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MINIMUM DEGREE, LEAF NUMBER AND TRACEABILITY

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Abstract. Let G be a finite connected graph with minimum degree δ. The leaf number $L(G)$ of G is defined as the maximum number of leaf vertices contained in a spanning tree of G. We prove that if $\delta \geqslant \frac{1}{2}(L(G)+1)$, then G is 2-connected. Further, we deduce, for graphs of girth greater than 4, that if $\delta \geqslant \frac{1}{2}(L(G)+1)$, then G contains a spanning path. This provides a partial solution to a conjecture of the computer program Graffiti.pc [DeLaViña and Waller, Spanning trees with many leaves and average distance, Electron. J. Combin. 15 (2008), 1–16]. For G claw-free, we show that if $\delta \geqslant \frac{1}{2}(L(G)+1)$, then G is Hamiltonian. This again confirms, and even improves, the conjecture of Graffiti.pc for this class of graphs.

Keywords: interconnection network, graph, leaf number, traceability, Hamiltonicity, Graffiti.pc

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1. Introduction

Let $G=(V,E)$ be a connected simple graph. Then G is *traceable* if it contains a spanning path, and is *Hamiltonian* if it contains a spanning cycle. The leaf number $L(G)$ of G is defined as the maximum number of end vertices contained in a spanning tree of G. Tree topologies appear when designing centralized terminal networks [6]. The constraint on the number of end vertices (i.e., "degree-1" terminals) arises because the software and hardware associated to each terminal differs accordingly with its position in the tree. Usually, the software and hardware associated to a leaf terminal is cheaper than the software and hardware used in the remaining terminals because for any intermediate terminal v one needs to check if the message arriving is

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destined to that terminal or to any other terminal located after v. For this reason, terminal v requires software and hardware for message routing, whereas leaf terminals do not require such equipment. Thus, if G represents the centralized terminal network, we then ask for a spanning tree solution containing as many leaf vertices as possible.

Several authors (see, for instance, [5], [8], [7]) have reported on sufficient conditions for a graph to be traceable. The search continues with various authors focussing their attention on sufficient conditions for traceability in particular classes of graphs. For instance, Ren [13] gave sufficient conditions for a 2-connected graph to be traceable while recently Čada, Flandrin and Kang [1] investigated sufficient conditions for traceability in locally claw-free graphs.

DeLaViña's computer program, Graffiti.pc (see, for example, [2] or [3]), which sorts through various graphs and looks for simple relations among parameters, posed the following attractive conjecture and posted it on the wall [2]. The conjecture speculates sufficient conditions for traceability based on minimum degree and the leaf number. Precisely,

Conjecture (Graffiti.pc 190)**.** *If G is a simple connected graph with more than one vertex such that $\delta \geqslant \frac{1}{2}(L(G)+1)$, then G is traceable.*

In this paper we prove that if G satisfies the hypothesis of the conjecture, then G is 2-connected. Moreover, we settle the conjecture for the class of graphs with girth greater than 4. Further, for all claw-free graphs, with the exception of a few from a forbidden family, we prove a strengthening of the conjecture.

We use the following terminology and notation. The distance between two vertices u and v in G, i.e., the length of a shortest u-v path in G, is denoted by $d_G(u,v)$. The neigbhourhood of a vertex u, i.e., the set $\{x \in V\colon\ d_G(x,u)=1\}$, is denoted by $N_G(u)$ whilst the closed neigbourhood of u, i.e., the set $\{x \in V\colon\ d_G(x,u) \leqslant 1\}$ is denoted by $N_G[u]$. The degree of vertex u in G, i.e., the cardinality of $N_G(u)$, is denoted by $\deg_G(u)$, and $\delta(G)=\delta$ denotes the minimum degree of G. Where there is no danger of confusion, we drop the subscript or argument G. A *cut vertex* of G is a vertex whose removal increases the number of components in G. We say that G is 2-*connected* if G has no cut vertex. A *block* of G is a maximal subgraph of G that has no cut vertex, and an *end block* of G is a block of G that contains exactly one cut vertex. If H is a subgraph of G, we write $H \leqslant G$. For vertex disjoint graphs $G_1, G_2, \ldots, G_k$, the *sequential join* $G_1 + G_2 + \ldots + G_k$ is the graph obtained from the union of $G_1, \ldots, G_k$ by joining every vertex of G_i to every vertex of G_{i+1} for $i = 1, 2, \ldots, k-1$. The complete graph and the cycle of order n is denoted by K_n and C_n, respectively.

2. Known results

Several authors have reported on sufficient conditions for a 2-connected graph to be traceable. We state below a result, due to Ren [13], which will be used later in this paper.

Theorem 2.1 (Ren [13]). *Let G be a 2-connected graph of order n. If $|N(u) \cup N(v)| \geqslant \frac{1}{2}(n-1)$ for all distinct vertices u, v with $d_G(u,v) = 2$, then G is traceable.*

Li [11] defines a family $\mathfrak{F}_1$ of graphs as follows: If G is in $\mathfrak{F}_1$, then G can be decomposed into three disjoint subgraphs, G_1, G_2 and G_3 such that for any $i \neq j$, $1 \leqslant i, j \leqslant 3$, $E_G(G_i, G_j) = \{u_i u_j, v_i v_j\}$, where $u_i, v_i \in V(G_i)$. We will make use of a theorem by Li.

Theorem 2.2 (Li [11]). *Let G be a 2-connected claw-free graph with minimum degree $\delta \geqslant \frac{1}{4}n$ which does not belong to $\mathfrak{F}_1$. Then G is Hamiltonian.*

Turning to the leaf number, its determination is known to be NP-hard. Lower bounds on the leaf number in terms of other parameters, for instance, order, independence number and maximum order of a bipartite graph [3], order and size [4] have been investigated. However, the first result on lower bounds seems to be a statement, without proof, by Storer [14] that every connected cubic graph G with n vertices has $L(G) \geqslant \frac{1}{4}n + 2$. Linial (see [4]) conjectured, more generally, that every connected graph G with n vertices and minimum degree δ satisfies

$$L(G) \geqslant \frac{\delta - 2}{\delta + 1} n + c_\delta,$$

where c_δ is a constant depending only on δ. Several authors have researched on this conjecture. Kleitman and West [10] introduced a heavy method, the dead leaves approach, with which they gave a proof of Linial's Conjecture for $\delta = 3$ with a best possible $c_\delta = 2$, and hence provided, for the first time, a rigorous proof to Storer's Theorem. Subsequently, Griggs and Wu [9], using the complicated dead leaves approach, settled Linial's Conjecture for $\delta = 4$ and 5. In this paper, we will make use of one of their theorems.

Theorem 2.3 (Griggs and Wu [9]). *If G is a connected simple graph with n vertices and minimum degree at least 5, then $L(G) \geqslant \frac{1}{2}n + 2$.*

The following simple lemma, which we also use in this paper, was proved in [12].

Lemma 2.1 (Mukwembi and Munyira [12]). *Let G be a connected graph and $T' \leqslant G$ a tree. Then there exists a spanning tree T of G such that $T' \leqslant T$ and $L(T) \geqslant L(T')$.*

3. Results

Given a connected graph G with minimum degree δ, it can easily be shown that $L(G) \geqslant \delta$ and that this bound is tight. In the next theorem we prove that the presence of cut vertices in G induces the existence of a spanning tree of G with a double number of end vertices to those in a general graph.

Theorem 3.1. *Let G be a connected graph with minimum degree δ. If G has a cut vertex, then $L(G) \geqslant 2\delta$. Moreover, the bound is tight.*

P r o o f. Suppose to the contrary that there is a counterexample to the theorem, and of such counterexamples, choose G to have the smallest order, n. Thus G has a cut vertex, minimum degree δ and

$$L(G) < 2\delta, \tag{3.1}$$

and $L(H) \geqslant 2\delta(H)$ for any graph H of order less than n with a cut vertex.

Claim 1. *G has no bridge.*

P r o o f of Claim 1. By contradiction, suppose that G has a bridge $e = uv$, and let G_1 and G_2 be the components of $G - e$ containing u and v, respectively. Let G' be the graph obtained from G_1 and G_2 by identifying u and v. Note that $\deg_{G'}(x) \geqslant \deg_G(x)$ for all x in G'. Hence $\delta(G') \geqslant \delta(G)$. Moreover, G' has a cut vertex u $(= v)$ and order $n - 1$. It follows, by our choice of G, that

$$L(G') \geqslant 2\delta(G') \geqslant 2\delta. \tag{3.2}$$

Let T' be a spanning tree of G' with $L(G') = L(T')$. We construct a spanning tree T of G from T' as follows. Since u is a cut vertex of G', u cannot be an end vertex in T' and so T' is a union of two trees T_1 and T_2, where T_1 spans G_1 and T_2 spans G_2. Let T be the tree obtained by taking disjoint copies of T_1 and T_2 and joining u and v by an edge. Then T is a spanning tree of G, and so from (3.2) we have

$$L(G) \geqslant L(T) = L(T') = L(G') \geqslant 2\delta;$$

a contradiction to (3.1), and so the claim is proven. □

We now find a lower bound on $L(G)$. Let G_1 be an end block of G, G_2 the union of the remaining blocks, and denote by n_i the order of G_i, $i = 1, 2$. Let w be the cut vertex of G in common between G_1 and G_2. For $i = 1, 2$, we construct a tree $T_i \leqslant G_i$ rooted at w such that if $T = T_1 \cup T_2$, then $L(T) \geqslant 2\delta$. First consider G_1. We show in each case that there is a tree $T_1 \leqslant G_1$, rooted at w, whose number of end vertices, excluding possibly w, is at least δ.

First assume that w is adjacent to every vertex in G_1, then let x, $x \neq w$, be a vertex in G_1. Note that all neighbours of x are in G_1; hence $n_1 \geqslant |N[x]| \geqslant \delta + 1$. Thus, w is adjacent to at least δ neighbours in G_1. Let T_1 be the tree with vertex set $V(G_1)$ and edge set $\{vw\colon v \in V(G_1) - \{w\}\}$. Then T_1 has at least δ end vertices excluding possibly w, as claimed.

From now onwards assume that there is a vertex y in G_1 which is not adjacent to w. Thus $n_1 \geqslant |N[y]| + |\{w\}| \geqslant \delta + 2$. Partition $V(G_1) - \{w\}$ as $V(G_1) - \{w\} = A \cup B$, where $A = \{u\colon d_{G_1}(w, u) = 1\}$ and $B = \{u\colon d_{G_1}(w, u) \geqslant 2\}$. Consider the set A. If on one hand there is a vertex x in A adjacent to every vertex in G_1, then let T_1 be the tree with vertex set $V(G_1)$ and edge set $\{xv\colon v \in V(G_1) - \{x\}\}$. Since x is adjacent to every vertex of G_1 and $n_1 \geqslant \delta + 2$, T_1 has at least δ end vertices excluding w, and we are done.

If on the other hand there is a vertex x in A which is not adjacent to some vertex x' in G_1, then we look at two cases separately:

Case 1: $x' \in A$. Let T_1 be the tree with vertex set $N[x] \cup \{x'\}$ and edge set $\{wx'\} \cup \{xv\colon v \in N(x)\}$. Then T_1 has at least $|\{x'\}| + |N(x) - \{w\}| \geqslant 1 + \delta - 1 = \delta$ end vertices, as required. Note that w is not an end vertex of T_1.

Case 2: $x' \in B$. Since G is bridgeless, by Claim 1, there is a w-x' path P not containing the edge wx. Of all such w-x' paths not containing the edge wx, choose P to be a shortest one. If on one hand x is not on P, then let T_1 be the tree with vertex set $V(P) \cup N[x'] \cup \{x\}$ and edge set $\{wx\} \cup E(P) \cup \{x'v\colon v \in N(x')\}$. Hence, since $N(x') \cap \{w, x\} = \emptyset$, T_1 has at least $|\{x\}| + |N(x')| - 1 \geqslant \delta$ end vertices, and w is not an end vertex of T_1, as required. If on the other hand x is on P, let $P = wu_1u_2 \dots u_kx'$, so that $x = u_t$ for some $t \in \{2, 3, \dots, k-1\}$. By our choice of P, x' cannot be adjacent to u_1. Now let T_1 be the tree with vertex set $\{u_1, w, x, u_{t+1}, u_{t+2}, \dots, u_k, x'\} \cup N(x')$ and edge set

$$\{wu_1, wx, xu_{t+1}, u_{t+1}u_{t+2}, u_{t+2}u_{t+3}, \dots, u_{k-1}u_k\} \cup \{vx'\colon v \in N(x')\}.$$

Hence, since $N(x') \cap \{w, u_1, x\} = \emptyset$, T_1 has at least $|\{u_1\}| + |N(x')| - 1 \geqslant \delta$ end vertices, and w is not an end vertex in T_1, as desired. We conclude that G_1 has a tree T_1, rooted at w, with at least δ end vertices excluding possibly w.

Analogously, there is a tree $T_2 \leqslant G_2$ rooted at w with, excluding possibly w, at least δ end vertices. The trees T_1 and T_2 have only w in common. Let $T' =$

$T_1 \cup T_2 \leqslant G$. Then $L(T') \geqslant \delta + \delta = 2\delta$. It follows, by Lemma 2.1, that G has a spanning tree T such that $T' \leqslant T$ and $L(T) \geqslant L(T')$. Thus, $L(T) \geqslant 2\delta$. Hence $L(G) \geqslant L(T) \geqslant 2\delta$, a contradiction to (3.1), and so the bound in the theorem is proven.

To see that the bound is tight, let δ be a positive integer. Let $G_{2\delta+1}$ be the graph $K_\delta + K_1 + K_\delta$ of order $2\delta + 1$. Then $G_{2\delta+1}$ has a cut vertex, minimum degree δ, and $L(G_{2\delta+1}) = 2\delta$. This completes the proof of the theorem. $\square$

Corollary 1. *Let G be a connected graph with minimum degree δ. If $\delta \geqslant \frac{1}{2}(L(G)+1)$, then G is 2-connected.*

Proof. Assume that $\delta \geqslant \frac{1}{2}(L(G)+1)$, and suppose to the contrary that G has a cut vertex. Then by Theorem 3.1,

$$L(G) \geqslant 2\delta \geqslant 2\Big(\frac{1}{2}(L(G)+1)\Big) = L(G)+1,$$

a contradiction. Hence G is 2-connected. $\square$

Theorem 3.2. *Let G be a connected graph with girth greater than 4 and minimum degree $\delta > 4$. If $\delta \geqslant \frac{1}{2}(L(G)+1)$, then G is traceable.*

Proof. Assume that $\delta \geqslant \frac{1}{2}(L(G)+1)$. Applying Theorem 2.3, we get

$$\delta \geqslant \frac{1}{4}(n+6). \tag{3.3}$$

Let u and v be arbitrary distinct vertices in G such that $d_G(u,v) = 2$. Since G has girth greater than 4, we have

$$|N(u) \cup N(v)| = |N(u)| + |N(v)| - |N(u) \cap N(v)| \geqslant \delta + \delta - 1 = 2\delta - 1.$$

This, in conjunction with (3.3), yields

$$|N(u) \cup N(v)| \geqslant 2\delta - 1 \geqslant 2\Big(\frac{1}{4}(n+6)\Big) - 1 = \frac{1}{2}(n+4).$$

Since u and v were arbitrary, by Theorem 2.1, G is traceable, as desired. $\square$

Theorem 3.3. *Let G be a connected claw-free graph not in $\mathfrak{F}_1$ with minimum degree $\delta > 4$. If $\delta \geqslant \frac{1}{2}(L(G)+1)$, then G is Hamiltonian.*

Proof. Assume that $\delta \geqslant \frac{1}{2}(L(G)+1)$. Then by Corollary 1, G is 2-connected. Further, applying Theorem 2.3, we get $\delta \geqslant \frac{1}{4}(n+6) > \frac{1}{4}n$. Hence by Theorem 2.2, G is Hamiltonian, as desired. $\square$

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