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Spanning paths in graphs

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ABSTRACT

The Conjecture, Graffiti.pc 190, of the computer program Graffiti.pc, instructed by DeLaviña, state that every simple connected graph G with minimum degree δ and leaf number $L(G)$ such that $\delta \geq \frac{1}{2}(L(G)+1)$, is traceable. Here, we prove a sufficient condition for a graph to be traceable based on minimum degree and leaf number, by settling completely, the Conjecture Graffiti.pc 190. We construct infinite graphs to show that our results are best in a sense. All graphs considered are simple. That is, they neither have loops nor multiple edges.

1. Introduction

Let $G = (V, E)$ be a simple, connected graph. The degree of a vertex $y \in V(G)$, denoted $deg_G(y)$, refers to the number of edges in G incident with y. The smallest of the degrees of vertices in G is called the minimum degree, denoted $\delta = \delta(G)$. The leaf number $L(G)$ whose applications are numerous in network designs, is the maximum number of leaf vertices contained in a spanning tree of G, where a leaf is a vertex of degree one and a tree refers to a connected graph without cycles. A Hamiltonian graph, which is crucial in solving data communication problems, is a graph that contains a spanning cycle. G is traceable if it contains a spanning path. Thus every Hamiltonian graph is traceable. The problem of determining whether spanning paths or cycles exist in a graph is NP-complete. More on applications of leaf number, traceability and Hamiltonicity can be found in [12,31].

To the best of our knowledge, Dirac [7] is the first to establish sufficient conditions for a graph to be Hamiltonian. His result is based on the order and minimum degree. Ore [28] generalised Dirac's theorem by considering degree sums for non-adjacent vertices in a graph. Later, Broersma and others [3] generalised Ore's result by involving large neighbourhood unions for non-adjacent vertices in G. Several authors researched on sufficient conditions for Hamiltonicity in G based on various invariants (see for instance [3,7,10,14,26,28]). Ren [30], Xiong and Zong [33] reported on sufficient conditions for a graph to be traceable.

However, no sufficient conditions for traceability or Hamiltonicity based on leaf number and minimum degree were known in literature until Mukwembi [23,24,22], started to solve conjectures posed by DeLaViña's computer program, Graffiti.pc [4]. Some of the conjectures of Graffiti.pc are the following:

Conjecture 1 ([4])**.** *Let G be a connected graph with leaf number $L(G)$ and minimum degree δ such that $\delta \geq L(G) - 1$. Then G is traceable.*

Conjecture 2 ([4] (Graffiti.pc 190))**.** *Let G be a connected graph with leaf number $L(G)$ and minimum degree δ such that $\delta \geq \frac{1}{2}(L(G)+1)$. Then G is traceable.*

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Conjecture 1 was solved completely in [24]. However, to date no complete solution has been presented for Conjecture 2. The purpose of this paper is to provide a complete solution to Conjecture 2 of Graffiti.pc. We mention here that, although no complete solution was found for Conjecture 2, attempts to solve it were made (see [21,23]). Of crucial importance in this paper are the following results:

Corollary 1 (*[23]*). *Let G be a connected graph with minimum degree* δ. *If* $\delta \geq \frac{1}{2}(L(G)+1)$, *then G is* 2*-connected.*

Theorem 1 (*[21]*). *Let G be a connected graph with* $diam(G) \neq 2$, *minimum degree* δ *and leaf number* $L(G)$ *such that* $\delta \geq \frac{1}{2}(L(G)+1)$. *Then G is traceable.*

Although a complete solution to Conjecture 2 was not found for graphs with $diam(G) = 2$, the upper bound on the order of such graphs was proved.

Theorem 2 (*[21]*). *Let G be a connected graph with* $diam(G) = 2$, *order n, minimum degree* δ *and leaf number* $L(G)$ *such that* $\delta \geq \frac{1}{2}(L(G)+1)$. *Then* $n \leq 3\delta$.

Corollary 2 (*[21]*). *Let G be a connected graph with minimum degree* δ *and leaf number* $L(G)$ *such that* $\delta \geq \frac{1}{2}(L(G)+1)$. *Then G has no tree with at least* 2δ *leaves.*

In this paper, we employ cycle related properties to settle Conjecture 190 of Graffiti.pc. Cycle and path related properties of graphs have been studied for a long time [2,7,26]. Many researchers have estimated the lower bound on the circumference of graphs by various invariants such as; minimum degree [7,8], minimum degree and girth [9,34], minimum degree and toughness [1,17], neighbourhood union [11,20] and longest path in $V(G) - V(C)$ [27], where C is a longest cycle in G. The order of a longest path and a longest cycle in G are denoted respectively by $p(G)$ and $c(G)$. The difference $p(G)-c(G)$ is known as the *relative length* of longest paths and cycles in graph G. The relative length in G is denoted by $diff(G)$. A vertex set $S \subset V$ is called an independent set if the elements of S are mutually nonadjacent. The maximum possible cardinality of S in G is called the independence number, denoted by α. The cycle C is called a dominating cycle if $V(G)-V(C)$ is an independent set. Relative length has been related to the concepts of Hamiltonicity and dominating cycles (see for instance [29]). The graph G is Hamiltonian if and only if $diff(G) = 0$, that is $c(G) = p(G)$. Furthermore, if $diff(G) \leq 1$ then every longest cycle in G is a dominating cycle. Ozeki and Yamashita [29] established a strong result that will help us settle Conjecture 2. We denote by σ_k, the minimum degree sum of an independent set of k vertices, provided the independence number is at least k, otherwise, we let $\sigma_k = +\infty$.

Independently, Dirac [7] proved the following result:

Theorem 3 (*[7]*). *Let G be a* 2*-connected graph and* C_k *a longest cycle in G. Then* $|V(C_k)| \geq \min(n, 2\delta)$, *where n is the order of G.*

Recently, Ozeki and Yamashita proved the following result:

Theorem 4 (*[29]*). *Let G be a* 2*-connected graph, with connectivity* κ *and minimum degree* δ. *If* $diff(G) \geq 2$ *then either* $c(G) \geq \sigma_3 - 3 \geq 3\delta - 3$ *or* $\kappa = 2$ *and* $p(G) \geq \sigma_3 - 1 \geq 3\delta - 1$.

Turning to leaf number, its determinant is known to be *NP*-hard [13]. Lower bounds on leaf number based on minimum degree [15,16,18,19,32], independence number, local independence number, bipartite number and average distance [5,6], minimum degree and diameter [25], were presented. Of vital importance in this paper is the following result:

Theorem 5 (*[16]*). *Let G be a simple connected graph with minimum degree at least* 5 *and order n. Then* $L(G) \geq \frac{1}{2}n + 2$.

Apart from the notation already defined, we shall use the following: If H is a subgraph of G then we denote by $V(\overline{H})$ vertices in G which are not in H, that is, $V(\overline{H}) = V(G) - V(H)$. The maximum number of leaf vertices in H is denoted by $L(H)$. A path joining vertices x and y in G is denoted by P_{xy}. We write $P_{xy} \subset P$ if the path P_{xy} is a subpath of P. The distance, $d_G(x, y)$, between vertices x and y in G is defined as the length of a shortest path joining x and y. The eccentricity $ecc(y)$ of a vertex $y \in V(G)$ is the distance from y to a vertex furthest from y in G, that is, $ecc(y) = \max_{x\in V(G)}(d_G(x, y))$. The diameter $diam(G)$ of G is the maximum eccentricity amongst eccentricities of all vertices in G. The neighbourhood, $N(x) = N_G(x)$, of a vertex x of graph G is the set $\{y \in V : d_G(x, y) = 1\}$. Further, we denote a complete bipartite graph with partite sets of order m and n, respectively, by $K_{m,n}$.

2. Results

We now settle Conjecture 2 completely. Precisely, we prove the following theorem:

Theorem 6. *Let G be a connected graph with minimum degree* δ *and leaf number* $L(G)$ *such that* $\delta \geq \frac{1}{2}(L(G)+1)$. *Then G contains a spanning path.*

Proof. Assume that G satisfies the hypothesis of the theorem. Clearly, there is no graph satisfying the conditions of the theorem if $\delta = 1$. Further, if $diam(G) \neq 2$ then we are done by Theorem 1. So it is enough to assume that $diam(G) = 2$.

If $\delta = 2$, then $L \leq 3$ and by Theorem 2, $n \leq 6$. We consider $n = 6$ the proof for $n \leq 5$ is easier. If $n = 6$, the are only two spanning trees that meet these requirements: A path, say, v_1, v_2, v_3, v_4, v_5 and a vertex v_6 such that v_6 is adjacent to v_2 or v_3, otherwise, G is traceable. If v_6 is adjacent to v_3, then if v_6 is adjacent to any other vertex in $V(G)$, G has a spanning path. Since $\delta = 2$ the result follows. If v_6 is adjacent to v_2, then since $\delta = 2$, we consider the case where v_1 and v_6 are adjacent to v_4 and v_5 is adjacent to v_2, otherwise, G is traceable. Then the leaf number is at least 4, which is a contradiction. Thus G must be traceable for $\delta = 2$.

The following claims and lemma help us settle the results in this paper for $\delta \geq 3$. Assume that $\delta \geq 3$.

Claim 1. *For $\delta \geq 3$, if G has a tree, T, say, such that $L(T) = 2\delta - 1$ then $\left|V(\overline{T})\right| \leq 2$. Further, if $u_1, u_2 \in V(\overline{T})$ then $u_1u_2 \in E(G)$.*

Proof of Claim 1. Since $L(T) = 2\delta - 1$, no interior vertex of T has a neighbour in $V(\overline{T})$ and each leaf of T has at most one neighbour in $V(\overline{T})$. Hence there are at most $2\delta - 1$ edges connecting leaf vertices of T and vertices in $V(\overline{T})$. Also, each vertex in $V(\overline{T})$ has at least $\delta - 1$ neighbours in T, otherwise, we obtain a tree of G that has 2δ leaves contradicting Corollary 2. Hence there are at most 2 vertices outside, otherwise, we would have at least $3(\delta - 1) > 2\delta - 1$ edges between vertices of T and vertices in $V(\overline{T})$. Further, if $u_1, u_2 \in V(G)$ then $u_1u_2 \in E(G)$, otherwise, each u_i has at least δ neighbours in T, a contradiction to the fact that there are at most $2\delta - 1$ edges between vertices of T and $V(\overline{T})$.

Claim 2. *For $\delta \geq 6$, if G has a tree, T, say, such that $L(T) = 2\delta - 2$ then $\left|V(\overline{T})\right| \leq 4$.*

Proof of Claim 2. If there is an interior vertex, say, x of T that has a neighbour, say, w in $V(\overline{T})$, then $T \cup \{xw\}$ is a tree with $2\delta - 1$ leaves. So the claim follows by application of Claim 1. Assume that no interior vertex of T has a neighbour in $V(\overline{T})$. If there is a leaf, say, y of T that has 2 neighbours, say, y_1 and y_2 in $V(\overline{T})$ then $T \cup \{yy_1, yy_2\}$ is a tree with $2\delta - 1$ leaves and the result follows by Claim 1. So, we assume that no interior vertex of T has a neighbour in $V(\overline{T})$ and that no leaf in T has at least 2 neighbours in $V(\overline{T})$. We notice first that every vertex in G has at most 2 neighbours outside T, otherwise, we get a tree with at least 2δ leaves. To be precise, if there is a vertex, say, w in G, that has neighbours, say, w_1, w_2 and w_3 in $V(\overline{T})$, then if w is in T, $T \cup \{ww_1, ww_2, ww_3\}$ has at least 2δ leaves, a contradiction. If $w \notin V(T)$ then let P_{xw} be a shortest $x - w$ path where $x \in V(T)$, then $T \cup P_{xw} \cup \{ww_1, ww_2, ww_3\}$ has 2δ leaves, a contradiction again. It follows that, each vertex in $V(\overline{T})$ has at least $\delta - 2$ neighbours in T. Thus $\left|V(\overline{T})\right| \leq 3$, otherwise, since each vertex in $V(\overline{T})$ has at least $\delta - 2$ neighbours in T, there would be at least $4(\delta - 2)$ leaves in T. This would imply that $4\delta - 8 \leq 2\delta - 2$ and $\delta \leq 3$, a contradiction. So, in all cases Claim 2 holds.

Lemma 1. *Let $\delta \geq 5$. Then $n \leq 3\delta - 1$.*

Proof of Lemma 1. For $\delta = 5$ we have $L(G) \leq 9$. This in conjunction with Theorem 5 implies that $n \leq 14$ as desired. Assume that $\delta \geq 6$. By Theorem 2, $n \leq 3\delta$. We show that $n \neq 3\delta$ and we are done. To show this assume on contrary that $n = 3\delta$. Pick a vertex y such that $deg(y) = \delta$. Let $N(y) = \{y_1, y_2, y_3, \ldots, y_\delta\}$. Then $N[y]$ forms a $K_{1,\delta}$ subgraph of G. Fix this $K_{1,\delta}$ to be the specific graph, say, $K^*_{1,\delta}$. By our assumption that $n = 3\delta$, $\left|V(\overline{K^*_{1,\delta}})\right| = 2\delta - 1$. We claim that each vertex not in $K^*_{1,\delta}$ has at least two neighbours in $K^*_{1,\delta}$. To prove this assume there is a vertex, x say, in $V(\overline{K^*_{1,\delta}})$ that has exactly one neighbour in $K^*_{1,\delta}$. Let y_1 be the only neighbour of x in $K^*_{1,\delta}$. Then, x has at least $\delta - 1$ neighbours in $V(\overline{K^*_{1,\delta}})$. Let $x_1, x_2, \ldots, x_{\delta-1}$ be neighbours of x, apart from, y_1. Then the tree $K^*_{1,\delta} \cup \{y_1x, xx_i : 1 \leq i \leq \delta - 1\}$ of G has $2\delta + 1$ vertices and $2\delta - 2$ leaves. Hence, outside that tree there are $\delta - 1 > 4$ vertices, since $n = 3\delta$. This is a contradiction to Claim 2, so the claim holds. Following this, each vertex in $V(\overline{K^*_{1,\delta}})$ has a neighbour in $N(y) - \{y_\delta\}$. This implies that $\left|V(\overline{K^*_{1,\delta}})\right| \leq 2\delta - 2$, otherwise, $\{y_\delta\} \cup V(\overline{K^*_{1,\delta}})$ forms at least 2δ leaves of a tree in G, which is obtained by joining each vertex of $V(\overline{K^*_{1,\delta}})$ to only one of its neighbours in $K^*_{1,\delta} - \{y_\delta\}$. So $n \leq 3\delta - 1$ as desired.

Claim 3. *G is traceable.*

Proof of Claim 3. Assume first that $\delta = 3$. By Theorem 2, $n \leq 9$. Let $C_k = v_1, v_2, v_3, \ldots, v_k, v_1$ be a longest cycle in G. By Corollary 1 and Theorem 3, $k \geq 6$. We assume there is at least a vertex not on C_k, otherwise, we are done. Since $V(\overline{C_k})$ has at most 3 vertices, there is a vertex in $V(\overline{C_k})$ such that all its neighbours are on C_k, otherwise, there is a path not on C_k containing all the at most 3 three vertices not on C_k and G would be traceable as desired. Let v be a vertex such that all its neighbours are on C_k. Let $v_{t_1}, v_{t_2}, v_{t_3} \in N_{C_k}(v)$ and $A = \left\{v_{t_1+1}, v_{t_2+1}, v_{t_3+1}\right\}$. Then A is an independent set, otherwise, we obtain a cycle longer than C_k, which is prohibited. If v is the only vertex not on C_k then we are done. Assume there is at least one vertex not on C_k, apart from, v.

Since $diam(G) = 2$, it implies that $ecc(v) \leq 2$. It is enough to consider the cases $n = 8$ and $n = 9$, since $k \geq 6$ and $n \leq 9$. Consider first $n = 8$ and assume that $\left|V(\overline{C_k})\right| = 2$, otherwise, we are done. Then $k = 6$. Let u be a vertex not on C_k, apart from v. If u has a neighbour, say v_{t_i+1} in A then we have a spanning path $v, v_{t_i}, v_{t_i-1}, \ldots, v_{t_i+1}, u$. So, assume that u has no neighbour in A. Then u is adjacent to all neighbours of v, since $deg(u) \geq 3$ (notice that u cannot be adjacent to v, since all neighbours of v are on C_k). By similar arguments, v_{t_3} is adjacent to all neighbours of v. Thus, in particular, u and v_{t_3+1} are adjacent to v_{t_2}. Hence u, v_{t_1}, v_{t_1+1}, v_{t_2+1}, v_{t_3} and v_{t_3+1} forms $2\delta = 6$ leaves of a tree whose interior vertices are v and v_{t_2}, a contradiction to Corollary 2.

Assume that $n = 9$. Notice here that if $deg(v) \geq 4$ then $k \geq 8$ and we are done. So assume that $deg(v) = 3$. Let u_1, u_2 be vertices in $V(G) - (N[v] \cup A)$ and assume that at least one of them is not on C_k. Since $ecc(v) \leq 2$, we can assume that each u_i, $i = 1,2$ has a neighbour in $N[v] - \{v_{t_3}\}$. We now look at possible neighbours of v_{t_3+1}, apart from v_{t_3}. If v_{t_3+1} is adjacent to either v_{t_1} or v_{t_2} then the set $\{v_{t_1+1}, v_{t_2+1}, v_{t_3+1}, u_1, u_2, v_{t_3}\}$ forms 6 leaves of a tree whose interior vertices are v, v_{t_1} and v_{t_2}, a contradiction to Corollary 2. So assume that v_{t_3+1} is neither adjacent to v_{t_1} nor v_{t_2}. Then v_{t_3+1} is adjacent to u_1 and u_2, since $deg(v_{t_3+1}) \geq 3$ and v_{t_3+1} has no neighbour in A. In this case, at least one of u_1, u_2, say u_2 is on C_k and $u_2 = v_{t_1-1} = v_{t_3+2}$. Hence the path $u_1, v_{t_3+1}, u_2, v_{t_1}, v_{t_1+1}, v_{t_2}, v_{t_2+1}, v_{t_3}, v$ spans G. So G is traceable for $\delta = 3$.

Assume that $\delta \geq 4$. We consider two main cases

Case 1. Assume first that $diff(G) \geq 2$. Then, by Corollary 1 and Theorem 4, $p(G) \geq 3\delta - 1$. Thus, for all $\delta \geq 5$, G is traceable, since $n \leq 3\delta - 1$.

Assume that $\delta = 4$. Then $p(G) \geq 11$. If $n \leq 11$ then we are done. Assume that $n = 12$ (notice that $n \leq 12$ by Theorem 2). Then $p(G) = 11$, otherwise, we are done. So, $c(G) \leq 9$. Let $P = w_1, w_2, w_3, \ldots, w_{11}$ be a longest path and x be a vertex not on P. Then all neighbours of x are on P. Further, x is neither adjacent to w_1 nor w_{11}, otherwise, we contradict our choice of P. In addition, neighbours of x are non-consecutive on P. Let $w_{s_1}, w_{s_2}, w_{s_3}, w_{s_4}$, $s_1 < s_2 < s_3 < s_4$ be neighbours of x on P. Then $d_P(w_{s_1}, w_{s_4}) \geq 6$, since no neighbours of x are consecutive on P. Further, $d_P(w_{s_1}, w_{s_4}) \leq 7$, otherwise, $P_{w_{s_1} w_{s_4}} \subset P$ together with edges xw_{s_1} and xw_{s_4} forms a cycle of length at least 10, a contradiction to $c(G) \leq 9$. Thus there exist two pairs of neighbours of x of the form $w_{s_i}, w_{s_{i+1}}$, such that for each of those pairs, there is a exactly one vertex, w_{s_i+1}, on P between w_{s_i} and $w_{s_{i+1}}$. Further, there is no pair $w_{s_i}, w_{s_{i+1}}$ that has at least 3 vertices on P between w_{s_i} and $w_{s_{i+1}}$. We can assume that w_{s_1+1} is the only vertex on P between w_{s_1} and w_{s_2} and that w_{s_2+1} is the only vertex between w_{s_2} and w_{s_3}, other cases are treated similarly. If either w_{s_1+1} or w_{s_2+1} has a neighbour outside the set $\{w_{s_1}, w_{s_2}, w_{s_3}, w_{s_4}\}$ then either G is traceable and we are done or we get a cycle of length at least 10, a contradiction. So, assume that all neighbours for w_{s_1+1} and w_{s_2+1} are in the set $\{w_{s_1}, w_{s_2}, w_{s_3}, w_{s_4}\}$. Then, in particular both w_{s_1+1} and w_{s_2+1} are adjacent to w_{s_4}, since $\delta = 4$. So the tree $\{xw_{s_i}, w_{s_4}w_{s_1+1}, w_{s_4}w_{s_2+1}, w_{s_4}w_{s_4-1}, w_{s_4}w_{s_4+1} : i = 1, 2, 3, 4\}$ has 9 vertices and 7 leaves. So by Claim 1, $n \leq 11$, a contradiction to $n = 12$. Thus in all sub-cases G must be traceable.

Case 2. Assume that $diff(G) \leq 1$. Let $C_k = v_1, v_2, v_3, \ldots, v_k, v_1$ be a longest cycle in G. Then C_k is a dominating cycle. Notice by Corollary 1 and Theorem 3 that $k \geq 2\delta$. If all vertices of G are on C_k then G is Hamiltonian and hence traceable as desired. So we assume that some vertices of G are not on C_k. Let $v \in V(G)$ be a vertex not on C_k. Then all its neighbours are on C_k. We denote its neighbours by $v_{t_1}, v_{t_2}, v_{t_3}, \ldots, v_{t_d}$, where $d = deg(v) = p + \delta$ for some integer $p : p \geq 0$. Throughout this paper, we let $A = \left\{v_{t_1+1}, v_{t_2+1}, v_{t_3+1}, \ldots, v_{t_d+1}\right\}$, $B = V(G) - (N[v] \cup A)$ and $l = |B|$. It follows that A is an independent set, otherwise, we obtain a cycle longer than C_k, which is prohibited. We show that v is the only vertex not on C_k and we are done. To do this, assume that there are some vertices not on C_k, apart from, v. Let u be a vertex not on C_k, apart from v. Clearly, u is neither adjacent to v_{t_i-1} nor v_{t_i+1}, otherwise, we have a path $v, v_{t_i}, v_{t_i+1}, \ldots, v_{t_i-1}, u$ or $v, v_{t_i}, v_{t_i-1}, \ldots, v_{t_i+1}, u$, a contradiction to $p(G) - c(G) \leq 1$.

Since $diam(G) = 2$, we have $ecc(v) \leq 2$. So, each of the l elements in B has a neighbour in $N(v)$. Recall that u has no neighbour in A. So, each element in A has at most $l - 1$ neighbours in B, since u cannot be a possible neighbour. Notice here that $l \leq \delta - 1$, otherwise, $A \cup B$ forms at least 2δ leaves of a tree in G, which is obtained by joining each element of $A \cup B$ to only one of its neighbours in the star graph formed by $N[v]$. Now, we construct a tree with at least 2δ leaves to get a contradiction.

Take v and join to it $p+l$ of its neighbours which are such that each element in B has at least a neighbour amongst these. To each of the aforementioned neighbours, v_{t_i} of v, add an edge $v_{t_i}v_{t_i+1}$ and join each element of B to only one of its neighbours amongst the aforementioned $p + l$ neighbours of v. This gives a tree, T_1 say, with $p + 2l$ leaves. Since A is an independent set, the minimum degree is δ and each element of A is adjacent to at most $l - 1$ leaves in T_1, it follows that each of the remaining $2(\delta - l)$ vertices in $N(v)$ and A, which are not in T_1 is adjacent to some interior neighbours of T_1. Joining each of these to only one of its neighbours, which is an interior vertex of T_1, yields a tree with $p + 2\delta \geq 2\delta$ leaves, which is not allowed. Hence G must be traceable.

Thus in all cases G contains a spanning path as needed. □

To see that our main result is best possible, let us show that for every δ and every L such that $L \geq 2\delta$, there exists a graph G of minimum degree δ and leaf number L which is not traceable. It is easy to see that the complete bipartite graph $K_{\delta,\delta+2+p}$, where $p \geq 0$, is a non-traceable graph of minimum degree δ. The leaf number of $K_{\delta,\delta+2+p}$ is $L = 2\delta + p$ if $\delta \geq 2$ and $L = p + 3$ if $\delta = 1$.

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