbb{R}^n$, and τ a stopping time w.r.t $\mathcal{F}_t$, $\tau < \infty$ a.s.
Then

$$\mathbb{E}^x[f(X_{\tau+h}) \mid \mathcal{F}_t] = \mathbb{E}^{X_\tau}[f(X_h)] \quad \forall \quad h \geq 0$$

The Generator of an Ito Diffusion

Definition 9 *Let $X(t) \in \mathbb{R}^n$ be a(time homogeneous) Ito diffusion. Then the generator A of X is defined on functions $f : \mathbb{R}^n \to \mathbb{R}$ by*

$$Af(x) = \lim_{t \to 0^+} \frac{1}{t}\{\mathbb{E}^x[f(X_t)] - f(x)\} \quad \textit{if the limit exists}$$

where $\mathbb{E}^x[f(X_t)] = \mathbb{E}[f(X_t^{(x)})]$, $\qquad X^{(x)}(0) = x$.

Theorem 4 *Let X_t be the Ito diffusion*

$$dX_t = \alpha(X(t))dt + \sigma(X(t))dB(t)$$

Suppose $f \in C_0^2(\mathbb{R}^n)$. Then the generator A of x is given by

$$Af(x) = \sum_{i=1}^{n} \alpha_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T)_{ij}(x))\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \tag{1.6}$$

Proof 1

$$df(X_t) = \sum_i \frac{\partial f(X_t)}{\partial X_i} dX_t^{(i)} + \frac{1}{2}\sum_{i,j}\frac{\partial^2 f(X_t)}{\partial X_i \partial X_j} dX_t^{(i)} dX_t^{(j)}$$

But $dX_t^{(i)} = \alpha^{(i)}(X_t)dt + \sum_l \sigma_{il} dB_t^{(i)}$. *This implies that*

$$\begin{aligned} df(X_t) &= \sum_i \frac{\partial f}{\partial X_i} dX_t^{(i)} + \frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial X_i \partial X_j} dX_t^{(i)} dX_t^{(j)} \\ &= \sum_i \frac{\partial f}{\partial X_i}\alpha^{(i)} dt + \sum_i \frac{\partial f}{\partial X_i}\sigma^{(i)} dB_t + \frac{1}{2}\sum_{i,j}\frac{\partial^2 f}{\partial X_i \partial X_j}(\sigma.\sigma^T)_{i,j} dt \end{aligned}$$

Since $dX_t^{(i)} \cdot dX_t^{(j)} = (\sigma\sigma^T)_{ij} dt$. *This implies that*

$$\begin{aligned} \frac{\mathbb{E}^x[f(X_t)] - f(x)}{t} &= \frac{1}{t}\left\{\mathbb{E}^x\left[f(x) + \int_0^t \sum_i \frac{\partial f(X_s)}{\partial X_i}\alpha^{(i)}(X_s) ds\right]\right\} \\ &+ \frac{1}{t}\left\{\mathbb{E}^x\left[\frac{1}{2}\int_0^T \sum_{i,j}\frac{\partial^2 f(X_s)}{\partial X_i \partial X_j}(\sigma.\sigma^T)_{i,j}(X_s) ds\right] - f(x)\right\} \end{aligned}$$

We observe that if $g(s)$ *is continuous, then*

$$\lim_{t\to 0}\frac{\int_0^t g(s)ds}{t} = \lim_{t\to 0}\frac{g(t)}{1} = g(0)$$

Thus to calculate $\lim_{t\to 0}\frac{\mathbb{E}^x[f(X_t)] - f(t)}{t}$, *we observe that* $X(0) = x$ *and from the above observation we obtain*

$$Af(x) = \sum_{i=1}^n \alpha^{(i)}(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^n (\sigma\sigma^T)_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

Dynkin's formula

Let $X(t) \in \mathbb{R}^n$ be a jump diffusion and let $f \in C_0^2(\mathbb{R}^n)$. Let τ be a stopping time i.e., τ is the first exit time such that $E^x[\tau] < \infty$.

Then

$$\mathbb{E}^x[f(X(\tau))] = f(x) + \mathbb{E}^x\left[\int_0^\tau Af(X_s)ds\right]$$

Proof 2 *Ito's formula on* $dX_t = \alpha(X_t)dt + \sigma(X_t)dB_t$ *yields*

$$df(X_t) = \sum_i \frac{\partial f}{\partial X_i}\alpha^{(i)} dt + \sum_i \frac{\partial f}{\partial X_i}\sigma^{(i)} dB_t + \frac{1}{2}\sum_{i,j}\frac{\partial^2 f}{\partial X_i \partial X_j}(\sigma.\sigma^T)_{i,j} dt$$

which gives

$$f(X_t) = f(X_0)+\int_0^t \sum_i \frac{\partial f}{\partial X_i}\alpha^{(i)}ds+\frac{1}{2}\int_0^t \sum_{i,j} \frac{\partial^2 f}{\partial X_i \partial X_j}(\sigma\sigma^T)_{ij}ds+\int_0^t \sum_i \frac{\partial f}{\partial X_i}\sigma^{(i)}dB_s$$

Now let $t = \tau(w)$ be a stopping time, then

$$\begin{aligned} f(X_\tau) &= f(X_0) + \int_0^\tau \sum_i \frac{\partial f}{\partial X_i}\alpha^{(i)}ds + \frac{1}{2}\int_0^\tau \sum_{i,j}(\sigma\sigma^T)_{ij}\frac{\partial^2 f}{\partial X_i \partial X_j}ds + \int_0^\tau \sum_i \frac{\partial f}{\partial X_i}\sigma^{(i)}dB_s \\ &= f(X_0) + \int_0^\tau Af(X_s)ds + \int_0^\tau \sum_i \frac{\partial f}{\partial X_i}\sigma^{(i)}dB_s \end{aligned}$$

Therefore

$$\mathbb{E}^x[f(X_\tau)] = f(x) + \mathbb{E}\left[\int_0^\tau Af(X_s)ds\right] + \mathbb{E}\left[\int_0^\tau \sum_i \frac{\partial f}{\partial X_i}\sigma^{(i)}dB_s\right]$$

Lemma

If $\tau(w)$ is a stopping time with $\mathbb{E}^x(\tau) < \infty$ and f is a bounded function, then

$$\mathbb{E}\left[\int_0^\tau \sum f(s,w)dB_s\right] = 0$$

Proof

Let t be any real number, then

$$\mathbb{E}\left[\int_0^{t_0} f(s,w)dB_s\right] = \mathbb{E}\left[\int_0^t f(s,w)\chi_{\{w;s\le\tau(w)\}}dB_s\right]$$

where $t_0 = \min\{\tau, t\}$.

From the definition of a stopping time, $\chi_{\{w;s\le\tau(w)\}}$ is $\mathcal{F}_s$-measurable. Then $\chi_{\{w;s\le\tau(w)\}}f(s,w)$ is adapted.

Therefore

$$\mathbb{E}\left[\int_0^t \chi_{\{w;s\le\tau(w)\}}f(s,w)dB_s\right] = 0$$

Thus Dynkin's formula is shown.

Model

If we assume that the price process is represented by the model

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

with the generator given by

$$Af = \sum_i b(x)\frac{\partial f}{\partial x_i} + \sum_{i,j}(\sigma\sigma^T)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}$$

Then we have two types of utility or reward functions.

(1) Instantaneous Reward : $g(x)$

Suppose that τ is a stopping or terminal time, then $g(X_\tau)$ is the terminal reward one gets when stopping at time τ where in this case $g(X_\tau)$ is given by

$$g(X_\tau) = \alpha_i X_i$$

α_i is the number of units in stock i, $\quad i = 1, 2, 3, ..., n$.

One can easily note that the terminal reward is the utility we get when we stop at time τ.

(2) Accumulated Reward: $f(x)$

$\int_0^\tau f(X_\tau)dt =$ utility of holding X_t from 0 to τ.

f is interpreted as the running reward or utility function from time $t = 0$ to time $t = \tau$.

Thus the total reward when stopping at some time τ is the sum of the two types of rewards i.e.,

$$\int_0^\tau f(X_\tau)dt + g(X_\tau)$$

Also, the price might be discounted back to the time of start of the production and thus, the running reward will change to include the discount factor $e^{-\delta t}$, where δ is the discount rate. Thus our function Φ will be such that

$$\Phi(y) = \sup_{\tau \leq T} \mathbb{E}^y \left[\int_0^\tau e^{-\delta t} f(X_t)dt + g(\tau, X_\tau) \right]$$

Instead of finding the supremum over all possible stopping times, we will find the supremum over all $\mathcal{F}_t$-adapted processes $\{u_t\}$ with values in U. Such a control u^* - if it exits - is called an optimal control and Φ is called the optimal performance or the value function.

Chapter 2

Stochastic Control

Let $S \in \mathbb{R}^k$ be a fixed domain which is the solvency region and consider a stochastic process $Y(t) = Y^u(t)$ of the form

$$dY(t) = b(Y(t), u(t))dt + \sigma(Y(t), u(t))dB(t)$$

$Y(0) = y \in \mathbb{R}^k$
where

$$b : \mathbb{R}^k \times u \longrightarrow \mathbb{R}^k \quad \sigma : \mathbb{R}^k \times u \longrightarrow \mathbb{R}^{k \times m}$$

are given functions and $u \in \mathbb{R}^k$ is a given Borel set. The process $u(t) = u(t, w)$ where $u(t, w) : [0, \infty) \times \varphi \longrightarrow u$ is the control process which is assumed to be adapted and cadlag. A process $u(t, w)$ is called $\mathcal{F}_t$-adapted if for each $t \geq 0$, the function $w \longrightarrow u(t, w)$ is $\mathcal{F}_t$-measurable. A cadlag process is one which is right continuous with left limits. We consider a performance criterion $J = J^{(u)}(y)$ of the form

$$J^{(u)}(y) = \mathbb{E}^{(y)}\left[\int_0^T f(Y(t), u(t))dt + g(Y(T))\right]$$

where $T = \inf\{t > 0 : Y^u(t) \notin S\}$ is the bankruptcy time and f and g are continuous functions.

Admissible Control

The control process u is said to be admissible if the corresponding stochastic differential equation(S.D.E)

$$dY(t) = b(Y(t), u(t))dt + \sigma(Y(t), u(t))dB(t)$$
$$Y(0) = y \in \mathbb{R}^k$$

has a unique strong solution $Y(t)$ for all $y \in S$ and

$$\mathbb{E}^{(y)}\left[\int_0^T f(Y(t), u(t))dt + g(Y(T))\right] < \infty$$

Thus if we let $\mathcal{A}$ be the set of all admissible controls, then the stochastic control problem is to find the value function $\Phi(y)$ and an optimal control $u^\star \in \mathcal{A}$ defined by $\Phi(y) = \sup_{u\in\mathcal{A}} J^{(u)}(y) = J^{(u^\star)}(y)$ where the supremum is taken over all $\mathcal{F}_t^{(m)}$-adapted process u_t with values in U.

Such a control $u^\star$ if it exists - is called an optimal control and Φ is called the optimal performance or the value function.

Examples of types of Controls

(1) Deterministic or Open Loop Controls:
In this type of control, we have functions of the form $u(t, w) = u(t)$ i.e., a function that does not depend on w. It is one in which the output of the system is not involved in its control.

(2) Closed Loop or Feedback Controls:
In this case, we have processes u_t which are $\mathcal{M}_t$-adapted i.e., for each t, the

function $w \longrightarrow U(t,w)$ is $\mathcal{M}_t$-measurable, where $\mathcal{M}_t$ is the σ-algebra generated by $X_s^u; s \leq t$. Generally, it compares the output(or some function of the output) with the input and forms an error actuating signal from their difference.

(3) Linear Filtering problem and Deterministic Controls:
In this type of control, the controller has only partial knowledge of the state of the system and these controls are the ones in which the stochastic control problem will be linked to the filtering problem and it forms the so called stochastic linear regulator problem. The controller only has (noisy) observations R_t of X_t given by an Ito process of the form

$$dR_t = a(t, X_t)dt + \gamma(t, X_t)d\hat{B}_t$$

where $\hat{B}$ is a Brownian motion(not necessarily related to B). Hence, the control process u_t must be adapted with respect to the σ-algebra $\mathcal{N}_t$ generated by $R_t; s \leq t$. If the above equation is linear and the performance function is integral quadratic, then the stochastic control problem splits into a linear filtering problem and a corresponding deterministic control problem. This is called the Separation Principle.

(4) Markov controls:
Under these controls, the functions $u(t,w)$ are of the form $u(t,w) = u_0(t, X_t(w))$ for some function $u_0 : \mathbb{R}^{n+1} \rightarrow u \subset \mathbb{R}^k$.
We will assume that u does not depend on the starting point $y = (s,x)$. The value that we choose at time t only depends on the state of the system at this time. These are called Markov controls because with such u, the corresponding process X_t becomes an Ito diffusion, in particular a Markov process.

2.0.4 Dynamic Programming

We are going to consider Markov controls only i.e., $u = u(t, w)$ and introducing $Y_t = (s + t, X_{s+t})$, the system equation becomes

$$dY(t) = b(Y(t), u(t))dt + \sigma(Y(t), u(t))dB(t)$$

Then for each choice of the function u, the solution $Y_t = Y_t^u$ is an Ito diffusion with generator A given by

$$\begin{aligned} A\phi(y) &= A^u\phi(y) \\ &= \frac{\partial \phi}{\partial t}(y) + \sum_{i=1}^{k} b_i(y, u(y))\frac{\partial \phi}{\partial y_i}(y) + \frac{1}{2}\sum_{i,j=1}^{k}(\sigma\sigma^T)_{ij}(y, u(y))\frac{\partial^2 \phi}{\partial y_i \partial y_j}(y) \end{aligned}$$

If only Markov controls are considered, the Hamilton Jacobi-Bellman(HJB) equations provide a very nice solution to the stochastic control problem. It is interesting to note that considering Markov controls only is too restrictive but fortunately one can always obtain as good performance with a Markov control as with an arbitrary $\mathcal{F}_t^{(m)}$-adapted control, at least if some extra conditions are satisfied.

In this section we will formulate a verification theorem for the optimal control problem which is analogous to the classical Hamilton-Jacobi-Bellman(HJB) for (continuous) Ito diffusions.

Theorem 5 *(HJB for optimal control)(See [2])*
Let ϕ be a bounded function and twice differentiable i.e., $\phi \in C^2(S) \cap C(\bar{S})$ and suppose the function ϕ satisfies the following:
(i)

$$A^v\phi(y) \;+\; f(y, v) \quad \leq 0 \quad \forall y \in S, \quad v \in U.$$

(ii)

$$\phi(Y(T)) \;\in\; \partial S \;\; if \;\; T < \infty \;\; \forall\, u \in \mathcal{A}$$

(iii)

$$\lim_{t\to T}\phi(Y(t)) \;=\; g(Y(T)) \;\; a.s \;\; \forall \;\; u\in\mathcal{A}, \;\; T<\infty$$

(iv)

$$\{\phi(Y(\tau))\}_{\tau\leq T}$$

is uniformly integrable $\forall\ u\in\mathcal{A}$ *and* $y\in S$

(v)

$$\mathbb{E}^y\left[\int_0^T\{|\ \sigma^T(Y(t))\bar{v}\phi(Y(t))\ |^2\}dt\right]<\infty$$

Then $\phi(y) \;\geq\; \Phi(y) \;\;\forall\;\; y\in S$

Moreover, suppose that for each $y\in S$*, there exists* $v=\hat{u}(y)\in U$ *such that*

(vi)

$$A^{\hat{u}(y)}\phi(y) \; + f(y,\hat{u}(y)) \; = 0$$

and

(vii)

$$\{\phi(Y^{(\hat{u})}(\tau))\}_{\tau\leq T}$$

is uniformly integrable.

Suppose $u^*(t) \;:=\; \hat{u}(Y(t)) \;\in\mathcal{A}$

Then u^* *is an optimal control and*

$$\phi(y) \quad = \quad \Phi(y) \quad = \quad J^{(u^*)}(y) \;\;\forall y\in S$$

Proof 3 *Assume that* ϕ *satisfies (i) and(iii).*

Let $u\in\mathcal{A}$*. Put* $T_n=\min(n,T)$ *where* $n=1,2,3,....$

Then by the Dynkin formula, we have

$$\mathbb{E}^y[\phi(Y(T_n))] = \phi(y) + \mathbb{E}^y\left[\int_0^{T_n} A^u\phi(Y(t))dt\right]$$

But from (i)

$$A^u\phi \leq -f^u$$

Thus

$$\mathbb{E}^y[\phi(Y(T_n))] \ \leq \ \phi(y) - \mathbb{E}^y\left[\int_0^{T_n} f(Y(t), u(t))dt\right]$$

This gives

$$\phi(y) \ \geq \ \mathbb{E}^y\left[\int_0^{T_n} f(Y(t), u(t))dt + \phi(Y(T_n))\right]$$

Taking the limit as $n \to \infty$, we get

$$\phi(y) \ \geq \ \lim_{n\to\infty} \inf \mathbb{E}^y\left[\int_0^{T} f(Y(t), u(t))dt + g(Y(T))\right] \quad = \quad J^{(u)}(y)$$

Since $u \in \mathcal{A}$ was arbitrary, we conclude that

$$\phi(y) \geq \Phi(y) \quad \forall y \in S...........(1)$$

Now if $u(t) \ = \ \hat{u}(Y(t))$ is such that (vi) holds, then by the above calculations we have

$$\phi(y) \ = \ J^{(\hat{u})}(y) \leq \Phi(y)...........(2)$$

Combining equations (1) and (2), we get

$$\phi(y) \quad = \quad \Phi(y) \quad = \quad J^{(u^*)}(y) \qquad \forall y \in S$$

Example 1 *See [3]*

Suppose the wealth $X(t) = X^{(u)}(t)$ of a person with consumption rate $u(t) \geq 0$ satisfies the following mean reverting Ornstein-Uhlenbeck stochastic differential equation

$$dX(t) = (\mu - \rho X(t) - u(t))dt + \sigma dB(t), \quad t > 0$$

$X(0) = x > 0$

Fix $T > 0$ *and define*

$$J^{(u)}(s,x) = \mathbb{E}^{s,x}\left[\int_0^{T_o - s} e^{-\delta t}\frac{u^\gamma(t)}{\gamma}dt + \lambda X(T_o - s)e^{-\delta T}\right]$$

where $\mu, \rho, \sigma, \theta, T, \delta > 0, \gamma \in (0,1)$ *and* $\lambda > 0$ *are constants.*

Solution

In this case we are going to use dynamic programming to find the value function $\Phi(s,x)$ *and the optimal consumption rate(control)* $u^\star(t)$ *such that*

$$\Phi(s,x) = \sup_{u(.)} J^{(u)}(s,x) = J^{(u^\star)}(s,x)$$

Firstly, note that

$$\mathbb{E}^{s,x}\left[\int_s^{T_o} e^{-\delta t}\frac{u^\gamma(t)}{\gamma}dt\right] = \mathbb{E}^{s,x}\left[\int_0^{T} e^{-\delta(s+t)}\frac{u^\gamma(t)}{\gamma}dt\right]$$

where $T = T_o - s = \inf\{t > 0; Y^{s,x}(t,x) \notin G\}$ *with* $G = \{(s,x); s < T_o\}$.

Let $Y(t) = [s+t, X(t)]$ *for* $t \geq 0$, $Y = (s,x)$.

We let A *be the differential operator which coincides with the generator of* $Y(t)$, *then the generator of* $Y(t)$ *is*

$$\begin{aligned} A^u\phi(y) &= A^u\phi(s,x) \\ &= \frac{\partial \phi}{\partial s} + (\mu - \rho x - u)\frac{\partial \phi}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 \phi}{\partial x^2} \end{aligned}$$

So the conditions of the HJB for optimal control get the form

(i) $A^u\phi(s,x) + e^{-\delta s}\frac{u^\gamma(t)}{\gamma} \leq 0 \quad \forall\ u \geq 0, s < T$

(ii) $\phi(Y(T)) \in \partial s$ *if* $T < \infty \quad \forall u \in \mathcal{A}$

(iii) $\mathbb{E}^y\left[\int_0^T\{\mid \sigma(Y(t))\nabla\phi(Y(t)) \mid^2\}dt\right] < \infty$

(iv) $\phi(T,x) = \lambda x e^{-\delta T}$

(v) $\{\phi(Y(\tau))\}_{\tau<T}$ *is uniformly integrable.*

(vi) $A^{\hat{u}}\phi(s,x) + e^{-\delta s}\frac{\hat{u}^\gamma}{\gamma} = 0$ *for* $s < T$

(vii) $\{\phi(Y^{(\hat{u})}(\tau))\}_{\tau\leq T}$ *is uniformly integrable.*

Now we try a function ϕ *of the form*

$$\phi(s,x) = h(s) + k(s)x$$

for suitable functions h(s), k(s). Then the conditions (i)-(vii) get the form

$$h'(s) + k'(s)x + (\mu - \rho x - u)k(s) + e^{-\delta s}\frac{u^\gamma}{\gamma} \leq 0$$

for all $s < T_o, \quad u \geq 0.$

(iv)' $k(T) = \lambda, \qquad h(T) = 0$

(vi)' $h'(s) + k'(s)x + (\mu - \rho x - \hat{u})k(s) + e^{-\delta s}\frac{\hat{u}^\gamma}{\gamma} = 0$ *where* $\hat{u}$ *is a possible candidate for optimal control.*

(vii)' The function $\{h(\tau) + k(\tau)X(\tau)\}_{\tau\leq T}$ *is uniformly integrable.*

Let

$$g(u) = h'(s) + k'(s)x + (\mu - \rho x - u)k(s) + e^{-\delta s}\frac{u^\gamma}{\gamma}$$

then the maximum of g is obtained at the critical points

i.e., when $\frac{\partial g}{\partial u} = 0.$

Thus we have

$$e^{-\delta s}u^{\gamma-1} - k(s) = 0$$

Therefore

$$u = \hat{u} = (k(s)e^{\delta s})^{\frac{1}{\gamma-1}}$$

From (vi)' we have

(1) $k'(s) - \rho k(s) = 0, \qquad k(T) = \lambda$

$\Rightarrow k(s) = Ae^{\rho s}$

$\Rightarrow k(s) = \lambda e^{\rho(s-T)}$

(2) $h'(s) + (\mu - \hat{u})k(s) + e^{-\delta s}\frac{\hat{u}^\gamma}{\gamma} = 0$

Therefore $h'(s) = e^{\frac{\delta s}{\gamma-1}}k(s)^{\frac{\gamma}{\gamma-1}}[1 - \frac{1}{\gamma}] - \mu k(s) \quad < \quad 0$

Hence, since $h(T_o) = 0$*, for* $s < T_o$ *we have* $h(s) > 0$

Therefore we conclude that

$$\phi(s, x) = h(s) + k(s)x \geq 0$$

and we are left to prove that the function ϕ *satisfies all the conditions of the theorem.*

i)

$$A^v\phi(y) + f(y, v) \leq 0$$

This is true by construction of the function.

ii)

$$\phi(Y(T)) = h(T) + k(T)x = 0 + \lambda x = \lambda x \in \partial S$$

iii)

$$\lim_{t\to T}\phi(Y(t)) = \lim_{t\to T}(h(t) + k(t)x) = h(T) + k(T)x = \lambda x = g(Y(T)) \ a.s \ \forall u$$

We show that (iv)-(vii) also hold and conclude that

$$\hat{u}(s) = \lambda^{\frac{1}{\gamma-1}} \exp\left\{\frac{(\delta + \rho)s - \rho T}{\gamma - 1}\right\} \quad s < T$$

is the optimal control.

Example 2 *Consider the stochastic control problem*

$$dX_t = audt + udB_t, \qquad X_0 = x > 0$$

where $B_t \in \mathbb{R}$ *and* $a \in \mathbb{R}$ *is a given constant and*

$$\Phi(s,x) = \sup_u \mathbb{E}^{s,x}[(X_T)^r]$$

where $0 < r < 1$ *and* T *is the minimum between* τ_0 *and* t_1 *where* $\tau_0 = \inf\{t > s; X_t = 0\}$ *and* t_1*, being a given future time(constant).*

Solution

Using the HJB theorem, we must have

$$\frac{\partial \phi}{\partial s} + au\frac{\partial \phi}{\partial x} + \frac{1}{2}u^2\frac{\partial^2 \phi}{\partial x^2} \leq 0$$

Since $F^u(Y_t) = 0$

$$\phi(Y_t) = (X_T)^r \quad \textit{for all} \quad y \in \partial_R G$$

Thus

$$\sup_v \left\{ \frac{\partial \phi}{\partial s} + av\frac{\partial \phi}{\partial x} + \frac{1}{2}v^2\frac{\partial^2 \phi}{\partial x^2} \right\} = 0$$

Define

$$h(v) = \frac{\partial \phi}{\partial s} + av\frac{\partial \phi}{\partial x} + \frac{1}{2}v^2\frac{\partial^2 \phi}{\partial x^2} \quad \textit{for fixed} \quad t \quad \textit{and} \quad x.....(1)$$

The optimal control is found by differentiating $h(v)$ *with respect to* v*. If* $\frac{\partial^2 \phi}{\partial x^2} > 0$*, no maximum exists.*

Assume $\frac{\partial^2 \phi}{\partial x^2} < 0$*, then*

$$h'(v) = a\phi_x + v\phi_{xx}$$

$h'(v) = 0$ *iff*

$$v = -a\frac{\phi_x}{\phi_{xx}} = u^*(Y_t)..........(2)$$

where $\phi_x = \frac{\partial \phi}{\partial x}$ *and* $\phi_{xx} = \frac{\partial^2 \phi}{\partial x^2}$

NB: $v = u^*(Y_t)$ *is our candidate for optimal control.*

We must check that $h(u^*) = 0$. *Thus*

$$h(u^*) = \frac{\partial \phi}{\partial s} + au^*\phi_x + \frac{1}{2}u^{*2}\phi_{xx}$$

Taking (1) and (2), then

$$\frac{\partial \phi}{\partial s} + a(\frac{-a\phi_x}{\phi_{xx}})\phi_x + \frac{1}{2}\left(\frac{-a\phi_x}{\phi_{xx}}\right)^2\phi_{xx} = 0$$

Which simplifies to

$$\frac{\partial \phi}{\partial s} - \frac{a^2}{2}\frac{(\phi_x)^2}{\phi_{xx}} = 0............(3)$$

Since $K(Y_t) = X_t^r$, *our trial solution is* $\phi(t, x) = g(t)x^r$. *Thus (3) becomes*

$$g'(t) - \frac{a^2 r}{2(r-1)}g(t) = 0$$

Solving and applying the terminal conditions $g(T) = 1$, *we get that*

$$g(t) = \exp\left(-\frac{a^2 r}{2(r-1)}(t-T)\right)$$

and thus

$$\phi(t, x) = x^r \exp\left(-\frac{a^2 r}{2(r-1)}(t-T)\right)$$

Thus our optimal control becomes

$$u^*(Y_t) = \frac{ax}{1-r}$$

Now to calculate the optimal performance. Our Stochastic differential equation becomes

$$dX_t = \frac{a^2 x}{1-r}dt + \frac{ax}{1-r}dB_t$$

Solving it we get

$$X_t = x\exp\left(\frac{(1-2r)a^2 t}{2(1-r)^2} + \frac{a}{1-r}B_t\right)$$

Now to calculate the value function,

$$\Phi(s, x) = \mathbb{E}[(X_T)^r]$$

$$\Phi(s,x) = \mathbb{E}\left[x^r \exp\left(\frac{(1-2r)a^2rT}{2(1-r)^2} + \frac{ar}{1-r}B_T\right)\right]$$

Which simplifies to

$$\Phi(s) = x^r \exp\left(\frac{a^2r(T-s)}{2(1-r)}\right)$$

Stochastic Control Problems with Terminal Conditions

Most types of Markov controls u that are considered in many applications have constraints. As an example, they have constraints in terms of the probabilistic behavior of Y_t^u at the terminal time $t = T$. If we have such problems, they can often be handled by applying a kind of "Lagrange multiplier" method as described below.

Consider the problem of finding the value function $\Phi(y)$ and the optimal control $u^*(y)$ such that

$$\Phi(y) = \sup_{u \in U} J^u(y) \tag{2.1}$$

where

$$J^u(y) = \mathbb{E}^y\left[\int_0^T F^u(Y_t^u)dt + K(Y_T^u)\right] \tag{2.2}$$

and where the supremum is taken over the space $\mathcal{K}$ of all Markov controls $u : \mathbb{R}^{n+1} \to U \subset \mathbb{R}^k$ such that

$$\mathbb{E}^y[M_i(Y_T^u)] = 0, \qquad i = 1, 2, ..., l$$

where $M = (M_1, ..., M_l) : \mathbb{R}^{n+1} \to \mathbb{R}^l$ is a given continuous function,

$$\mathbb{E}^y[| \; M_i(Y_T^u) \; |] < \infty$$

for all y, u.

Now we introduce a related unconstrained problem as follows:

For each $\lambda \in \mathbb{R}^l$ and each Markov control u define

$$J_\lambda^u(y) = \mathbb{E}^y\left[\int_0^T F^u(Y_t^u)dt + K(Y_T^u) + \lambda \cdot M(Y_T^u)\right] \tag{2.3}$$

where $\cdot$ denotes the inner product in $\mathbb{R}^l$. The problem will be to find $\Phi_\lambda(y)$ and $u^*_\lambda(y)$ such that

$$\Phi_\lambda(y) = \sup_{u \in U} J^u_\lambda(y) = J^{u^*_\lambda}_\lambda(y) \tag{2.4}$$

without terminal conditions.

Theorem 6 *Suppose that for all $\lambda \in \Lambda \subset \mathbb{R}^l$ we can find $\Phi_\lambda(y)$ and u^*_λ solving the unconstrained stochastic control problem (2.3-2.4). Moreover, suppose that there exists $\lambda_o \in \Lambda$ such that*

$$\mathbb{E}^y[M(Y_T^{u^*_{\lambda_o}})] = 0$$

Then $\Phi(y) := \Phi_{\lambda_o}(y)$ and $u^* := u^*_{\lambda_o}$ solves the constrained stochastic control problem (2.1-2.2).

Proof 4 *Let u be a Markov control and $\lambda \in \Lambda$. Then by the definition of u^*_λ we have*

$$\mathbb{E}^y\left[\int_0^T F^{u^*_\lambda}(Y_t^{u^*_\lambda})dt + K(Y_T^{u^*_\lambda}) + \lambda M(Y_T^{u^*_\lambda})\right] = J^{u^*_\lambda}_\lambda(y)$$

which follows that

$$\begin{aligned} J^{u^*_\lambda}_\lambda(y) &\geq J^u_\lambda(y) \\ &= \mathbb{E}^y\left[\int_0^T F^{u_\lambda}(Y_t^{u_\lambda})dt + K(Y_T^{u_\lambda}) + \lambda M(Y_T^{u_\lambda})\right] \text{.............}(a) \end{aligned}$$

In particular, if $\lambda = \lambda_o$ and $u \in \mathcal{K}$ then

$$\mathbb{E}^y\left[M(Y_T^{u^*_{\lambda_o}})\right] = 0 = \mathbb{E}^y\left[M(Y_T^u)\right]$$

and hence by (a), we have

$$J^{u^*_{\lambda_o}}(y) \geq J^u(y) \quad \textit{for all} \quad u \in \mathcal{K}$$

2.0.5 The Maximum Principle For Stochastic Control

The maximum principle states that "Any optimal control along with optimal state trajectory must solve the Hamiltonian system, which is a two point boundary value problem plus a maximum condition of a function called the Hamiltonian". The mathematical significance of the maximum principle lies in that maximising the Hamiltonian is much easier than the original control problem that is infinite dimensional. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a given filtered probability space satisfying the usual conditions on which an m-dimensional standard Brownian motion $B(t)$ is given. We consider the following controlled system

$$dX(t) = b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dB(t), \quad t \in [0, T]$$

$$X(0) = x \tag{2.5}$$

with the performance criterion $J(u)$ of the form

$$J(u(.)) = \mathbb{E}\Big[\int_0^T f(t, x(t), u(t))dt + g(X(T))\Big] \tag{2.6}$$

where

$$b : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n$$

$$\sigma : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^{n\times m}$$

$$f : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}$$

and

$$g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

The process $u(t) = u(t, w), \quad t \in [0, T], \quad w \in \Omega$ is our control process, and has values in a given closed set $U \subset \mathbb{R}^k$. We also require that $u(t, w)$ gives rise to a unique strong solution $X(t) = X^{(u)}(t)$ of (1) for $t \in [0, T]$.

If

$$\mathbb{E}\Big[\int_0^T \mid f(t, X(t), u(t)) \mid dt + \max\{0, g(X(T))\}\Big] < \infty$$

then such controls are admissible and the set of all admissible controls is denoted by $\mathcal{A}$. Thus, if $u \in \mathcal{A}$ and $X(t) = X^{(u)}(t)$ is the corresponding solution of (2.5), we call (u, X) an admissible pair.

We define

$$b(t,x,u) = (b_1(t,x,u), ..., b_n(t,x,u))'$$
$$\sigma(t,x,u) = (\sigma^1(t,x,u), ..., \sigma^m(t,x,u))$$

where

$$\sigma^j(t,x,u) = (\sigma_{1j}(t,x,u), ..., \sigma_{nj}(t,x,u))', \quad 1 \leq j \leq n$$

We now make the following assumptions:

a) $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by B_t augmented by all the P-null sets in f.

b) (U, d) is a separable metric space and $T > 0$.

c) The maps $b, \sigma, f$and h are measurable and there exists a constant $L > 0$ and a modulus of continuity $\overline{w} : [0, \infty) \to [0, \infty)$ such that for

$$\varphi(t,x,u) = b(t,x,u), \sigma(t,x,u), f(t,x,u), h(x)$$

we have

$$| \varphi(t,x,u) - \varphi(t,\hat{x},\hat{u}) | \leq L \mid x - \hat{x} \mid + \overline{w}(d(u,\hat{u})) \;\; \forall \; t \in [0,T], \quad x, \hat{x} \in \mathbb{R}^n, \quad u, \hat{u} \in U$$

$$| \varphi(t,0,u) | \leq L; \qquad \forall \; (t,u) \in [0,T] \times U$$

d) The maps b, σ, f and h are C^2 in x. Moreover there exist a constant $L > 0$ and a modulus of continuity $\overline{w} : [0, \infty) \to [0, \infty)$ such that for $\varphi = b, \sigma, f, h$ we have

$$| \varphi_x(t,x,u) - \varphi_x(t,\hat{x},\hat{u}) | \;\; \leq \;\; L \mid x - \hat{x} \mid + \overline{w}(d(u,\hat{u}))$$

$$| \varphi_{xx}(t,x,u) - \varphi_{xx}(t,\hat{x},\hat{u}) | \leq \overline{w}(| x - \hat{x} | + d(u,\hat{u}))$$

The first assumption signifies that the system noise is the only source of uncertainty in the problem and the past information about the noise is available to the controller. Now we define $U[0,T] := \{u : [0,T] \times \Omega \to u : u \text{ is } \mathcal{F}_t - \text{adapted}\}$. For any $u(.) \in U[0,T]$, the state equation (2.5) admits a unique solution $x(.) = x(.,u(.))$ and the cost functional (2.6) is well defined. In the case that $x(.)$ is the solution of (2.5) corresponding to $u(.) \in u[0,T]$, we call $(x(.),u(.))$ an admissible pair and $x(.)$ an admissible state process.

Now we can state our problem as to maximize (2.5) over $u[0,T]$ and any $\bar{u}(.) \in U[0,T]$ satisfying

$$J(u^*) = \sup_{u \in \mathcal{A}} J(u)$$

is called an optimal control.

If $X^* = X^{(u^*)}$ is the corresponding solution of (2.5), then (X^*,u^*) is called an optimal pair.

We now introduce the Hamiltonian $H : [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^{n \times m} \to \mathbb{R}$ which is given by

$$H(t,x,u,p,q) = f(t,x,u) + b^T(t,x,u)p + tr(\sigma^T(t,x,u)q)$$

We will assume that H is differentiable with respect to x. The adjoint equation(corresponding to the admissible pair (u,X) in the unknown adapted processes $p(t) \in \mathbb{R}^n, q(t) \in \mathbb{R}^{n \times m}$ is the backward stochastic differential equation

$$dp(t) = -\nabla_x H(t,X(t),u(t),p(t),q(t))dt + q(t)dB(t)$$

with terminal conditions

$$p(T) = \nabla_x g(X(T)) \qquad t < T \tag{2.7}$$

The above equation is the first order adjoint equation and p(.) is the first order adjoint process. From now on, we will assume that

$$\mathbb{E}\left[\int_0^T\{\sigma\sigma^T(t,X(t),u(t))\}dt\right] \;<\; \infty \;\; \forall u\in\mathcal{A}$$

From equation (2.7), the unknown is a pair of $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $(p(.),q(.))$. We call the equation a backward stochastic differential equation(BSDE). The key issue is that the equation is to be solved backwards since the terminal value is given and the solution $(p(.),q(.))$ is required to be $\{\mathcal{F}_t\}_{t\geq 0}$-adapted.

The adjoint variable $p(.)$ corresponds to the so called price or the marginal value of the resource represented by the state variable in economic theory. In this section, we have noted that the maximum principle is nothing but the so called duality principle : **Minimising the total cost amounts to maximising the total contribution of the marginal value**.

Theorem 7 *A Sufficient Maximum Principle.*
Let $(\hat{u},\hat{X})$ be an admissible pair and suppose that there exists an adapted solution $(\hat{p}(t),\hat{q}(t))$ of the corresponding adjoint equation (4) satisfying

$$\mathbb{E}\left[\int_0^T\{\hat{q}\hat{q}^T(t)\}dt\right] < \infty$$

Moreover, suppose that

$$H(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t)) = \sup_{u\in U} H(t,\hat{X}(t),u(t),\hat{p}(t),\hat{q}(t))$$

for all $t\in[0,T]$ and that

$$\hat{H}(x) := \max_{u\in U} H(t,x,u,\hat{p}(t),\hat{q}(t))$$

exists and is a concave function of x for all $t\in[0,T]$.
Then the pair $(\hat{u},\hat{X})$ is an optimal pair.

For more on this and the proof of the theorem, details in [6]

Relation to Dynamic Programming

Without proof, we just state the relation between the Maximum Principle and Dynamic Programming. In the diffusion case, the relation between the maximum principle and Dynamic Programming is well known.

Theorem 8 *Suppose the state $X(t) = X^{(u)}(t)$ of a controlled diffusion in $\mathbb{R}^n$ is given by*

$$dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dB(t)$$

Let $X^*(t)$ be the solution of the above stochastic differential equation corresponding to an optimal control $u^*(t)$. Then, under some conditions the two adjoint processes $p(t), q(t)$ for the jump diffusion case are given by

$$p_i(t) = \frac{\partial v}{\partial x_i}(t, X^*(t))$$

$$q_{ik}(t) = \sum_{j=1}^{n} \sigma_{jk}(t, X^*(t), u^*(t)) \frac{\partial^2 v}{\partial x_i \partial x_j}(t, X^*(t))$$

where $V(t, x)$is the value function.

Example 3 *Suppose the wealth $X(t) = X^{(u)}(t)$ of a person with consumption rate $u(t) \geq 0$ satisfies the following Levy type mean reverting Ornstein-Uhlenbeck stochastic differential equation*

$$dX(t) = (\mu - \rho X(t) - u(t))dt + \sigma dB(t) \quad t > 0$$

$X(0) = x > 0$

Fix $T > 0$ and define

$$J^{(u)}(s, x) = \mathbb{E}^{s,x}\left[\int_s^{T_o} e^{-\delta t}\frac{u^\gamma(t)}{\gamma}dt + \lambda X(T_o - s)\right]$$

where $\mu, \rho, \sigma, \theta, T, \delta > 0, \gamma \in (0, 1)$ and $\lambda > 0$ are constants. The Hamiltonian gets the form

$$\begin{aligned} H(t, x, u, p, q) &= f(t, x, u) + b^T(t, x, u)p + tr(\sigma^T(t, x, u)q) \\ &= e^{-\delta t}\frac{u^\gamma}{\gamma} + (\mu - \rho x - u)p + \sigma q \qquad 0 \leq t \leq T_0 \end{aligned}$$

and the adjoint equation becomes

$$dp(t) = -(-\rho p)dt + q(t)dB(t)$$
$$p(T) = \lambda$$

This implies that

$$dp(t) = \rho p dt + q(t)dB(t)$$
$$p(T) = \lambda$$

From the above equations, λ *and* ρ *are deterministic and so we guess that* $q(t) = 0$ *and this gives*

$dp(t) = \rho p dt$

$\Rightarrow lnp(t) = \rho t + c$ *where* c *is a constant of integration*

$\Rightarrow p(t) = Ae^{\rho t}$ *where* $A = e^c$

Applying boundary conditions

$p(T) = \lambda \Rightarrow \lambda = Ae^{\rho T} \Rightarrow A = \lambda e^{-\rho T}$

Therefore

$p(t) = \lambda e^{\rho(t-T)}$

Now

$$H(t, \hat{X}(t), u, \hat{p}(t), \hat{q}(t)) = e^{-\delta t}\frac{u^\gamma}{\gamma} + (\mu - \rho\hat{X}(t) - u)\hat{p}(t)$$

This is maximal when: $\frac{\partial H}{\partial u} = 0$

This implies that $e^{-\delta t}\hat{u}^{\gamma-1} - \hat{p}(t) = 0$

$$\begin{aligned} \Rightarrow \hat{u}^{\gamma-1} &= \hat{p}(t)e^{\delta t} \\ &= \lambda e^{\rho(t-T)+\delta t} \\ &= \lambda e^{(\rho+\delta)t-\rho T} \end{aligned}$$

Therefore

$$\hat{u}(t) = \lambda^{\frac{1}{\gamma-1}} e^{\frac{(\rho+\delta)t-\rho T}{\gamma-1}}$$

With $\hat{p}(t), \hat{q}(t)$ as given above, we see that all the conditions of the maximum principle are satisfied.

Chapter 3

The Problem

Suppose that the fortune at time t is given by $dX_t = u(t)dt + \sigma dB(t)$ where u_t is the consumption rate. The performance criterion $J^u(s,x)$ is the expected total discounted consumption i.e.,

$$J^u(s,x) = E\left[\int_0^T e^{-\delta t}(cX(t) + \theta u(t)^2)dt\right]$$

with the boundary conditions $X(0) = 0$, $E[X(T)] = M$ where $T, \delta, c, \theta, \sigma$ and M are constants.
We firstly note that the above problem is a minimisation problem.

$$\begin{aligned}\Phi(y) &= \inf_u J^u(y) = \inf_u E\left[\int_0^T e^{-\delta t}(cX(t) + \theta u(t)^2)dt\right] \\ &= -\sup_u \left\{E\left[\int_0^T -e^{-\delta t}(cX(t) + \theta u(t)^2)dt\right]\right\}\end{aligned}$$

Thus, the problem becomes a maximisation problem.

Solution

3.0.6 The Maximum Principle

In this case the problem can be reformulated as

$$-\sup_u E\left[\int_0^T -e^{-\delta t}(cX(t)+\theta u(t)^2)dt - e^{-\delta T}\lambda(X(T)-M)\right]$$

In this case, the Hamiltonian gets the form

$$H(t,x,u,p,q) = -cxe^{-\delta t} - \theta u^2 e^{-\delta t} + \sigma q + up \tag{3.1}$$

Hence the adjoint equations become

$$dp(t) = ce^{-\delta t}dt + q(t)dB(t)$$

$$p(T) = -\lambda e^{-\delta T} \tag{3.2}$$

Since λ and c are deterministic, we guess that $\hat{q}=0$ and this gives

$$\hat{p}(t) = \frac{-ce^{-\delta t} - (\delta\lambda - c)e^{-\delta T}}{\delta} \tag{3.3}$$

Let $\hat{u} \in \mathcal{A}$ be a candidate for the optimal control with corresponding $\hat{X}$ and $\hat{p}, \hat{q}$. Then

$$H(t,\hat{X}_t,u_t,\hat{p}_t,\hat{q}_t) = -c\hat{X}_t e^{-\delta t} - \theta u^2 e^{-\delta t} + u\hat{p} + \sigma\hat{q}$$

This is maximal when

$$\frac{\partial H}{\partial u} = 0 \Rightarrow -2\theta\hat{u}e^{-\delta t} + \hat{p} = 0$$

$$\Rightarrow \hat{u}_t = \frac{\hat{p}_t e^{\delta t}}{2\theta} \tag{3.4}$$

$$= \frac{-c-(\delta\lambda - c)e^{\delta(t-T)}}{2\delta\theta} \tag{3.5}$$

But $dX_t = u_t dt + \sigma dB_t$

Substituting for u_t and solving where $X(0)=0$, we get

$$X_t = -\frac{ct}{2\delta\theta} - \frac{(\delta\lambda - c)e^{\delta(t-T)}}{2\delta^2\theta} + \frac{(\delta\lambda - c)e^{-\delta T}}{2\delta^2\theta} + \sigma B_t \tag{3.6}$$

But we are given that $E[X(T)] = M$, thus

$$M = -\frac{cT}{2\delta\theta} - \frac{(\delta\lambda - c)}{2\delta^2\theta} + \frac{(\delta\lambda - c)e^{-\delta T}}{2\delta^2\theta} + \sigma T$$

Solving for λ, we get

$$\lambda = \frac{c}{\delta} + \frac{2\delta^2\theta M + c\delta T - 2\delta^2\theta\sigma T}{\delta(e^{-\delta T} - 1)}$$

Substituting for λ into (3.5), we get

$$\begin{aligned}\hat{u}(t) &= -\frac{c}{2\delta\theta} - \frac{e^{\delta(t-T)}}{2\delta\theta}\left(\frac{2\delta^2\theta M + c\delta T - 2\delta^2\theta\sigma T}{e^{-\delta T} - 1}\right) \\ &= \frac{c(1 - e^{\delta T}) + \delta e^{\delta t}(2\delta\theta M + cT - 2\delta\theta\sigma T)}{2\delta\theta(-1 + e^{\delta T})} \text{.........................}(a)\end{aligned}$$

Also

$$\begin{aligned}X_t &= -\frac{ct}{2\delta\theta} - \frac{e^{\delta(t-T)}}{2\delta^2\theta}\left(\frac{2\delta^2\theta M + c\delta T - 2\delta^2\theta\sigma T}{e^{-\delta T} - 1}\right) \\ &+ \frac{e^{-\delta T}}{2\delta^2\theta}\left(\frac{2\delta^2\theta M + c\delta T - 2\delta^2\theta\sigma T}{e^{-\delta T} - 1}\right) + \sigma B_t \\ &= \frac{-ct(-1 + e^{\delta T}) + (2\delta\theta M + cT - 2\delta\theta\sigma T)(-1 + e^{\delta t}) + 2\delta\theta\sigma B_t(-1 + e^{\delta T})}{2\delta\theta(-1 + e^{\delta T})} \text{...}(b)\end{aligned}$$

and

$$\begin{aligned}p(t) &= -\frac{c}{\delta}e^{-\delta t} - \frac{e^{-\delta T}}{\delta}\left(\frac{2\delta^2\theta M + c\delta T - 2\delta^2\theta\sigma T}{e^{-\delta T} - 1}\right) \\ &= -\frac{c}{\delta}e^{-\delta t} + \frac{2\delta\theta M + cT - 2\delta\theta\sigma T}{-1 + e^{\delta T}} \text{.................}(c)\end{aligned}$$

We are also going to consider the dynamic programming approach in solving the above problem.

3.0.7 Dynamic Programming Approach

Since we are given the probabilistic behavior of X_t^u at the terminal time $t = T$, we apply a kind of "Lagrange Multiplier" method and our value function becomes

$$\begin{aligned}\Phi(y) &= \inf_{u\in\mathcal{A}} J^{(u)}(y) = -\sup_{u\in\mathcal{A}}\{-J^{(u)}(y)\} \\ &= -\sup_{u\in\mathcal{A}} E\left[\int_0^T -e^{-\delta t}(cX(t) + \theta u(t)^2)dt - \lambda(X(T) - M)e^{-\delta T}\right]\end{aligned}$$

where $\lambda \in \mathbb{R}$.

Thus from the HJB's

$$\begin{aligned}0 &= \sup_v \left\{F^v(t,x) + (L^v\psi)(t,x)\right\} \\ &= \frac{\partial\psi}{\partial t} + \sup_v \left\{ -cxe^{-\delta t} - \theta v^2 e^{-\delta t} + v\frac{\partial\psi}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2\psi}{\partial x^2}\right\}\end{aligned}$$

and $\psi(T,x) = -\lambda(x - M)e^{-\delta T}$

We try to find a solution ψ of the form

$$\psi(t,x) = a(t)x + b(t)...........(1)$$

where $a(t), b(t)$ are deterministic continuous and differentiable functions. We need to find $a(t)$ and $b(t)$ such that

$$\sup_v \left\{F^v(t,x) + (L^v\psi)(t,x)\right\} = 0 \quad \text{for} \quad t < T$$

and $\psi(T,x) = -\lambda(x - M)e^{-\delta T}$............(2)

where $a(T) = -\lambda e^{-\delta T}$ and $b(T) = \lambda M e^{-\delta T}$ for us to get (2).

Therefore, for each (t,x), we try to find the value $v = u(t,x)$ which maximizes the function

$$\begin{aligned}H(v) &= F^v(t,x) + (L^v\psi)(t,x) \\ &= a'(t)x + b'(t) - cxe^{-\delta t} - \theta v^2 e^{-\delta t} + va(t).....(3)\end{aligned}$$

The maximum of this expression is obtained when $\frac{\partial H}{\partial v} = 0$

i.e when $-2\theta v e^{-\delta t} + a(t) = 0$

Therefore

$$v = \hat{u}(t,x) = \frac{a(t)}{2\theta} e^{\delta t} ....(4)$$

If we substitute (4) into the HJB equation (3), we get the following

$$\begin{aligned} 0 &= a'(t)x + b'(t) - cxe^{-\delta t} - \theta(\frac{a(t)}{2\theta}e^{\delta t})^2 e^{-\delta t} + (\frac{a(t)}{2\theta}e^{\delta t})a(t) \\ &= a'(t)x + b'(t) - cxe^{-\delta t} + \frac{a(t)^2}{4\theta}e^{\delta t} \end{aligned}$$

This is zero when

(1) $a'(t) - ce^{-\delta t} = 0 \quad \Rightarrow \quad a'(t) = ce^{-\delta t}$

$$a(t) = -\frac{c}{\delta}e^{-\delta t} + E_1 \quad \text{But} \quad a(T) = -\lambda e^{-\delta T}$$

Thus

$$E_1 = -\lambda e^{-\delta T} + \frac{c}{\delta}e^{-\delta T}$$

Therefore

$$a(t) = \frac{-ce^{-\delta t} + (c - \delta\lambda)e^{-\delta T}}{\delta} ....(5)$$

(2) $b'(t) + \frac{a^2(t)e^{\delta t}}{4\theta} = 0$

If we substitute for $a(t)$ and solving for $b(t)$ we get

$$b(t) = \frac{1}{4\theta\delta^2}\left(-\frac{c^2}{\delta}e^{-\delta t} - 2c^2te^{-\delta T} + \frac{c^2}{\delta}e^{\delta(t-2T)} + 2c\delta\lambda t - 2c\lambda e^{\delta(t-T)} + \delta\lambda^2 e^{\delta t}\right) + E_2$$

But $b(T) = \lambda M e^{-\delta T}$

Therefore

$$E_2 = \lambda M e^{-\delta T} - \frac{1}{4\theta\delta^2}(2c\lambda(\delta T - 1) + \delta\lambda^2 e^{\delta T} - 2c^2 T e^{-\delta T})$$

Thus from (4)

$$\begin{aligned} u(t) &= \frac{-c(e^{-\delta t} - e^{-\delta T})e^{\delta t} - \lambda\delta e^{\delta(t-T)}}{2\delta\theta} \\ &= -\frac{c}{2\delta\theta} - \frac{e^{\delta(t-T)}(\delta\lambda - c)}{2\delta\theta}, \quad t < T. \end{aligned}$$

With this value of u, we solve the differential equation for X_t

$$dX_t = u_t dt + \sigma dB_t$$

$$\Rightarrow X_t = -\frac{ct}{2\delta\theta} - \frac{(\delta\lambda - c)}{2\delta\theta} e^{\delta(t-T)} + \sigma B_t + E_5$$

But $X(0) = 0$

$$\Rightarrow E_5 = \frac{e^{-\delta T}}{2\delta^2\theta}(\delta\lambda - c)$$

Therefore

$$X_t = \frac{-ct}{2\delta\theta} + \frac{e^{-\delta T}}{2\delta^2\theta}(\delta\lambda - c)(1 - e^{\delta t}) + \sigma B_t$$

In order to solve for λ, applying Theorem (6), we have $E[X^{\lambda_o}(T)] = M$

$$\Rightarrow M = -\frac{cT}{2\delta\theta} + \frac{e^{-\delta T}}{2\delta^2\theta}(\delta\lambda_o - c)(1 - e^{\delta T}) + \sigma T$$

Thus we have for $\lambda = \lambda_o$

$$\lambda = \frac{c}{\delta} + \frac{2\delta\theta M + cT - 2\delta\theta\sigma T}{e^{-\delta T} - 1}$$

With this value of λ, then X_t becomes

$$X_t = (1 - e^{\delta t}) \left(\frac{2\delta M\theta}{2\delta\theta(1 - e^{\delta T})} - \frac{c(1 - \delta T - e^{-\delta T})}{2\delta^2\theta(1 - e^{\delta T})} - \frac{ce^{-\delta T}}{2\delta^2\theta} \right) - \frac{ct}{2\delta\theta} + \sigma B_t$$

$$= \frac{1}{2\delta\theta(-1 + e^{\delta T})} \left[(1 - e^{\delta t})(-cT - 2\delta M\theta) - ct(-1 + e^{\delta T}) + 2\delta\theta(-1 + e^{\delta T})\sigma B_t \right]$$

Now $u(t)$ becomes

$$\begin{aligned} u(t) &= \frac{1}{2\delta\theta(1 - e^{\delta T})}[-c + ce^{\delta T} - 2\delta^2 M\theta e^{\delta t} - c\delta T e^{\delta t} + 2\delta^2\theta\sigma B_T e^{\delta t}] \\ &= \frac{1}{2\delta\theta(1 - e^{\delta T})} \left[-c(1 - e^{\delta T}) + \delta e^{\delta t}(-cT - 2\delta M\theta + 2\delta\theta\sigma B_T) \right] \text{.............(6)} \end{aligned}$$

Since $u(t)$ is the consumption process, it is always positive and we also expect that its increasing with increasing time. We check these conditions on equations (a) and (6). Equation (6) simplifies to

$$\begin{aligned} u(t) &= \frac{1}{2\delta\theta(-1 + e^{\delta T})} \left[c(1 - e^{\delta T}) + \delta e^{\delta t}(cT + 2\delta M\theta - 2\delta\theta\sigma B_T) \right] \\ &= \frac{1}{2\delta\theta(-1 + e^{\delta T})} \left[c - ce^{\delta T} + c\delta T e^{\delta t} + 2\delta^2 M\theta e^{\delta t} - 2\delta^2\theta\sigma B_T e^{\delta t} \right] \\ &\geq 0 \text{ for } 2\delta^2 M\theta e^{\delta t} \geq 2\delta^2\theta\sigma B_T e^{\delta t} + c(e^{\delta T} - 1 - \delta T e^{\delta t}) \end{aligned}$$

Also

$$
\begin{aligned}
u'(t) &= \frac{\delta e^{\delta t}(cT + 2\delta M\theta - 2\delta\theta\sigma B_T}{2\theta(-1+e^{\delta T})} \\
&= \frac{2\delta^2 M\theta e^{\delta t} + c\delta T e^{\delta t} - 2\delta^2\theta\sigma B_T e^{\delta t}}{2\theta(-1+e^{\delta T})} \\
&\geq 0 \;\text{ since }\; 2\delta^2 M\theta \geq 2\delta^2\theta\sigma B_T - c\delta T
\end{aligned}
$$

Since the derivative is positive, it shows that the consumption is increasing with increasing time.

Now, we consider the deterministic case i.e., when $\sigma = 0$.

$$
\begin{aligned}
u(t) &= \frac{1}{2\delta\theta(-1+e^{\delta T})}\left[c(1-e^{\delta T}) + \delta e^{\delta t}(cT + 2\delta M\theta)\right] \\
&= \frac{1}{2\delta\theta(-1+e^{\delta T})}\left[c - ce^{\delta T} + c\delta T e^{\delta t} + 2\delta^2 M\theta e^{\delta t}\right] \\
&\geq 0 \;\text{ for }\; 2\delta^2 M\theta e^{\delta t} \geq c(e^{\delta T} - 1 - \delta T e^{\delta t})
\end{aligned}
$$

At $t = 0$, we have

$$
\begin{aligned}
u(t) &= \frac{1}{2\delta\theta(-1+e^{\delta T})}\left[2\delta^2 M\theta - c(e^{\delta T} - 1 - \delta T)\right] \\
&\geq 0 \;\text{ for }\; 2\delta^2 M\theta \geq c(e^{\delta T} - 1 - \delta T)
\end{aligned}
$$

For $t > 0$, we have

$$
\begin{aligned}
u'(t) &= \frac{\delta e^{\delta t}(cT + 2\delta M\theta)}{2\theta(-1+e^{\delta T})} \\
&\geq 0 \;\text{ for }\; 2\delta M\theta + cT \geq 0
\end{aligned}
$$

At $t = 0$, we have positive consumption yet we are starting with an initial fortune of zero. This means that the consumer will have to start consuming at some time $0 < s < T$. Thus we are faced with a new optimisation problem. We now solve the problem in the interval $[s, T_o]$.

In this case our performance criterion becomes

$$E^{s,x}[\int_s^{T_o} -e^{-\delta t}(cX(t) + \theta u(t)^2)dt - \lambda(X(T_o) - M)e^{-\delta T}]$$

$$= E[\int_0^{T} -e^{-\delta(s+t)}(cX(t) + \theta u(t)^2)dt - \lambda(X(T) - M)e^{-\delta T}]$$

where $T = T_o - s = \inf\{t > 0; Y^{s,x}(t,x) \notin G\}$ with $G = \{(s,x); s < T_o\}$

Just like in the previous case our u becomes

$$u(s) = \frac{-c + ce^{\delta(s-T)} - \delta\lambda e^{\delta s}}{2\delta\theta}$$

Now $dX_s = u_s ds + \sigma dB_s$

Solving, we get

$$X_s = \frac{-cs}{2\delta\theta} + \frac{c}{2\delta^2\theta}e^{\delta(s-T)} - \frac{\lambda}{2\delta\theta}e^{\delta s} + \sigma B_s + E_1$$

But in this case $X(0) = x_1 > 0$

Thus $E_1 = x_1 - \frac{c}{2\delta^2\theta}e^{-\delta T} + \frac{\lambda}{2\delta\theta}$

Therefore

$$X_s = x_1 + (\frac{\lambda}{2\delta\theta} - \frac{c}{2\delta^2\theta}e^{-\delta T})(1 - e^{\delta s}) - \frac{cs}{2\delta\theta} + \sigma B_s$$

Now $E[X^{\lambda_o}(T)] = M$, thus

$$M = x_1 + (\frac{\lambda_o}{2\delta\theta} - \frac{c}{2\delta^2\theta}e^{-\delta T})(1 - e^{\delta T}) - \frac{cT}{2\delta\theta} + \sigma B_T$$

Solving for λ_o and since $\lambda = \lambda_o$, we have

$$\lambda = \frac{1}{\delta(1 - e^{\delta T})}[2\delta^2\theta M - 2\delta^2\theta x_1 + c\delta T - 2\delta^2\theta\sigma B_T] + \frac{c}{\delta}e^{-\delta T}$$

Substituting this value of λ into u and simplifying, we get

$$u(s) = \frac{1}{2\delta\theta(1 - e^{\delta T})}[c(e^{\delta T} - 1) - c\delta T e^{\delta s} + 2\delta^2\theta(x_1 + \sigma B_T - M)e^{\delta s}]............(7)$$

Also X_s simplifies to

$$X_s = x_1 + \frac{1}{2\delta\theta(1 - e^{\delta T})}[(2\delta\theta M - 2\delta\theta x_1 + cT - 2\delta\theta\sigma B_T)(1 - e^{\delta s}) - cs(1 - e^{\delta T}) + 2\delta\theta\sigma B_s(1 - e^{\delta T})]$$

Considering equation (7)

$$u(s) = \frac{1}{2\delta\theta(-1+e^{\delta T})}[-c(-1+e^{\delta T}) + c\delta T e^{\delta s} + 2\delta^2\theta(M - x_1 - \sigma B_T)e^{\delta s}]$$

$$\begin{aligned} u'(s) &= \frac{1}{2\delta\theta(-1+e^{\delta T})}[c\delta^2 T e^{\delta s} + 2\delta^3\theta(M - x_1 - \sigma B_T)e^{\delta s}] \\ &= \frac{1}{2\delta\theta(-1+e^{\delta T})}[c\delta T e^{\delta s} + 2\delta^2\theta(M - x_1 - \sigma B_T)e^{\delta s}] \\ &\geq 0 \quad \text{for} \quad c \geq 0, \quad \theta > 0 \quad \text{and} \quad M \geq x_1 + \sigma B_T \end{aligned}$$

We shall summarise the results of the above problem as follows:

Theorem 9 *Let X_t^u be the controlled process and consider the following stochastic control problem*

$$dX_t = u_t dt + \sigma dB_t$$

with the performance criterion defined by

$$J^u(s,x) = E\left[\int_0^T e^{-\delta t}(cX(t) + \theta u(t)^2)dt\right]$$

where $X(0) = 0, E[X(T)] = M$ and $T, \delta, c, \theta, \delta, M$ are constants.
(a) The optimal value function is given by

$$\begin{aligned} \Phi(t,x) &= \frac{-ce^{-\delta t} + (c - \delta\lambda)e^{-\delta T}}{\delta}x \\ &+ \frac{1}{4\theta\delta^2}\left(-\frac{c^2}{\delta}e^{-\delta t} - 2c^2 t e^{-\delta T} + \frac{c^2}{\delta}e^{\delta(t-2T)} + 2c\delta\lambda t - 2c\lambda e^{\delta(t-T)} + \delta\lambda^2 e^{\delta t}\right) \\ &+ \lambda M e^{-\delta T} - \frac{1}{4\theta\delta^2}(2c\lambda(\delta T - 1) + \delta\lambda^2 e^{\delta T} - 2c^2 T e^{-\delta T}) \end{aligned}$$

(b) The control u^* is given by

$$u^*(t) = \frac{1}{2\delta\theta(1-e^{\delta T})}\left[-c(1-e^{\delta T}) + \delta e^{\delta t}(-cT - 2\delta M\theta + 2\delta\theta\sigma B_T)\right]$$

It is also of interest to investigate the behaviour of the control process and the value function for large values of T. Thus we come up with the following corollary.

Corollary 1 *For large values of T, i.e., as $T \to \infty$, our control process gets the form*

$$
\begin{aligned}
(a) \lim_{T\to\infty} u(t) &= \lim_{T\to\infty} \frac{1}{2\delta\theta(1-e^{\delta T})}\left[-c(1-e^{\delta T}) + \delta e^{\delta t}(-cT - 2\delta M\theta + 2\delta\theta\sigma B_T)\right] \\
&= -\frac{c}{2\delta\theta}, \quad c \geq 0, \quad \theta > 0.
\end{aligned}
$$

and is negative.

(b)

$$\lim_{T\to\infty} \Phi(t,x) \to \infty$$

The value function diverges for large values of T.

Remark

Now we consider the connection between the Dynamic Programming and the Maximum Principle.

$$p(t) = \frac{\partial \Phi}{\partial x} = \frac{-ce^{-\delta t} + (c-\delta\lambda)e^{-\delta T}}{\delta}$$

which is the same with p(t) in equation (3.3).

Also $q(t) = \frac{\partial^2\phi}{\partial x^2} = 0$ which compares well with our value of $q = 0$.

Conclusion

We note that for T small the consumption value or control is dependent on time only. But for large values of T the condition that $u(t) \geq 0$ is violated and our control in this case becomes constant. A well known case (see [1]), Chapter 11, Exercise 11.12 where both the state and control are quadratic i.e.,

$$J^u(s,x) = E\left[\int_s^\infty e^{-\delta t}(X^2(t) + \theta u^2(t))dt\right],$$

the control is feedback in form i.e., depends on the state and this is the only striking difference with the problem under consideration.

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