LOW-FREQUENCY ASYMMETRIC VIBRATIONS OF A THIN SHELL WITH A SIGN-CHANGING CURVATURE

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Abstract

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Declaration

No portion of the work in this thesis has been submitted for another degree or qualification of this or any other university or another institution of learning.

We are considering the low frequency vibrations of a thin shell of revolution with a curvature which changes sign. Integrals of the equilibrium equations and stress-strain relations are represented in the form of asymptotic series and their solutions as a combination of the Airy function and its derivative. In application known exact analytical solutions are rather rare, in most cases approximate analytic solutions, i.e. asymptotic methods based on expansions in powers of small or large parameters occupy a central role.

Acknowledgments

This thesis grew out of a series of dialogues with my supervisor and would not have been possible without the kind support, the probing questions and remarkable patience of my supervisor, Professor M.B. Petrov. I cannot thank him enough.

I do thank my colleagues for their overwhelming support and criticism they have given me over my thesis building.

I remain indebted to my parents and lecturers for providing me the means to learn and understand.

Introduction

Most problems in Applied Mathematics involving difficulties such as nonlinear governing equations and boundary conditions, variable coefficients and complex boundary shapes preclude exact solutions. Consequently exact solutions are approximated with ones using numerical techniques, analytical techniques or a combination of both. We need to obtain some insight into the character of the solutions and their dependence on certain parameters. Often one or more of the parameters becomes either very large or very small. Typically these are very difficult situations to treat by straight-forward numerical procedure. The analytical method that can provide an accurate approximation is by asymptotic expansions.

This thesis is largely influenced by Professor M.B. Petrov and P.E. Tovstik [12] paper, which examines the role of turning points on shells of revolution with low frequency vibrations and how the resulting solutions behave.

The thesis focuses on the general idea introduced by Langer, where he realized that any attempt to express the asymptotic expansions of the solutions of turning point problems in terms of elementary functions must fail in regions containing the turning point. A uniformly valid expansion must be expressed in terms of the solution of non-elementary functions which have the same qualitative features as the equation, for example, Airy equations and the exploration of the shell of revolution with emphasis on the negative Gaussian curvature region where the instability occurs due to low frequency vibrations.

This thesis consists of 4 Chapters. Historical developments, refinements and definitions are provided in Chapter 1. This Chapter also contains a number of examples and applications illustrating the practical use of shells. The concept of surfaces is introduced and the basic equilibrium and stress-strain relations are derived using the Love-Kirchhoff assumptions.

The concept of Airy functions as solutions of the Airy equation is introduced in Chapter 2. Using Bessel functions of the first kind, it is shown that the Airy functions have an oscillatory character for negative values of the argument. The Chapter ends with a discussion of the Liouville's differential equation and turning points.

Chapter 3 deals with a shell of revolution with low frequency vibrations of sign-changing curvature. It begins with a brief account of the derivation of the governing equations and the use of separating variables. Principal sections in this Chapter are asymptotic solutions, zeroth approximation, first approximation, Airy solutions and the variational approach to the boundary conditions. The Chapter ends with the conclusion, where it has been shown that, in the region with negative Gaussian curvature both solutions oscillate and in the remaining region, where the curvature is positive, both Airy functions and the unknown displacements and stresses exponentially increase or decrease.

Chapter 4 deals with the numerical and asymptotic aspects of the theory. It begins with an example, where different values of the eigenvalue are found for different shell thickness and the results tabulated and compared.

Chapter 1

Two-Dimensional Shell Theories

1.1 Historical Developments in Linear Shell Theory

An investigation of the general theory of shells, based on the Kirchhoff hypothesis concerning the deformation of plates, was first attempted by Aaron in 1874, when he considered bending behaviour. On the basis of the same Kirchhoff assumptions, Love derived in 1888, the basic equations that govern the behaviour of thin elastic shells. Subsequent theoretical efforts have been directed towards improvements of Love's formulation and the solution of the associated differential equations [2].

1.1.1 Refinements

Fl*ü*gge in 1934 and Byrne in 1944 independently developed a more general set of shell equations by retaining all the first-approximation assumptions of Love except, one on thinness. Further refinements in the theory, from the period 1948-1958 included the incorporation of the effects of transverse normal stress and transverse shear deformations. These additional refinements are justifiable in the case of thin shells. In 1963, the refined theories of E.Reissner and Naghdi were made invariant under different coordinate system, by Naghdi, this being more of a mathematical refinement of the formulation than an extension of the physical validity of the theory [2].

1.1.2 What is a Shell?

To quote Fl*ü*gge, a shell is the "... materialization of a curved surface." So it is, in definition a matter of geometry and not of material, for example, a parachute, a concrete roof, a bubble, or even the surface of a liquid can all be treated as shells [1].

1.1.3 Early Shell Structures

Man-made shell structures have been in existence for many centuries. One of the earliest applications of the shell as a structural form is represented by the several domes that have been constructed for the purpose of providing roofing for temples, cathedrals, monuments and other buildings. Notable historical examples include the Pantheon of ancient Rome, St Peter's Cathedral in Rome and the Taj Mahal of India, built in the seventeenth century by the Mogul Emperor, Shah Jahan [2]. Shells were first used by the creator of the earth and its inhabitants. The list of natural shell-like structures is long, and the strength properties of some of them are remarkable. Egg shells range in size from the smallest insects to the large ostrich eggs, and cellular structures are the building blocks for both plants and animals. Bamboo is basically a thin-walled cylindrical structure, as is the root section of a bird's feather. The latter structural element develops remarkable load-carrying abilities in both bending and torsion [7].

1.1.4 Modern Shell Applications

The general high strength-to-weight ratio of the shell form, combined with its inherent stiffness, has formed the basis of modern applications of shell structures. Among these are thin concrete shell roofs, thin-walled hyperbolic concrete cooling towers at a thermal power station, cylindrical concrete silos for the storage of grain, elevated conical concrete water reservoir and large storage vessels for oil and industrial chemicals. In industry, boilers, pressure vessels and associated piping are further examples of shell structures in metal construction. Hollow members of large industrial steel structures, such as offshore oil platforms are another example of shell applications, as are bodies of transportation structures such as motor vehicles, ships, aircrafts, missiles and spacecrafts. The essential property of all the above shell structures that distinguishes them from other structural forms is, as Calladine points out, the possession of both 'surface' and 'curvature'. This combination endows shells with their characteristic strength and stiffness [2].

1.1.5 General Aspects of the Membrane Theory of Shells

The membrane theory of shells had its origins in the work of Lamé and Clapeyron who in 1828, had considered shells of revolution loaded symmetrically with respect to their axes. The theory is applicable to either completely flexible membranes (for example, an inflated tyre tube or toy balloon), which have negligible bending stiffness, or shells with finite bending rigidity but in which the moments that are developed are so small as to be negligible (i.e. the state of stress is essentially momentless), owing to the geometry of the shell, the nature of the boundary conditions at the shell edges and the manner in which the applied loading is distributed. The bending theory of shells, takes account of both extensional (i.e. in-plane) effects and flexural (i.e. bending, twisting and shearing) effects within the shell material [2].

1.2 Elements of the Theory of Surfaces

The behaviour of a shell is usually modelled on the basis of its middle surface (alternatively referred to as midsurface), which is the locus of interior points equidistant from the two bounding surfaces of the shell. As coordinate lines we will use the 'lines of curvature' of the undisplaced middle surface of the shell wall, together with normals to this surface. These lines of curvature are defined as lines along which the twist is zero, and is shown in the theory of continuous surfaces that there are always at least two such systems of lines, and that these systems are orthogonal to each other, that is tangents to two such lines at the point where they intersect will be at right angles to each other. The shell boundaries follow or are normal to these lines of symmetry and hence coincide with the lines of curvature and the coordinate lines [2].

We introduce the system of orthogonal curvilinear coordinates α_1 and α_2 which coincide with the lines of curvature of the middle surface, S, of the shell, which is a body bounded primarily by two closely spaced curved surfaces. Let a point M on S be determined by the radius vector $\mathbf{r} = \mathbf{r}(\alpha_1, \alpha_2)$, where $\mathbf{r}$ is the position vector from the origin O to points on the surface, (α_1, α_2)-plane [8].

The shell fills the volume

$$(\alpha_1, \alpha_2) \in G, \quad |z| \leq \frac{h}{2},$$

where z is the distance along the normal to the middle surface and h is the shell thickness. Let $\Gamma = \partial G$ be the boundary of the domain G. The shell is said to be thin if its relative

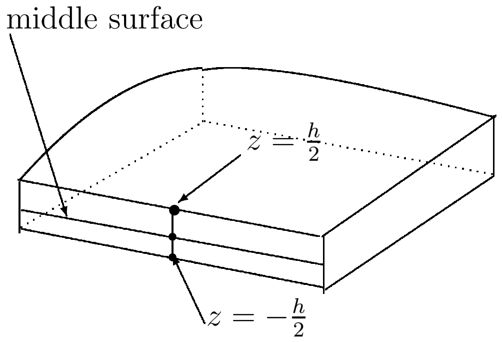


Figure 1.1: Middle surface of a shell element

thickness $\dfrac{h}{R}$ is small, where R is the characteristic length [8].

For shells, the relative proportions of extensional and flexural (bending) effects at a given point depend on several factors such as type of shell surface (synclastic, anticlastic, or developable), support conditions, loading configuration and the proximity of edges and certain discontinuities. Synclastic surfaces are those with positive Gaussian curvature, while anticlastic surfaces are those that possess negative Gaussian curvature, and developable surfaces are those that can be flattened into a plane surface, either directly or after making a single straight-line cut in the surface, they are characterized by zero Gaussian curvature [2].

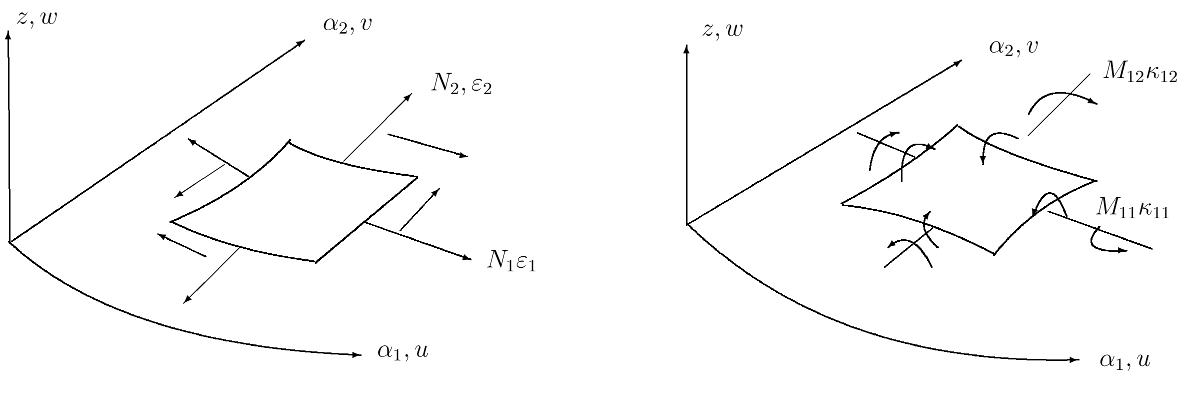
For synclastic shells, if both the shell geometry (i.e. shell thickness, middle surface slope in any arbitrary direction and principal radii of curvature) and surface loads are smoothly varying (i.e. exhibiting no discontinuities in the variation of the shell geometrical components of the shell, nor in their first derivative with respect to arc length along a given direction), then extensional effects generally predominate in the interior regions of the shell while in the edge zones bordering the supports, extensional and flexural effects become equally significant for most practical constructions of supports [2].

Consider base vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ of the surface which are to be defined by

$$\mathbf{a}_1 = \frac{\partial \mathbf{r}}{\partial \alpha_1}, \qquad \mathbf{a}_2 = \frac{\partial \mathbf{r}}{\partial \alpha_2}.$$

The infinitesimal vector connecting two points on the surface with coordinates α_i and $(\alpha_i + d\alpha_i)$ for $i = 1, 2$ is given by

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \alpha_1} d\alpha_1 + \frac{\partial \mathbf{r}}{\partial \alpha_2} d\alpha_2,$$



(a) Middle surface forces

(b) Flexural forces

Figure 1.2: Coordinate directions and stress resultants on a curved thin shell differential element

the length of this vector is therefore determined by

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = \sum_{i,j} \mathbf{a}_i \mathbf{a}_j d\alpha_i d\alpha_j,$$

where the quadratic form $(ds)^2$ which determines the line element ds on the surface is called the *first fundamental form* of the surface. The length of the base vectors are denoted by A_1 and A_2 and is given by

$$A_i = |\mathbf{a}_i| = \sqrt{(\mathbf{a}_i \cdot \mathbf{a}_i)}, \quad \text{for } i = 1, 2 \quad [3].$$

We introduce local orthogonal system of coordinates by means of the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{n}$, where

$$\mathbf{e}_1 = \frac{1}{A_1}\frac{\partial \mathbf{r}}{\partial \alpha_1}, \quad A_1 = \left|\frac{\partial \mathbf{r}}{\partial \alpha_1}\right|, \quad (1 \leftrightarrow 2), \quad \mathbf{n} = \mathbf{e_1} \times \mathbf{e_2}.$$

In this section $(1 \leftrightarrow 2)$ means the formula with 1 and 2 interchanged is also valid. The first and second quadratic forms of the surface are

$$\begin{aligned} I &= ds^2 = A_1^2 d\alpha_1^2 + A_2^2 d\alpha_2^2, \\ II &= \frac{A_1^2}{R_1} d\alpha_1^2 + \frac{A_2^2}{R_2} d\alpha_2^2, \end{aligned}$$

where ds is the arc length on the surface, A_1 and A_2 are Lamé's coefficients, R_1 and R_2 are the principal radii of curvature. Geometry of the middle surface is determined by the changes of the coefficients of the first and second fundamental forms of the surface, I and II. The moment intensities are satisfied everywhere on the middle surface [8]. We also use the notation $k_i = R_i^{-1}$, where $k = k_1 k_2$ denotes the Gaussian curvature of the surface at a point. If $k_2 = 0, k_1 \neq 0$, then the surface is said to have zero Gaussian curvature (parabolic). For $k > 0$, positive Gaussian curvature (elliptic) and if $k < 0$, negative Gaussian curvature (hyperbolic) [2].

The variables u, v, w are the displacements of the middle surface point, which are caused directly by the loading. The w displacements usually tend to produce v displacements which are in general much smaller than the w displacements. In turn the v displacements tend to produce u displacements which are much smaller than the v displacements and hence even smaller compared to the w displacements [2].

For the chosen coordinate system A_1, A_2, R_1 and R_2 are related as follows

$$\frac{\partial}{\partial \alpha_2}\Big(\frac{A_1}{R_1}\Big) = \frac{1}{R_2}\frac{\partial A_1}{\partial \alpha_2}, \quad (1 \leftrightarrow 2), \tag{1.1}$$

$$\frac{1}{A_1 A_2}\left[\frac{\partial}{\partial \alpha_1}\Big(\frac{1}{A_1}\frac{\partial A_2}{\partial \alpha_1}\Big) + \frac{\partial}{\partial \alpha_2}\Big(\frac{1}{A_2}\frac{\partial A_1}{\partial \alpha_2}\Big)\right] = -\frac{1}{R_1 R_2}, \tag{1.2}$$

and the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{n}$ are also related as follows

$$\frac{\partial \mathbf{e}_1}{\partial \alpha_1} = -\frac{1}{A_2}\frac{\partial A_1}{\partial \alpha_2}\mathbf{e}_2 + \frac{A_1}{R_1}\mathbf{n}, \tag{1.3}$$

$$\frac{\partial \mathbf{e}_2}{\partial \alpha_1} = \frac{1}{A_2}\frac{\partial A_1}{\partial \alpha_2}\mathbf{e}_1, \tag{1.4}$$

$$\frac{\partial \mathbf{n}}{\partial \alpha_1} = -\frac{A_1}{R_1}\mathbf{e}_1, \quad (1 \leftrightarrow 2) \quad [3]. \tag{1.5}$$

1.2.1 The Love-Kirchhoff Assumptions

1. The shell thickness is negligibly small in comparison with the least radius of curvature of the shell middle surface.

2. Strains and displacements that arise within the shell are small which implies that products of deformation quantities in the derivation theory may be neglected, ensuring that the system is described by a set of geometrically linear equations. This makes it possible to formulate the equilibrium conditions of the deformed middle surface with reference to the original position of the middle surface prior to deformation.

3. Linear elements normal to the unstrained middle surface remain straight during deformation and suffer no extensions. This means that if the initial and final positions on

the middle surface are known, the initial and final positions of all points of the shell wall will also be known, hence the strains everywhere can be calculated in terms of the displacements of the middle surface alone.

4. The component of stress normal to the middle surface is small compared with other components of stress and may be neglected in the stress-strain relationships. It permits the displacement of every point in the shell wall, hence the strains and stresses at every point, to be defined in terms of the displacement of one surface such as the middle surface of the shell wall. This represents in effect the reduction of the problem from a three- to a two-dimensional one. Errors due to this approximation are negligible for thin shells of homogeneous materials under most loading conditions for practical interest [2].

1.2.2 Geometry of the Middle Surface and its Deformations

In the linear approximation, the tangential (membrane) surface deformations, ε_1, ε_2 and ω, are as by

$$\varepsilon_1 = \frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}u_2 - \frac{w}{R_1}, \quad (1 \leftrightarrow 2), \tag{1.6}$$

$$\omega_1 = \frac{1}{A_1}\frac{\partial u_2}{\partial \alpha_1} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}u_1, \quad \omega = \omega_1 + \omega_2, \tag{1.7}$$

the angles of rotation γ_1 and γ_2 of the normal $\mathbf{n}$ are equal to

$$\gamma_1 = -\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} - \frac{u_1}{R_1}, \quad (1 \leftrightarrow 2) \quad [8], \tag{1.8}$$

where for an isotropic, homogeneous elastic solid, the in-plane constitutive law is given by the plane stress equations

$$\begin{aligned} \varepsilon_1 &= \frac{1}{E}(\sigma_1 - \nu\sigma_2), \\ \varepsilon_2 &= \frac{1}{E}(\sigma_2 - \nu\sigma_1), \end{aligned}$$

or

$$\begin{aligned} \sigma_1 &= \frac{E}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2), \\ \sigma_2 &= \frac{E}{1-\nu^2}(\varepsilon_2 + \nu\varepsilon_1), \end{aligned}$$

and ε_1 is the elongation per unit length (extension) in the direction of the α_1-curve and E is the Young's modulus [1]. The bending surface deformations, κ_1, κ_2 and τ, are as by

$$\kappa_1 = -\frac{1}{A_1}\frac{\partial \gamma_1}{\partial \alpha_1} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\gamma_2, \quad (1 \leftrightarrow 2), \tag{1.9}$$

$$\tau = -\frac{1}{A_2}\frac{\partial \gamma_1}{\partial \alpha_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\gamma_2 + \frac{\omega_1}{R_2} \quad [8]. \tag{1.10}$$

1.2.3 Equilibrium Equations and Elasticity Relations

The equations of equilibrium of an element of the middle surface for small deformations are

$$\frac{\partial(A_2 T_1)}{\partial \alpha_1} - \frac{\partial A_2}{\partial \alpha_1}T_2 + \frac{\partial(A_1 S_2)}{\partial \alpha_2} + \frac{\partial A_1}{\partial \alpha_2}S_1 - \frac{A_1 A_2}{R_1}Q_1 + A_1 A_2 F_1 = 0, \quad (1 \leftrightarrow 2), \tag{1.11}$$

$$\frac{\partial(A_2 Q_1)}{\partial \alpha_1} + \frac{\partial(A_1 Q_2)}{\partial \alpha_2} + A_1 A_2\left(\frac{T_1}{R_1} + \frac{T_2}{R_2} + F_n\right) = 0, \tag{1.12}$$

$$A_1 A_2 Q_2 + \frac{\partial(A_2 H_1)}{\partial \alpha_1} + \frac{\partial(A_1 M_2)}{\partial \alpha_2} + \frac{\partial A_2}{\partial \alpha_1}H_2 = 0, \tag{1.13}$$

$$S_1 - S_2 + \frac{H_1}{R_1} - \frac{H_2}{R_2} = 0, \tag{1.14}$$

where T_i, S_i and Q_i are the projections of the stress-resultant of the internal forces acting in the cross-section $\alpha_i = \text{const}$, on the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{n}$, H_i and M_i are the projections of the stress-couples of the internal forces, and F_1, F_2 and F_n are the projections of the distributed external load. The Kirchhoff-Love's hypotheses, which generally assume that a shell element, normal to the middle surface before deformation, does not change length and remains normal to the middle surface after deformation, lead to the elasticity relations introduced by V.V. Novozhilov

$$T_1 = K(\varepsilon_1 + \nu\varepsilon_2), \qquad S_1 = \frac{K(1-\nu)}{2}\left(\omega + \frac{h^2\tau}{6R_2}\right), \quad (1 \leftrightarrow 2), \tag{1.15}$$

$$M_1 = D(\kappa_1 + \nu\kappa_2), \qquad H_1 = H = D(1-\nu)\tau, \tag{1.16}$$

$$K = \frac{Eh}{1-\nu^2}, \qquad D = \frac{Eh^3}{12(1-\nu^2)}, \tag{1.17}$$

where K is extensional rigidity, D is bending rigidity and ν is Poisson's ratio (here it is supposed that the shell material is homogeneous and isotropic). The F_i are proportional to the eigenvalue λ. In case of vibrations $F_i = -\lambda u_i, \quad \omega^2 = \frac{E\lambda}{\rho}$, where ρ is the density and ω is the unknown natural frequency [8].

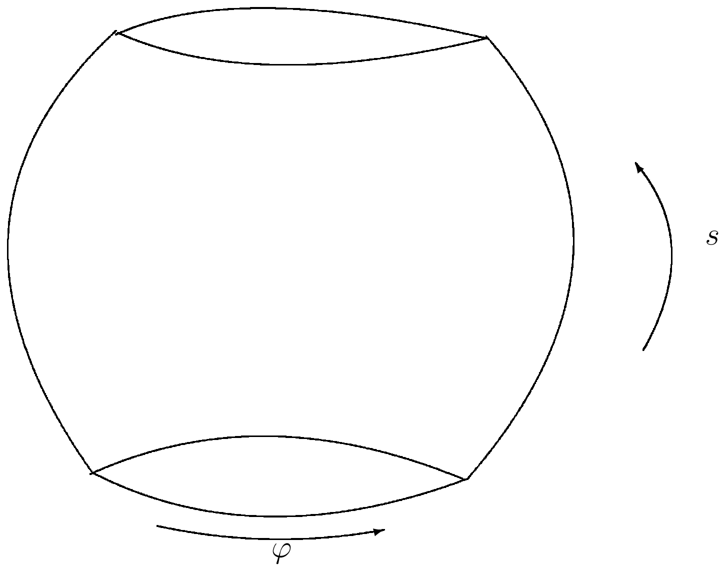


Figure 1.3: A shell of revolution

1.2.4 Shell of Revolution

The middle surface is a surface of revolution which is formed by rotating a plane curve (the meridian) about a straight line in the plane of the curve (the axis of rotation). It will be assumed that the axis of rotation is vertical, so that the parallel circles of latitude of the shell, which are the intersections of the shell middle surface with planes perpendicular to its axis of revolution, lie in horizontal planes. Shells of revolution find application in the design of pressure vessels, liquid-filled tanks, roof domes and cooling towers. For these structures, the principal loading conditions are usually axisymmetric, so that every meridian of the shell of revolution deforms in the same manner, and there is no relative transverse shearing between adjacent portions of the shell when viewed in the plane of a circle of latitude of the shell of revolution. In this axisymmetrical case, the loading, the constant forces, couples and the displacements are assumed to be functions of φ only (complete rotational symmetry) [2].

To describe the vibrations of shells of revolution, it is convenient to introduce as curvilinear coordinates, α_1 and α_2, the arc length, s, of the generator and the angle, φ, in the circumferential direction (see Fig 1.3) [8].

In this case

$$A_1 = 1, \quad A_2 = B, \quad R_1 = \left(\frac{d\theta}{ds}\right)^{-1}, \quad R_2 = \frac{B}{\sin\theta},$$

where B is the distance between a point on the middle surface and the axis of symmetry and θ is the angle between the normal to the surface and the axis of symmetry. The functions B, R_1 and R_2 depend only on s, and do not depend upon φ, where R_1 is the distance from a point on the surface to the corresponding centre of curvature and R_2 is the radius of curvature of the second principal section and is the length measured on a normal to the meridian between its intersection with the axis of rotation and the middle surface [8].

For a conical shell, B and R_2 are linear functions of s ($B = R_2 \cos\alpha$, where α is the angle at the top of the cone), and for a cylindrical shell we may take $B = R_2 = 1$ (by properly choosing the non-dimensional variables) [8].

If the shell is bounded by two parallels, which are formed when we have the intersection of the surface with planes perpendicular to the axis of rotation and are parallel circles, s_1 and s_2 or is in a form of a cupola, then since all the coefficients do not depend upon φ, we can separate the variables,

$$w(s,\varphi) = w(s)e^{im\varphi}, \quad m = 0, 1, 2, \ldots, \tag{1.18}$$

where m is the number of waves in the circumferential direction [8].

1.2.5 The Construction of a Shell Theory

In elastic theory there are three basic sets of equations namely equilibrium, kinematic (strain-displacement), and constitutive (Hooke's law). In shell theory, analysis of structures which physically have three-dimensional but which can be modelled as two-dimensional surfaces, for this we shall develop the equilibrium relations and limited to static isothermal loading of isotropic shells [1].

1.2.6 Deriving the Equations of Equilibrium

To formulate a complete set of equilibrium equations, we should write down a complete Lagrangian

$$L = T - (U_e + V)$$

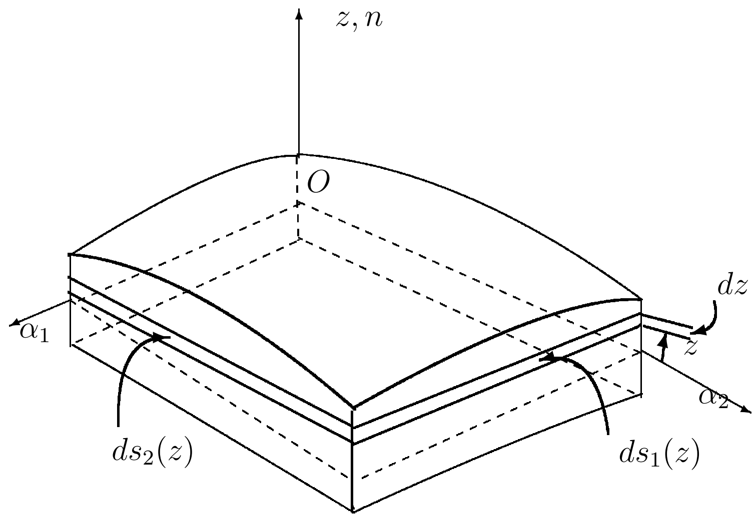


Figure 1.4: Shell differential element

where T=kinetic energy, U_e=strain energy, V= potential of applied edge and surface loads, and then properly apply Hamilton's principle,

$$\delta \int_{t_1}^{t_2} L \, dt = 0.$$

Now we switch to a consideration of the strain energy in a shell. By definition

$$U_e = \frac{1}{2} \int_V \Big(\sigma_1 \varepsilon_1 + \sigma_2 \varepsilon_2 + \tau_{12} \gamma_{12} \Big) \, dV, \tag{1.19}$$

then

$$\delta U_e = \int_V \Big(\sigma_1 \delta\varepsilon_1 + \sigma_2 \delta\varepsilon_2 + \tau_{12} \delta\gamma_{12} \Big) \, dV, \tag{1.20}$$

as a consequence of the linearity of the constitutive law applied by equation (1.19). For the volume element

$$dV = A_1 A_2 d\alpha_1 d\alpha_2 dz, \tag{1.21}$$

and then substituting we find

$$\delta U_e = \iiint_{\text{surf. z}} \left[\left(\frac{\sigma_1}{1 + \frac{z}{R_1}} \right) (\delta\varepsilon_1^0 + z\delta\kappa_1) + \left(\frac{\sigma_2}{1 + \frac{z}{R_2}} \right) (\delta\varepsilon_2^0 + z\delta\kappa_2) + \left(\frac{\tau_{12}}{1 + \frac{z}{R_2}} \right) (\delta\omega_2 + z\delta\tau_2) \right] dV.$$

If we substitute for the volume element (1.21)

$$\delta U_e = \iint (N_1\delta\varepsilon_1^0+M_1\delta\kappa_1+N_2\delta\varepsilon_2^0+M_2\delta\kappa_2+N_{12}\delta\omega_1+N_{21}\delta\omega_2+M_{12}\delta\tau_1+M_{21}\delta\tau_2)\, A_1A_2d\alpha_1d\alpha_2.$$

We shall ignore the kinetic energy (thus restricting ourselves to static problems) and include only a normal surface loading, i.e.,

$$V = +\iint q_n(\alpha_1,\alpha_2)w(\alpha_1,\alpha_2)\, A_1A_2d\alpha_1d\alpha_2,$$

so that

$$\delta V = +\iint q_n\delta w\, A_1A_2d\alpha_1d\alpha_2. \tag{1.22}$$

But,

$$\delta U_e = \iint \left(N_1\delta\varepsilon_1^0 + M_1\delta\kappa_1 + N_2\delta\varepsilon_2^0 + M_2\delta\kappa_2 + S\delta\omega + 2H\delta\tau^*\right)\, A_1A_2d\alpha_1d\alpha_2.$$

Thus the strain energy due to the shell deformation is now reduced to a surface integral involving the energy of stretching and bending the surface.

Now by use of the principle of minimum potential energy, we see that

$$\delta(U_e+V) = \iint (N_1\delta\varepsilon_1^0 + N_2\delta\varepsilon_2^0 + S\delta w + M_1\delta\kappa_1 + M_2\delta\kappa_2 + 2H\delta\tau^* + q_n\delta w)\, A_1A_2d\alpha_1d\alpha_2,$$

where

$$\tau^* = \tau_1 + \frac{\omega_2}{R_1} \equiv \tau_2 + \frac{\omega_1}{R_2},$$

and defining two new stress variables,

$$\begin{aligned} S &= N_{12} - \frac{M_{21}}{R_2} = N_{21} - \frac{M_{12}}{R_1}, \\ H &= \frac{1}{2}(M_{12}+M_{21}), \end{aligned}$$

and, defining a system of stress resultants, as follows

$$[N_1, M_1] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_1 \left(1+\frac{z}{R_2}\right) dz, \tag{1.23a}$$

$$[N_2, M_2] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_2 \left(1+\frac{z}{R_1}\right) dz, \tag{1.23b}$$

$$[N_{12}, M_{12}] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{12} \left(1+\frac{z}{R_2}\right) dz, \tag{1.23c}$$

$$[N_{21}, M_{21}] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{21} \left(1+\frac{z}{R_1}\right) dz. \tag{1.23d}$$

Thus we would define shear force resultants as follows

$$
\begin{aligned}
Q_1 &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{1n} \left(1 + \frac{z}{R_2}\right) dz, \\
Q_2 &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{2n} \left(1 + \frac{z}{R_1}\right) dz.
\end{aligned} \tag{1.24}
$$

Then, the term that we would add to the strain energy variation is

$$
\delta U_e|_{\text{add}} = \iiint \left(\tau_{1n}\delta\gamma_{1n} + \tau_{2n}\delta\gamma_{2n}\right) A_1 A_2 d\alpha_1 d\alpha_2 dz.
$$

then,

$$
\begin{aligned}
\delta\gamma_{1n} &= \frac{1}{A_1}\left(\frac{\partial \delta w}{\partial \alpha_1} - \frac{A_1}{R_1}\delta u_1 + A_1 \delta\beta_1\right), \\
\delta\gamma_{2n} &= \frac{1}{A_2}\left(\frac{\partial \delta w}{\partial \alpha_2} - \frac{A_2}{R_2}\delta u_2 + A_2 \delta\beta_2\right),
\end{aligned}
$$

so that using the definitions (1.24), the additional energy variation terms are

$$
\delta U_e|_{\text{add}} = \iint\limits_{\text{surf.}} \left[Q_1 A_2\left(\frac{\partial \delta w}{\partial \alpha_1} - \frac{A_1}{R_1}\delta u_1 + A_1\delta\beta_1\right) + Q_2 A_1\left(\frac{\partial \delta w}{\partial \alpha_2} - \frac{A_2}{R_2}\delta u_2 + A_2\delta\beta_2\right)\right] d\alpha_1 d\alpha_2.
$$

We shall conveniently leave the shear forces and twisting moments in their original untransformed state, i.e. as in the following equation

$$
\delta U_e = \iint \left(N_1\delta\varepsilon_1^0 + M_1\delta\kappa_1 + N_2\delta\varepsilon_2^0 + M_2\delta\kappa_2 + N_{12}\delta\omega_1 + N_{21}\delta\omega_2 + M_{12}\delta\tau_1 + M_{21}\delta\tau_2\right) A_1 A_2 d\alpha_1 d\alpha_2.
$$

Thus we shall vary

$$
\begin{aligned}
\delta U_e = \iint \Bigg[& N_1 \left(A_2 \frac{\partial \delta u_1}{\partial \alpha_1} + \frac{\partial A_1}{\partial \alpha_2} \delta u_2 + A_1 A_2 \frac{\delta w}{R_1} \right) \\
& + M_1 \left(A_2 \frac{\partial \delta \beta_1}{\partial \alpha_1} + \frac{\partial A_1}{\partial \alpha_2} \delta \beta_2 \right) + N_2 \left(A_1 \frac{\partial \delta u_2}{\partial \alpha_2} \right. \\
& \left. + \frac{\partial A_2}{\partial \alpha_1} \delta u_1 + A_1 A_2 \frac{\delta w}{R_2} \right) + M_2 \left(A_1 \frac{\partial \delta \beta_2}{\partial \alpha_2} + \frac{\partial A_2}{\partial \alpha_1} \delta \beta_1 \right) \\
& + N_{12} \left(A_2 \frac{\partial \delta u_2}{\partial \alpha_1} - \frac{\partial A_1}{\partial \alpha_2} \delta u_1 \right) + N_{21} \left(A_1 \frac{\partial \delta u_1}{\partial \alpha_2} - \frac{\partial A_2}{\partial \alpha_1} \delta u_2 \right) \\
& + M_{12} \left(A_2 \frac{\partial \delta \beta_2}{\partial \alpha_1} - \frac{\partial A_1}{\partial \alpha_2} \delta \beta_1 \right) + M_{21} \left(A_1 \frac{\partial \delta \beta_1}{\partial \alpha_2} - \frac{\partial A_2}{\partial \alpha_1} \delta \beta_1 \right) \\
& + Q_1 \left(A_2 \frac{\partial \delta w}{\partial \alpha_1} + A_1 A_2 \left(\delta \beta_1 - \frac{\delta u_1}{R_1} \right) \right) \\
& + Q_2 \left(A_1 \frac{\partial \delta w}{\partial \alpha_2} + A_1 A_2 \left(\delta \beta_2 - \frac{\delta u_2}{R_2} \right) \right) \Bigg] \, d\alpha_1 d\alpha_2.
\end{aligned}
\tag{1.25}
$$

Then variation of equations (1.22),(1.25) yields the equations of equilibrium (1.11) to (1.14) [1].

Chapter 2

Airy Functions

The solutions of the second order linear differential equation

$$\frac{d^2u}{dz^2} = zu, \tag{2.1}$$

are called *Airy functions*. These functions are closely related to the cylindrical functions namely Bessel functions, and play an important role in the theory of asymptotic representations of various special functions arising as solutions of linear differential equations [5].

By a cylindrical function we mean a solution of the second-order linear differential equation

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \Big(1 - \frac{\nu^2}{z^2}\Big)u = 0, \tag{2.2}$$

where z is a complex variable and ν is a parameter which can take the arbitrary real or complex values [5].

2.1 Integral Representations of $Ai(z)$ and $Bi(z)$

For real values of z the Airy integral is defined by

$$Ai(z) = \frac{2\sqrt{z}}{3\pi}\int_0^\infty \cos\Big(\frac{2z^{\frac{3}{2}}}{3}\sinh y\Big)\cosh\frac{y}{3}\,dy, \quad z > 0. \tag{2.3}$$

After making the substitution $\sinh\dfrac{y}{3} = \dfrac{1}{2}z^{-\frac{1}{2}t}$ this gives the following representation

$$Ai(z) = \frac{1}{\pi}\int_0^\infty \cos\Big(\frac{1}{3}t^3 + zt\Big)\,dt, \quad z \geq 0, \tag{2.4}$$

where the integrand does not decay as $t \to \infty$ making the integral converge because of the negative and positive parts of the rapid oscillations tend to cancel one another out and that for real values of z and the Airy integral of $Bi(z)$ is

$$Bi(z) = \frac{1}{\pi} \int_0^\infty \left[\exp\Big(-\frac{1}{3}t^3 + zt \Big) + \sin\Big(\frac{1}{3}t^3 + zt \Big) \right] dt, \quad z \geq 0 \quad [5]. \tag{2.5}$$

The Airy equation has no singular points except at $z = \infty$. Every other point in the z-plane is an ordinary point and so two linearly independent series expansions about it (formally with indicial values $\sigma = 0$ and $\sigma = 1$) can be found. Those about $z = 0$ take the forms $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^{n+1}$. The corresponding recurrence relations are

$$\begin{aligned} u_1(z) &= 1 + \frac{z^3}{(3)(2)} + \frac{z^6}{(6)(5)(3)(2)} + \cdots, \\ u_2(z) &= z + \frac{z^4}{(4)(3)} + \frac{z^7}{(7)(6)(4)(3)} + \cdots. \end{aligned}$$

The ratios of successive terms for the two series are thus

$$\begin{aligned} \frac{a_{n+3} z^{n+3}}{a_n z^n} &= \frac{z^3}{(n+3)(n+2)}, \quad \text{and} \\ \frac{b_{n+3} z^{n+4}}{b_n z^{n+1}} &= \frac{z^3}{(n+4)(n+3)}. \end{aligned}$$

It follows from the ratio test that both series are absolutely convergent for all z. A similar argument shows that the series for their derivatives are also absolutely convergent for all z. Any solution of the Airy equation is representable as a superposition of the two series and so is analytic for all finite z, it is therefore an integral function with only singularity at infinity [11].

Another form of solution of the Airy equation is one that takes the form of a contour integral in which z appears as a parameter in the integral. Consider the contour integral

$$u(z) = \int_a^b f(t) \exp(zt)\, dt, \tag{2.6}$$

in which a, b and $f(t)$ to be chosen. Substituiting (2.6) into (2.1) yields

$$\begin{aligned} \int_a^b t^2 f(t) \exp(zt)\, dt &= \int_a^b z f(t) \exp(zt)\, dt, \\ &= [f(t)\exp(zt)]_a^b - \int_a^b \frac{df(t)}{dt} \exp(zt)\, dt. \end{aligned}$$

If we could choose the limits a and b so that the end-point contributions vanish, then the Airy equation would be satisfied by (2.6), provided $f(t)$ satisfies

$$\frac{df(t)}{dt} + t^2 f(t) = 0, \quad \Rightarrow f(t) = A \exp\left(-\frac{1}{3}t^3\right),$$

where A is any constant. To make the end-point contributions vanish we must choose a and b such that

$$\exp\Big(-\frac{1}{3}t^3 + zt\Big) = 0 \quad \text{for both values of } t.$$

This can only happen if $|a| \to \infty$ and $|b| \to \infty$ and, even then, only if the real part of t^3 is positive. Setting $t = is$, where s is real and $-\infty < s < \infty$, converts the integral representation of $Ai(z)$ to

$$Ai(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[i\left(\frac{1}{3}s^3 + zs\right)\right] ds.$$

Now the exponent in this integral is an odd function of s and so the imaginary part of the integrand contributes nothing to the integral. What is left is therefore

$$Ai(z) = \frac{1}{\pi}\int_{0}^{\infty} \cos\left(\frac{1}{3}s^3 + zs\right) ds.$$

This shows explicitly that when z is real, so is $Ai(z)$. All solutions except the one called $Ai(z)$ tend to $\pm\infty$ as z (real) takes on increasingly large positive values. Its behaviour for negative real values of z, is that $Ai(z)$ oscillates almost sinusoidally in this region, except for a relatively slow increase in frequency and an even slower decrease in amplitude as $-z$ increases. The solution $Bi(z)$ is chosen to be the particular function that exhibits the same behaviour as $Ai(z)$ except that it is in quadrature with $Ai(z)$, i.e., it is $\frac{\pi}{2}$ out of phase with it [11].

General solutions of (2.1) can be expressed in terms of Bessel functions of imaginary argument of order $\nu = \pm\frac{1}{3}$. Two linearly independent solutions of (2.1) are

$$u_1 = Ai(z) \;=\; \frac{1}{3}\sqrt{z}\Big[I_{-\frac{1}{3}}(\zeta) - I_{\frac{1}{3}}(\zeta)\Big] \equiv \frac{1}{\pi}\sqrt{\frac{z}{3}}K_{\frac{1}{3}}(\zeta), \quad |\text{arg}z| < \frac{2\pi}{3}, \tag{2.7}$$

$$u_2 = Bi(z) \;=\; \sqrt{\frac{z}{3}}\Big[I_{-\frac{1}{3}}(\zeta) + I_{\frac{1}{3}}(\zeta)\Big], \quad |\text{arg}z| < \frac{2\pi}{3}, \tag{2.8}$$

$$\tag{2.9}$$

where $\zeta = \frac{2}{3}z^{\frac{3}{2}}$, are called Airy functions of the first and second kind respectively, where $Bi(z)$ is defined as the solution with the same amplitude of oscillations as $Ai(z)$ as $z \to -\infty$ [5].

Replacing I_ν by the series expansion

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(\frac{z}{2})^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \quad |z| < \infty, |\arg z| < \pi, \tag{2.10}$$

we obtain the expansions

$$\begin{aligned} Ai(z) &= \sum_{k=0}^{\infty} \frac{z^{3k}}{3^{2k+\frac{2}{3}} k!\Gamma(k+\frac{2}{3})} - \sum_{k=0}^{\infty} \frac{z^{3k+1}}{3^{2k+\frac{4}{3}} k!\Gamma(k+\frac{4}{3})}, \quad |z| < \infty, & (2.11) \\ Bi(z) &= \sqrt{3}\left[\sum_{k=0}^{\infty} \frac{z^{3k}}{3^{2k+\frac{2}{3}} k!\Gamma(k+\frac{2}{3})} + \sum_{k=0}^{\infty} \frac{z^{3k+1}}{3^{2k+\frac{4}{3}} k!\Gamma(k+\frac{4}{3})}\right], \quad |z| < \infty, & (2.12) \end{aligned}$$

where $\Gamma(x)$ is the gamma function and may be regarded as the generalization of the factorial function to non-integer and/or negative arguments and which shows that the Airy functions are entire functions of z and are real for real z [5].

A complex function is said to be analytic, if it is complex differentiable at every point and if it is analytic at all finite points of the complex plane, is said to be *entire* [9].

We can write the first expansion (2.11) as

$$Ai(z) = 3^{-\frac{2}{3}} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{k+1}{3}) \sin \frac{2\pi}{3}(k+1)}{k!} (\sqrt{3}z)^k, \quad |z| < \infty. \tag{2.13}$$

Asymptotic representations of the Airy functions for large $|z|$ are

$$\begin{aligned} Ai(z) &= \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\zeta}[1 + \underline{O}(|z|^{-\frac{3}{2}})], \quad |\arg z| \leq \frac{2\pi}{3} - \delta, & (2.14) \\ Bi(z) &= \sqrt{\pi} z^{-\frac{1}{4}} e^{\zeta}[1 + \underline{O}(|z|^{-\frac{3}{2}})], \quad |\arg z| \leq \frac{\pi}{3} - \delta, & (2.15) \end{aligned}$$

where $\zeta = \dfrac{2}{3} z^{\frac{3}{2}}$ [5].

The standard solutions $Ai(z)$ and $Bi(z)$ have the initial values

$$\begin{aligned} Ai(0) &= \frac{1}{\sqrt{3}} Bi(0) = \frac{1}{3^{\frac{2}{3}}\Gamma(\frac{2}{3})}, & (2.16) \\ Ai'(0) &= -\frac{1}{\sqrt{3}} Bi'(0) = -\frac{1}{3^{\frac{1}{3}}\Gamma(\frac{1}{3})}, & (2.17) \end{aligned}$$

and satisfy the Wronskian relation

$$Ai(z)Bi'(z) - Ai'(z)Bi(z) = \frac{1}{\pi} \quad [4]. \tag{2.18}$$

When z is positive $Ai(z),\ Bi(z),\ -Ai'(z)$ and $Bi'(z)$ are all positive and monotonic, when z is negative these functions are oscillatory, with diminishing period as $z \to -\infty$ [4].

The Airy functions of argument $-z$ can be expressed in terms of Bessel functions of the first kind, J_ν, of order $\nu = \pm\frac{1}{3}$ as

$$\begin{aligned} Ai(-z) &= \frac{1}{3}\sqrt{z}\Big[J_{-\frac{1}{3}}(\zeta) + J_{\frac{1}{3}}(\zeta)\Big], \quad |\arg z| \le \frac{2\pi}{3}, && (2.19)\\ Bi(-z) &= \Big(\frac{z}{3}\Big)^{\frac{1}{3}}\Big[J_{-\frac{1}{3}}(\zeta) - J_{\frac{1}{3}}(\zeta)\Big], \quad |\arg z| \le \frac{2\pi}{3}. && (2.20) \end{aligned}$$

$$\begin{aligned} Ai(-z) &\approx \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}}\cos(\zeta - \frac{\pi}{4}), \quad z \to \infty, && (2.21)\\ Bi(-z) &\approx -\frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}}\sin(\zeta - \frac{\pi}{4}), \quad z \to \infty, && (2.22) \end{aligned}$$

where $\zeta = \frac{2}{3}z^{\frac{3}{2}}$, which shows that the Airy functions have an oscillatory character for large negative values of the argument [5].

2.2 Asymptotic Behaviour of Airy Functions

The precise asymptotic behaviour is given by

$$\begin{aligned} Ai(z) &= \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\zeta}\Big[1 + O(z^{-\frac{3}{2}})\Big],\\ Ai'(z) &= -\frac{\sqrt{\pi}}{2} z^{\frac{1}{4}} e^{-\zeta}\Big[1 + O(z^{-\frac{3}{2}})\Big],\\ Bi(z) &= \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}} e^{\zeta}\Big[1 + O(z^{-\frac{3}{2}})\Big],\\ Bi'(z) &= \frac{1}{\sqrt{\pi}} z^{\frac{1}{4}} e^{\zeta}\Big[1 + O(z^{-\frac{3}{2}})\Big], \end{aligned}$$

as $z \to \infty$, where $\zeta = \frac{2}{3}z^{\frac{3}{2}}$ and

$$\begin{aligned} Ai(z) &= \frac{1}{\sqrt{\pi}} |z|^{-\frac{1}{4}}\Big[\cos\theta + O(|z|^{-\frac{3}{2}})\Big],\\ Ai'(z) &= \frac{1}{\sqrt{\pi}} |z|^{\frac{1}{4}}\Big[\sin\theta + O(|z|^{-\frac{3}{2}})\Big],\\ Bi(z) &= -\frac{1}{\sqrt{\pi}} |z|^{-\frac{1}{4}}\Big[\sin\theta + O(|z|^{-\frac{3}{2}})\Big],\\ Bi'(z) &= \frac{1}{\sqrt{\pi}} |z|^{\frac{1}{4}}\Big[\cos\theta + O(|z|^{-\frac{3}{2}})\Big], \end{aligned}$$

as $z \to -\infty$, where $\theta = \frac{2}{3}|z|^{\frac{3}{2}} - \frac{\pi}{4}$ [4].

2.3 Liouville's Differential Equation

Consider

$$\frac{d^2y}{dx^2} + [\lambda^2 p(x) + r(x)]y = 0, \tag{2.23}$$

for large positive λ, where x is a real variable and $a \leq x \leq b$, has solution

$$y = \frac{a_1 \cos[\lambda \int \sqrt{p(x)}\, dx] + b_1 \sin[\lambda \int \sqrt{p(x)}\, dx]}{\sqrt[4]{p(x)}}, \tag{2.24}$$

for positive $p(x)$ and

$$y = \frac{a_2 \exp[\lambda \int \sqrt{-p(x)}\, dx] + b_2 \exp[-\lambda \int \sqrt{-p(x)}\, dx]}{\sqrt[4]{-p(x)}}, \tag{2.25}$$

for negative $p(x)$ [6]. These approximations are valid as long as x is away from the zeros of $p(x)$. Equations (2.24) and (2.25) show that y is oscillatory on one side of a zero of $p(x)$ while it is exponential on the other side, hence such a zero is called a *transition point.* It is also called a *turning point* because in classic mechanics it is the point at which the kinetic energy of an incident particle is equal to its potential energy and that the particle therefore reverses direction. The point $x = c$ is called a turning point or a transition point of order α where α is the order of the zero of $p(x)$ at $x = c$ [6].

Assuming $p(x)$ has a zero in (a, b) and therefore a zero of $p(x)$ will be called a transition point of the differential equation, and also that $p(x)$ has a simple pole at $x = c$, another zero in $a \leq x \leq b$, then $p'(x) > 0$, $\;p(c) = 0$, $\;p'(c) \neq 0$, so that

$$\begin{aligned} p(x) < 0, &\quad \text{when} \;\; a \leq x < c, \\ p(x) > 0, &\quad \text{when} \;\; c < x \leq b, \end{aligned}$$

so there seems to be no simple elementary functions for which describes the transition from monotonic to oscillatory behaviour, then the asymptotic forms will involve some higher transcendental functions, for example, Airy functions [10].

Chapter 3

Low Fequency Vibrations of Shell of Revolution of Sign-Changing Curvature

3.1 Governing Equations

We now consider the vibration of a shell of revolution rotating around its axis of symmetry with a generating line which has a point of inflection as a result the sign of curvature of the middle surface changes.

On the middle surface we have curvilinear coordinates (s, φ), where s is the arc length of the generating line for which $s_1 \leq s \leq s_2$ and φ is the angle in the circumferential direction.

In this case

$$A_1 = 1, \quad A_2 = B,$$

where A_1 and A_2 are Lamé's coefficients as introduced in Chapter 1. The shell is bounded by two parallels and all the coefficients do not depend upon φ, and therefore we can separate the variables,

$$u, v, w(s, \varphi) = w(s)e^{im\varphi}, \quad m = 0, 1, 2, \ldots, \tag{3.1}$$

where m is the number of waves in the circumferential direction.

We introduce the following relations for the forces (T_1, T_2, S) and displacement components

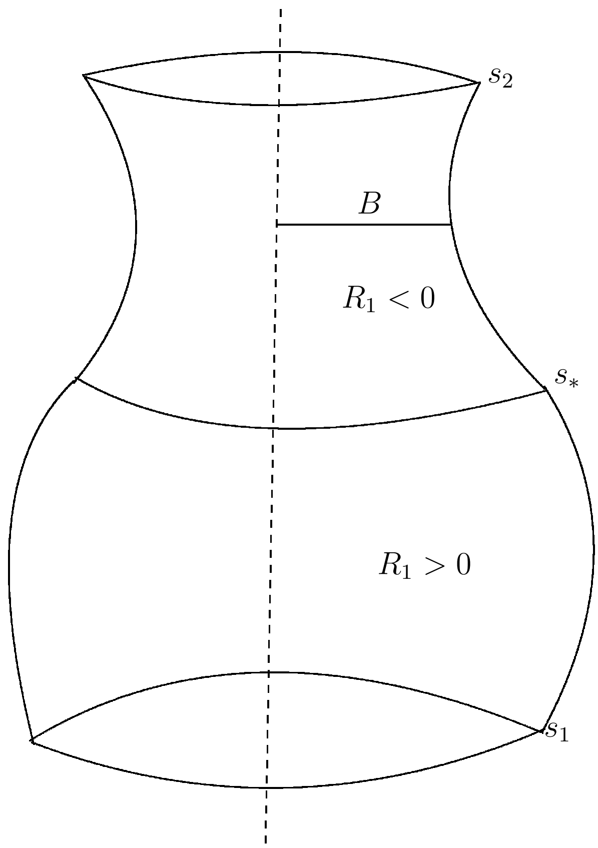


Figure 3.1: A shell of sign-changing curvature

(u, v, w) after separating the variables

$$T_1(s,\varphi) = T_1(s)\cos m\varphi, \qquad u(s,\varphi) = u(s)\cos m\varphi, \tag{3.2}$$

$$T_2(s,\varphi) = T_2(s)\cos m\varphi, \qquad v(s,\varphi) = v(s)\sin m\varphi, \tag{3.3}$$

$$S(s,\varphi) = S(s)\sin m\varphi, \qquad w(s,\varphi) = mw(s)\cos m\varphi. \tag{3.4}$$

Since we consider the shell dynamics, more precisely free vibrations, then the load projections are simply the forces of inertia, which are proportional to the frequency parameter λ, where

$$\lambda = \frac{\rho\omega^2}{E},$$

where ρ is the density, E is Young's modulus and ω is the unknown natural frequency.

The equations of equilibrium are from (1.11) and substituting $A_1 = 1, A_2 = B, \alpha_1 = s$ and $\alpha_2 = \varphi$, we get

$$\frac{\partial(BT_1)}{\partial s} - \frac{\partial B}{\partial s}T_2 + \frac{\partial S}{\partial\varphi} - \frac{B}{R_1}Q_1 + BF_1 = 0,$$

from (3.2) and (3.3), we have

$$\frac{\partial(B(s)T_1(s)\cos m\varphi)}{\partial s} - \frac{\partial B(s)}{\partial s}T_2(s)\cos m\varphi + \frac{\partial S(s)\sin m\varphi}{\partial\varphi} + \frac{\partial(1)}{\partial\varphi}S(s)\sin m\varphi = \frac{B}{R_1}Q_1 - BF_1,$$

Differentiating term by term, we have

$$\frac{dB(s)}{ds}T_1(s)\cos m\varphi + B(s)\frac{dT_1(s)}{ds}\cos m\varphi - \frac{dB(s)}{ds}T_2(s)\cos m\varphi + S(s)m\cos m\varphi = \frac{B}{R_1}Q_1 - BF_1,$$

dividing by $\cos m\varphi \neq 0$, we have

$$\frac{dB}{ds}T_1 + B\frac{dT_1}{ds} - \frac{dB}{ds}T_2 + mS = \frac{B}{R_1}Q_1 - BF_1,$$

grouping terms and dividing by $B \neq 0$,

$$\frac{dT_1}{ds} + \frac{1}{B}\frac{dB}{ds}(T_1 - T_2) + \frac{m}{B}S = \frac{Q_1}{R_1} - F_1,$$

denoting $\dfrac{dB}{ds}$ to be B', giving

$$\frac{dT_1}{ds} + \frac{B'}{B}(T_1 - T_2) + \frac{m}{B}S = \frac{Q_1}{R_1} - F_1.$$

We have by definition that $F_1 = Eh\lambda u$ and $Q_1 = \dfrac{h^2}{12}L_1$ and therefore combining we have

$$\frac{dT_1}{ds} + \frac{B'}{B}(T_1 - T_2) + \frac{m}{B}S = \frac{h^2}{12}L_1 - Eh\lambda u. \tag{3.5}$$

From (1.11) interchanging the indices we have

$$\frac{\partial T_2}{\partial \varphi} + \frac{\partial (BS)}{\partial s} + \frac{\partial B}{\partial s} S = \frac{B}{R_2} Q_2 - BF_2,$$

after separating variables, we have

$$\frac{\partial (T_1(s)\cos m\varphi)}{\partial \varphi} + \frac{\partial (B(s)S(s)\sin m\varphi}{\partial s} = \frac{\partial B(s)}{\partial s} S(s)\sin m\varphi = \frac{B}{R_2} Q_2 - BF_2,$$

differentiating term by term yields

$$-T_2(s) m \sin m\varphi + \frac{dB(s)}{ds} S(s)\sin m\varphi + B(s)\frac{dS(s)}{ds}\sin m\varphi + \frac{dB(s)}{ds} S(s)\sin m\varphi = \frac{B}{R_2} Q_2 - BF_2,$$

and dividing by $\sin m\varphi \neq 0$ and $B \neq 0$, we have

$$\frac{dS}{ds} + 2\frac{B'}{B} S - \frac{m}{B} T_2 = \frac{Q_2}{R_2} - F_2,$$

we have by definition that $F_2 = Eh\lambda v$ and $Q_2 = \frac{h^2}{12} L_2$ and therefore combining we have

$$\frac{dS}{ds} + 2\frac{B'}{B} S - \frac{m}{B} T_2 = \frac{h^2}{12} L_2 - Eh\lambda v. \tag{3.6}$$

Simplifying we have

$$B(s)\left(\frac{T_1(s)\cos m\varphi}{R_1} + \frac{T_2(s)\cos m\varphi}{R_2} + F_n\right) = 0,$$

dividing by $B \neq 0$ and $\cos m\varphi \neq 0$, we have

$$\frac{T_1}{R_1} + \frac{T_2}{R_2} = \frac{h^2}{12} L_3 - Eh\lambda w. \tag{3.7}$$

By definition, $\varepsilon_1 = \frac{1}{Eh}(T_1 - \nu T_2)$, we have

$$\frac{\partial (u(s)\cos m\varphi)}{\partial s} - \frac{w(s)\cos m\varphi}{R_1} = \frac{1}{Eh}(T_1(s)\cos m\varphi - \nu T_2(s)\cos m\varphi),$$

differentiating and dividing by $\cos m\varphi \neq 0$, we have

$$\frac{du}{ds} - \frac{w}{R_1} = \frac{T_1 - \nu T_2}{Eh}. \tag{3.8}$$

Interchanging indices and defining $\varepsilon_2 = \frac{1}{Eh}(T_2 - \nu T_1)$, we have

$$\frac{1}{B(s)}\frac{\partial (v(s)\sin m\varphi)}{\partial \varphi} + \frac{1}{B}\frac{\partial B(s)}{\partial s} u(s)\cos m\varphi - \frac{w(s)\cos m\varphi}{R_2} = \frac{1}{Eh}(T_2(s)\cos m\varphi - \nu T_1(s)\cos m\varphi),$$

differentiating and dividing by $\cos m\varphi \neq 0$, we have

$$\frac{m}{B}v + \frac{B'}{B}u - \frac{w}{R_2} = \frac{1}{Eh}(T_2 - \nu T_1). \tag{3.9}$$

By definition, $\omega = \omega_1 + \omega_2$ and $\omega = \dfrac{2(1+\nu)S}{Eh}$, we have

$$\frac{\partial(v(s)\sin m\varphi}{\partial s} - -\frac{1}{B(s)}\frac{\partial B(s)}{\partial s}v(s)\sin m\varphi + \frac{1}{B(s)}\frac{\partial(u(s)\cos m\varphi)}{\partial\varphi} = \frac{2(1+\nu)S}{Eh},$$

differentiating and dividing by $\sin m\varphi \neq 0$, yields

$$\frac{dv}{ds} - \frac{B'}{B} - \frac{m}{B}u = \frac{2(1+\nu)S}{Eh},$$

grouping terms yields

$$-\frac{m}{B}u + B\frac{d}{ds}\left(\frac{v}{B}\right) = \frac{2(1+\nu)S}{Eh}. \tag{3.10}$$

By definition, $L_k, k = 1, 2, 3$ are the momentum terms.

The equilibrium equations are therefore

$$\begin{aligned} \frac{dT_1}{ds} + \frac{B'}{B}(T_1 - T_2) + \frac{m}{B}S &= \frac{h^2}{12}L_1 - Eh\lambda u, \\ \frac{dS}{ds} + 2\frac{B'}{B}S - \frac{m}{B}T_2 &= \frac{h^2}{12}L_2 - Eh\lambda v, \\ \frac{T_1}{R_1} + \frac{T_2}{R_2} &= \frac{h^2}{12}L_3 - Eh\lambda w, \end{aligned} \tag{3.11}$$

and the deformation-displacement relations together with the elasticity relations are given by

$$\begin{aligned} \frac{du}{ds} - \frac{w}{R_1} &= \frac{T_1 - \nu T_2}{Eh}, \\ \frac{m}{B}v + \frac{B'}{B}u - \frac{w}{R_2} &= \frac{T_2 - \nu T_1}{Eh}, \\ -\frac{m}{B}u + B\frac{d}{ds}\left(\frac{v}{B}\right) &= \frac{2(1+\nu)S}{Eh}. \end{aligned} \tag{3.12}$$

Together the governing equations are

$$\begin{aligned}
\frac{dT_1}{ds} + \frac{B'}{B}(T_1 - T_2) + \frac{m}{B}S &= \frac{h^2}{12}L_1 - Eh\lambda u, \\
\frac{dS}{ds} + 2\frac{B'}{B}S - \frac{m}{B}T_2 &= \frac{h^2}{12}L_2 - Eh\lambda v, \\
\frac{T_1}{R_1} + \frac{T_2}{R_2} &= \frac{h^2}{12}L_3 - Eh\lambda w, \\
\frac{du}{ds} - \frac{w}{R_1} &= \frac{T_1 - \nu T_2}{Eh}, \\
\frac{m}{B}v + \frac{B'}{B}u - \frac{w}{R_2} &= \frac{T_2 - \nu T_1}{Eh}, \\
-\frac{m}{B}u + B\frac{d}{ds}\left(\frac{v}{B}\right) &= \frac{2(1+\nu)S}{Eh}.
\end{aligned}$$

The system of ordinary differential equations (3.11) and (3.12) has order eight and is complete with respect to six unknown variables u, v, w, T_1, T_2 and S.

Assume that the middle surface satisfies the following conditions

$$R_2 > 0, \qquad R_1^{-1}(s_*) = 0, \qquad (R_1^{-1}(s_*))' < 0, \tag{3.13}$$

and

$$\begin{aligned}
R_1 > 0 \quad &\text{if} \quad s_1 \leq s < s_*, \\
R_1 < 0 \quad &\text{if} \quad s_* < s \leq s_2,
\end{aligned}$$

where s_* is a turning point and $R_1^{-1} \neq 0$ and $s \neq s_*$.

On the negative Gaussian part of the shell

$$m \sim h^{-\frac{1}{3}}, \qquad \lambda \sim h^{\frac{2}{3}}, \tag{3.14}$$

and for the positive Gaussian part of the shell

$$\lambda \sim 1, \tag{3.15}$$

and the boundary conditions are strong enough to exclude pure bending.

3.1.1 Asymptotic Solutions

We now consider the integrals of the systems (3.11) and (3.12) assuming (3.14). Four solutions of (3.11) and (3.12) are boundary effect integrals and the other four integrals describe the main stress state, where the index of variation is $\frac{1}{3}$ (from the shell of negative Gaussian curvature).

The general idea of the method belongs to Langer, who realized that any attempt to express the asymptotic expansions of the solutions of turning point problems in terms of elementary functions must fail in regions containing the turning points. A uniformly valid expansion for all s must be expressed in terms of the solution of non-elementary functions which have the same qualitative features as the equation, for example, Airy functions [6].

The functions $\dfrac{1}{R_1}, \dfrac{1}{R_2}$ and B are sufficiently smooth.

We are going to use x for any of the variables in (3.11) and (3.12) i.e. u, v, w, T_1, T_2 and S.

For $s > s_*$, we have the representation

$$x_n = m^{\gamma(x)} \sum_{k=0}^{\infty} m^{-k} x_n^{(k)}(s) \exp\Big\{m \int_{s_0}^{s} q_n \, ds\Big\}, \quad n = 1, 2, 3, 4. \tag{3.16}$$

As for $s \neq s_*$, we have

$$q_1(s) \equiv q_2(s) = \sqrt{\frac{R_2}{B^2 R_1}}, \qquad q_3(s) \equiv q_4(s) = -\sqrt{\frac{R_2}{B^2 R_1}},$$

and for $s = s_*$,

$$q_1(s) = q_3(s),$$

hence s_* is a turning point, so the coefficients $x_n^{(k)}$ in (3.16) are irregular.

For the asymptotic representation of solutions, let all unknowns $x(s, m)$ from (3.11) and (3.12) be represented in the form of the combination of the Airy function $U(\eta)$ and its derivative $\dfrac{dU(\eta)}{d\eta}$ as

$$x(s, m) = x^{(1)}(s, m) U(\eta) + x^{(2)}(s, m) \frac{dU(\eta)}{d\eta}, \tag{3.17}$$

where $U(\eta)$ is solution of the Airy equation

$$\frac{d^2 U}{d\eta^2} + \eta U = 0, \quad \eta = m^{\frac{2}{3}} \xi(s), \tag{3.18}$$

and functions $x^{(k)}$ are formal series of (3.18),

$$x^{(k)} = m^{\gamma_k(x)} \sum_{l=0}^{\infty} m^{-2l} x^{(k,l)}(s), \quad k = 1, 2, \tag{3.19}$$

i.e. if they are substituted into the differential equation as if the infinite series were convergent, the differential equation is satisfied. However, the infinite series involved in the formal series are in general divergent.

We have the unknowns $\xi, x^{(k,l)}$ and index of intensity γ_k, which are found by substituting (3.17) into (3.11) and (3.12) and equating coefficients by m^pU and $m^p\dfrac{dU}{d\eta}$.

3.1.2 Zeroth Approximation

We consider the zeroth approximation solution for the case when $l = 0$ in the series (3.19) i.e.

$$\begin{aligned} x^{(1)}(s,m) &= m^{\gamma_1(x)}x^{(1,0)} + \underline{O}\Big(\frac{1}{m^2}\Big), \quad \text{when} \quad k = 1, \\ x^{(2)}(s,m) &= m^{\gamma_2(x)}x^{(2,0)} + \underline{O}\Big(\frac{1}{m^2}\Big), \quad \text{when} \quad k = 2, \end{aligned}$$

then the asymptotic series (3.17) becomes

$$x(s,m) = m^{\gamma_1(x)}x^{(1,0)}U(\eta) + m^{\gamma_2(x)}x^{(2,0)}\frac{dU(\eta)}{d\eta}, \tag{3.20}$$

where the terms $x^{(k,0)}$ are functions of s for $k = 1, 2$.

Differentiating (3.17) noting that $\eta = m^{\frac{2}{3}}\xi(s)$ and $\dfrac{d^2U}{d\eta^2} = -\eta U$, we have

$$\begin{aligned} \frac{dx(s)}{ds} &= m^{\gamma_1(x)}\frac{dx^{(1,0)}}{ds}U + m^{\gamma_1(x)}x^{(1,0)}\frac{dU}{ds} \\ &+ m^{\gamma_2(x)}\frac{dx^{(2,0)}}{ds}\frac{dU}{d\eta} + m^{\gamma_2(x)}x^{(2,0)}\frac{d}{ds}\Big(\frac{dU}{d\eta}\Big) \\ &= m^{\gamma_1(x)}\frac{dx^{(1,0)}}{ds}U + m^{\gamma_1(x)}x^{(1,0)}\frac{dU}{d\eta}\frac{d\eta}{ds} + m^{\gamma_2(x)}\frac{dx^{(2,0)}}{ds}\frac{dU}{d\eta} \\ &+ m^{\gamma_2(x)}x^{(2,0)}\frac{d}{d\eta}\Big(\frac{dU}{d\eta}\Big)\frac{d\eta}{ds} \\ &= m^{\gamma_1(x)}\frac{dx^{(1,0)}}{ds}U + m^{\gamma_1(x)}x^{(1,0)}\frac{dU}{d\eta}\frac{d\eta}{ds} + m^{\gamma_2(x)}\frac{dx^{(2,0)}}{ds}\frac{dU}{d\eta} \\ &+ m^{\gamma_2(x)}x^{(2,0)}\frac{d^2U}{d\eta^2}\frac{d\eta}{ds}, \end{aligned}$$

since $\eta = m^{\frac{2}{3}}\xi(s)$, differentiating gives

$$\begin{aligned} d\eta &= m^{\frac{2}{3}}\xi'(s)ds, \\ \frac{d\eta}{ds} &= m^{\frac{2}{3}}\xi'(s), \end{aligned} \tag{3.21}$$

substituting (3.21) into the above gives

$$\frac{dx}{ds} = m^{\gamma_1(x)}\frac{dx^{(1,0)}}{ds}U + m^{\gamma_1(x)}x^{(1,0)}m^{\frac{2}{3}}\xi'\frac{dU}{d\eta} + m^{\gamma_2(x)}\frac{dx^{(2,0)}}{ds}\frac{dU}{d\eta} - m^{\gamma_2(x)}x^{(2,0)}m^{\frac{2}{3}}\xi\xi' m^{\frac{2}{3}}U,$$

and grouping terms, we have

$$\frac{dx}{ds} = m^{\gamma_1(x)}\frac{dx^{(1,0)}}{ds}U + \xi' m^{\gamma_1(x)+\frac{2}{3}}x^{(1,0)}\frac{dU}{d\eta} - m^{\gamma_2(x)+\frac{4}{3}}\xi\xi' x^{(2,0)}U + m^{\gamma_2(x)}\frac{dx^{(2,0)}}{ds}\frac{dU}{d\eta}. \quad (3.22)$$

The membrane integrals are the solutions of the membrane equations (3.11) when they are homogeneous equations. For low-frequency vibrations of shell of revolution with m waves in circumferential direction, the following expressions are obtained when equating coefficients of U and $\frac{dU}{d\eta}$.

For

$$\frac{dT_1}{ds} + \frac{B'}{B}(T_1 - T_2) + \frac{m}{B}S = 0, \quad (3.23)$$

substituting (3.20)and (3.22) into (3.23), we have

$$\begin{aligned}0 = {} & m^{\gamma_1(T_1)}\frac{dT_1^{(1,0)}}{ds}U + \xi' m^{\gamma_1(T_1)+\frac{2}{3}}T_1^{(1,0)}\frac{dU}{d\eta} - m^{\gamma_2(T_1)+\frac{4}{3}}\xi\xi' T_1^{(2,0)}U + m^{\gamma_2(T_1)}\frac{dT_1^{(2,0)}}{ds}\frac{dU}{d\eta} \\ & + \frac{B'}{B}\left(m^{\gamma_1(T_1)}T_1^{(1,0)}U + m^{\gamma_2(T_1)}T_1^{(2,0)}\frac{dU}{d\eta} - m^{\gamma_1(T_2)}T_1^{(1,0)}U - m^{\gamma_2(T_2)}T_2^{(2,0)}\frac{dU}{d\eta}\right) \\ & + \frac{m}{B}\left(m^{\gamma_1(S)}S^{(1,0)}U + m^{\gamma_2(S)}S^{(2,0)}\frac{dU}{d\eta}\right),\end{aligned}$$

equating coefficients of U, gives

$$m^{\gamma_1(T_1)}\frac{dT_1^{(1,0)}}{ds} - m^{\gamma_2(T_1)+\frac{4}{3}}\xi\xi' T_1^{(2,0)} + \frac{1}{B}m^{\gamma_1(S)+1}S^{(1,0)} + \frac{B'}{B}\left(m^{\gamma_1(T_1)}T_1^{(1,0)} - m^{\gamma_1(T_2)}T_1^{(1,0)}\right) = 0,$$

and those after equating coefficients of $\frac{dU}{d\eta}$ are

$$\xi' m^{\gamma_1(T_1)+\frac{2}{3}}T_1^{(1,0)} + m^{\gamma_2(T_1)}\frac{dT_1^{(2,0)}}{ds} + \frac{B'}{B}\left(m^{\gamma_2(T_1)}T_1^{(2,0)} - m^{\gamma_2(T_2)}T_2^{(2,0)}\right) + \frac{1}{B}m^{\gamma_2(S)+1}S^{(2,0)} = 0.$$

For

$$\frac{dS}{ds} + 2\frac{B'}{B}S - \frac{m}{B}T_2 = 0, \quad (3.24)$$

substituting (3.20)and (3.22) into (3.24), we have

$$0 = m^{\gamma_1(S)}\frac{dS^{(1,0)}}{ds}U + \xi' m^{\gamma_1(S)+\frac{2}{3}}S^{(1,0)}\frac{dU}{d\eta} - m^{\gamma_2(S)+\frac{4}{3}}\xi\xi' S^{(2,0)}U + m^{\gamma_2(S)}\frac{dS^{(2,0)}}{ds}\frac{dU}{d\eta}$$
$$+ 2\frac{B'}{B}\left(m^{\gamma_1(S)}S^{(1,0)}U + m^{\gamma_2(S)}S^{(2,0)}\frac{dU}{d\eta}\right) - \frac{1}{B}\left(m^{\gamma_1(T_2)+1}T_2^{(1,0)}U + m^{\gamma_2(T_2)+1}T_2^{(2,0)}\frac{dU}{d\eta}\right),$$

equating coefficients of U,

$$m^{\gamma_1(S)}\frac{dS^{(1,0)}}{ds} - m^{\gamma_2(S)+\frac{4}{3}}\xi\xi' S^{(2,0)} + 2\frac{B'}{B}m^{\gamma_1(S)}S^{(1,0)} - \frac{1}{B}m^{\gamma_1(T_2)+1}T_2^{(1,0)} = 0,$$

and equating the coefficients of $\dfrac{dU}{d\eta}$ we have

$$\xi' m^{\gamma_1(S)+\frac{2}{3}}S^{(1,0)} + m^{\gamma_2(S)}\frac{dS^{(2,0)}}{ds} + 2\frac{B'}{B}m^{\gamma_2(S)}S^{(2,0)} - \frac{1}{B}m^{\gamma_2(T_2)+1}T_2^{(2,0)} = 0.$$

For

$$\frac{T_1}{R_1} + \frac{T_2}{R_2} = 0, \tag{3.25}$$

substituting (3.20)and (3.22) into (3.25), we have

$$\frac{1}{R_1}\left(m^{\gamma_1(T_1)}T_1^{(1,0)}U + m^{\gamma_2(T_1)}T_1^{(2,0)}\frac{dU}{d\eta}\right) + \frac{1}{R_2}\left(m^{\gamma_1(T_2)}T_2^{(1,0)}U + m^{\gamma_2(T_2)}T_2^{(2,0)}\frac{dU}{d\eta}\right) = 0,$$

equating coefficients of U,

$$m^{\gamma_1(T_1)}\frac{T_1^{(1,0)}}{R_1} + m^{\gamma_1(T_2)}\frac{T_2^{(1,0)}}{R_2} = 0,$$

and equating the coefficients of $\dfrac{dU}{d\eta}$, we have

$$m^{\gamma_2(T_1)}\frac{T_1^{(2,0)}}{R_1} + m^{\gamma_2(T_2)}\frac{T_2^{(2,0)}}{R_2} = 0.$$

The last group of integrals called the bending ones, may be obtained from the bending equations after we put all deformations $\varepsilon_1, \varepsilon_2$ and ω to be equal to zero in equations (3.12). Hence their zeroth approximations are, for

$$\frac{du}{ds} - \frac{w}{R_1} = 0, \tag{3.26}$$

substituting (3.20)and (3.22) into (3.26), we have

$$0 = m^{\gamma_1(u)}\frac{du^{(1,0)}}{ds}U + \xi' m^{\gamma_1(u)+\frac{2}{3}}u^{(1,0)}\frac{dU}{d\eta} - m^{\gamma_2(u)+\frac{4}{3}}\xi\xi' u^{(2,0)}U + m^{\gamma_2(u)}\frac{du^{(2,0)}}{ds}\frac{dU}{d\eta}$$
$$+ \frac{B'}{B}\left(m^{\gamma_1(u)}u^{(1,0)}U + m^{\gamma_2(u)}u^{(2,0)}\frac{dU}{d\eta}\right) - \frac{1}{R_1}\left(m^{\gamma_1(w)}w^{(1,0)}U + m^{\gamma_2(w)}w^{(2,0)}\frac{dU}{d\eta}\right),$$

equating coefficients of U,

$$m^{\gamma_1(u)}\frac{du^{(1,0)}}{ds} - m^{\gamma_2(u)+\frac{4}{3}}\xi\xi' u^{(2,0)} - \frac{1}{R_1}m^{\gamma_1(w)}w^{(1,0)} + \frac{B'}{B}m^{\gamma_1(u)}u^{(1,0)} = 0,$$

and equating the coefficients of $\frac{dU}{d\eta}$, we have

$$\xi' m^{\gamma_1(u)+\frac{2}{3}}u^{(1,0)} + m^{\gamma_2(u)}\frac{du^{(2,0)}}{ds} - \frac{1}{R_1}m^{\gamma_2(w)}w^{(2,0)} + \frac{B'}{B}m^{\gamma_2(u)}u^{(2,0)} = 0.$$

For

$$\frac{m}{B}v + \frac{B'}{B}u - \frac{w}{R_2} = 0, \tag{3.27}$$

coefficients of U are

$$\frac{1}{B}m^{\gamma_1(v)+1}v^{(1,0)} - \frac{1}{R_2}m^{\gamma_1(w)}w^{(1,0)} = 0,$$

coefficients of $\frac{dU}{d\eta}$ are

$$m^{\gamma_2(v)+1}v^{(2,0)} - \frac{1}{R_2}m^{\gamma_2(w)}w^{(2,0)} = 0,$$

and for

$$-\frac{m}{B}u + B\frac{d}{ds}\left(\frac{v}{B}\right) = 0, \tag{3.28}$$

coefficients of U are

$$-\frac{1}{B}m^{\gamma_1(u)+1}u^{(1,0)} + m^{\gamma_1(v)}\frac{dv^{(1,0)}}{ds} - m^{\gamma_2(v)+\frac{4}{3}}\xi\xi' v^{(2,0)} + \frac{B'}{B}m^{\gamma_1(v)}v^{(1,0)} = 0,$$

and coefficients of $\frac{dU}{d\eta}$ are

$$-\frac{1}{B}m^{\gamma_2(u)+1}u^{(2,0)} + m^{\gamma_1(v)+\frac{2}{3}}\xi' v^{(1,0)} + m^{\gamma_2(v)}\frac{dv^{(2,0)}}{ds} + \frac{B'}{B}m^{\gamma_2(v)}v^{(2,0)} = 0.$$

We now consider the equilibrium conditions for the shell of revolution and free vibrations for the zeroth approximation after equating coefficients, thus these relations are obtained

$$-\xi\xi' T_1^{(2,0)} + \frac{1}{B} S^{(1,0)} = 0, \tag{3.29}$$

$$\begin{aligned}
\xi' T_1^{(1,0)} + \frac{1}{B} S^{(2,0)} &= 0, \\
\xi\xi' S^{(2,0)} + \frac{1}{B} T_2^{(1,0)} &= 0, \\
\xi' S^{(1,0)} - \frac{1}{B} T_2^{(2,0)} &= 0.
\end{aligned} \tag{3.30}$$

Equating the terms in powers of m, we have

$$m^{\gamma_1(T_1)} = m^{\gamma_1(T_2)} \quad \Longrightarrow \gamma_1(T_1) = \gamma_1(T_2) = \alpha_1,$$

$$\begin{aligned}
m^{\gamma_1(T_1)+\frac{2}{3}} &= m^{\gamma_2(S)+1}, \\
\gamma_1(T_1) + \frac{2}{3} &= \gamma_2(S) + 1, \\
\alpha_1 + \frac{2}{3} - 1 &= \gamma_2(S), \\
\gamma_2(S) &= \alpha_1 - \frac{1}{3}.
\end{aligned}$$

$$m^{\gamma_2(T_1)} = m^{\gamma_2(T_2)} \quad \Longrightarrow \gamma_2(T_1) = \gamma_2(T_2) = \alpha_2,$$

$$\begin{aligned}
m^{\gamma_1(S)+1} &= m^{\gamma_2(T_1)+\frac{4}{3}}, \\
\gamma_1(S) + 1 &= \gamma_2(T_1) + \frac{4}{3}, \\
\gamma_1(S) &= \alpha_2 + \frac{4}{3} - 1, \\
\gamma_1(S) &= \alpha_2 + \frac{1}{3}.
\end{aligned}$$

Considering the bending conditions for the shell of revolution and free vibrations for the zeroth approximation after equating coefficients, we have the following relations

$$\begin{aligned}
\frac{1}{R_1}T_1^{(1,0)} + \frac{1}{R_2}T_2^{(1,0)} &= 0,\\
\frac{1}{R_1}T_1^{(2,0)} + \frac{1}{R_2}T_2^{(2,0)} &= 0,\\
\xi\xi' u^{(2,0)} + \frac{1}{R_1}w^{(1,0)} &= 0,\\
\xi' u^{(1,0)} - \frac{1}{R_1}w^{(2,0)} &= 0,\\
\frac{1}{B}v^{(1,0)} - \frac{1}{R_2}w^{(1,0)} &= 0,\\
\frac{1}{B}v^{(2,0)} - \frac{1}{R_2}w^{(2,0)} &= 0,\\
\frac{1}{B}u^{(1,0)} - \xi\xi' v^{(2,0)} &= 0,\\
-\frac{1}{B}u^{(2,0)} - \xi' v^{(1,0)} &= 0.
\end{aligned}$$

Equating the terms in powers of m, the following results are obtained ,

$$\begin{aligned}
\text{let } \gamma_1(u) &= \alpha_1,\\
\gamma_2(u) &= \alpha_2,
\end{aligned}$$

$$\begin{aligned}
m^{\gamma_2(u)+\frac{4}{3}} &= m^{\gamma_1(w)},\\
\gamma_2(u) + \frac{4}{3} &= \gamma_1(w),\\
\gamma_1(w) &= \alpha_2 + \frac{4}{3}.
\end{aligned}$$

$$\begin{aligned}
m^{\gamma_1(u)+\frac{2}{3}} &= m^{\gamma_2(w)},\\
\gamma_1(u) + \frac{2}{3} &= \gamma_2(w),\\
\gamma_2(w) &= \alpha_1 + \frac{2}{3}.
\end{aligned}$$

$$\begin{aligned}
m^{\gamma_1(u)+1} &= m^{\gamma_2(v)+\frac{4}{3}},\\
\gamma_1(u) + 1 &= \gamma_2(v) + \frac{4}{3},\\
\gamma_1(u) + 1 - \frac{4}{3} &= \gamma_2(v),\\
\gamma_2(v) &= \alpha_1 - \frac{1}{3}.
\end{aligned}$$

$$\begin{aligned}
m^{\gamma_2(u)+1} &= m^{\gamma_1(v)+\frac{2}{3}}, \\
\gamma_2(u)+1 &= \gamma_1(v)+\frac{2}{3}, \\
\gamma_2(u)+1-\frac{2}{3} &= \gamma_1(v), \\
\gamma_1(v) &= \alpha_2+\frac{1}{3}.
\end{aligned}$$

Collectively we have

$$\gamma_1(u)=\alpha_1, \qquad \gamma_2(u)=\alpha_2, \qquad \gamma_1(w)=\alpha_2+\frac{4}{3}, \tag{3.31}$$

$$\gamma_2(w)=\alpha_1+\frac{2}{3}, \qquad \gamma_2(v)=\alpha_1-\frac{1}{3}, \qquad \gamma_1(v)=\alpha_2+\frac{1}{3}, \tag{3.32}$$

$$\gamma_2(S)=\alpha_1-\frac{1}{3}, \qquad \gamma_1(S)=\alpha_2+\frac{1}{3}, \qquad \gamma_1(T_1)=\gamma_1(T_2)=\alpha_1. \tag{3.33}$$

From these zeroth approximation, the equation for $\xi(s)$ may be obtained in the following way. Using (3.29) and rearranging to have

$$\begin{aligned}
\frac{1}{B}S^{(1,0)} &= \xi\xi' T_1^{(2,0)}, \\
S^{(1,0)} &= B\xi\xi' T_1^{(2,0)},
\end{aligned} \tag{3.34}$$

and substituting (3.34) into (3.30), we have

$$\begin{aligned}
\xi(\xi')^2 B T_1^{(2,0)} - \frac{1}{B}T_2^{(2,0)} &= 0, \\
\frac{1}{R_1}T_1^{(2,0)} + \frac{1}{R_2}T_2^{(2,0)} &= 0.
\end{aligned}$$

In matrix form, these two equations are

$$\begin{pmatrix} \xi(\xi')^2 B & -\frac{1}{B} \\ \frac{1}{R_1} & \frac{1}{R_2} \end{pmatrix} \begin{pmatrix} T_1^{(2,0)} \\ T_2^{(2,0)} \end{pmatrix} = 0.$$

To obtain non-trivial solutions the determinant of the system should be zero i.e.

$$\begin{vmatrix} \xi(\xi')^2 B & -\frac{1}{B} \\ \frac{1}{R_1} & \frac{1}{R_2} \end{vmatrix} = 0,$$

cross-multiplying gives

$$\begin{aligned}
\frac{\xi(\xi')^2 B}{R_2} + \frac{1}{BR_1} &= 0, \\
\frac{\xi(\xi')^2 B}{R_2} &= -\frac{1}{BR_1}, \\
\xi(\xi')^2 &= -\frac{R_2}{B^2 R_1} = r^2.
\end{aligned}$$

Since we are dealing with the part containing the negative Gaussian curvature, $R_1R_2 < 0$, but $R_2 > 0$, therefore $R_1 < 0$. Hence $-\dfrac{R_2}{R_1} > 0$, therefore $-\dfrac{R_2}{B^2R_1} > 0$ and making $r > 0$ $\Longrightarrow \xi > 0$ and $s > s_*$ on the negative Gaussian curvature part.

$$\xi(\xi')^2 = r^2,$$

squaring both sides since $r > 0$, $\xi > 0$ and $\sqrt{r^2} = |r|$, but $r > 0$, hence $\sqrt{r^2} = |r| = r$,

$$\sqrt{\xi}\xi' = r,$$

but,

$$\sqrt{\xi}\frac{d\xi}{ds} = r,$$

integrating both sides on the negative Gaussian curvature part,

$$\begin{aligned}
\int \xi^{\frac{1}{2}}\, d\xi &= \int_{s_*}^{s} r\, ds,\\
\frac{2}{3}\xi^{\frac{3}{2}} &= \int_{s_*}^{s} r\, ds,\\
\xi^{\frac{3}{2}} &= \frac{3}{2}\int_{s_*}^{s} r\, ds,\\
\xi &= \Big(\frac{3}{2}\int_{s_*}^{s} r ds\Big)^{\frac{2}{3}}.
\end{aligned}$$

3.1.3 First Approximation

When considering the first approximation, the set of equilibrium equations are as follows

$$\begin{aligned}
\frac{dT_1}{ds} + \frac{B'}{B}(T_1 - T_2) + \frac{m}{B}S = 0,\\
\frac{dS}{ds} + 2\frac{B'}{B}S - \frac{m}{B}T_2 = 0,\\
\frac{T_1}{R_1} + \frac{T_2}{R_2} = \frac{m}{Eh}\Big(Eh\lambda w'' - Eh\lambda\frac{m^2}{B^2}w\Big) + \frac{h^2m}{12(1-\nu^2)}\Big(\frac{d^2}{ds^2} - \frac{m^2}{B^2}\Big)^2 w,
\end{aligned}$$

and the equations for the bending ones are

$$\begin{aligned}
\frac{du}{ds} - \frac{w}{R_1} &= T_1 - \nu T_2,\\
\frac{m}{B}v + \frac{B'}{B}u - \frac{w}{R_2} &= T_2 - \nu T_1,\\
-\frac{m}{B}u + B\frac{d}{ds}\left(\frac{v}{B}\right) &= 2(1+\nu)S.
\end{aligned}$$

Consider the case when $l = 1$ and substitute into (3.19), we have

$$\begin{aligned} x^{(1)} &= m^{\gamma_1(x)} x^{(1,0)} + m^{\gamma_1(x)-2} x^{(1,1)} + \underline{O}\Big(\frac{1}{m^4}\Big), \\ x^{(2)} &= m^{\gamma_2(x)} x^{(2,0)} + m^{\gamma_2(x)-2} x^{(2,1)} + \underline{O}\Big(\frac{1}{m^4}\Big), \end{aligned}$$

and hence (3.17) becomes

$$x(s,m) = \Big(m^{\gamma_1(x)} x^{(1,0)} + m^{\gamma_1(x)-2} x^{(1,1)}\Big) U + \Big(m^{\gamma_2(x)} x^{(2,0)} + m^{\gamma_2(x)-2} x^{(2,1)}\Big) \frac{dU}{d\eta}. \tag{3.35}$$

Differentiating expression (3.35) with respect to s, the expression obtained is

$$\begin{aligned} \frac{dx}{ds} &= \Big(m^{\gamma_1(x)} \frac{dx^{(1,0)}}{ds} + m^{\gamma_1(x)-2} \frac{dx^{(1,1)}}{ds}\Big) U + \Big(m^{\gamma_1(x)} x^{(1,0)} + m^{\gamma_1(x)-2} x^{(1,1)}\Big) \frac{dU}{d\eta}\frac{d\eta}{ds} \\ &+ \Big(m^{\gamma_2(x)} \frac{dx^{(2,0)}}{ds} + m^{\gamma_2(x)-2} \frac{dx^{(2,1)}}{ds}\Big) \frac{dU}{d\eta} + \Big(m^{\gamma_2(x)} x^{(2,0)} + m^{\gamma_2(x)-2} x^{(2,1)}\Big) \frac{d}{d\eta}\Big(\frac{dU}{d\eta}\Big)\frac{d\eta}{ds} \\ &= \Big(m^{\gamma_1(x)} \frac{dx^{(1,0)}}{ds} + m^{\gamma_1(x)-2} \frac{dx^{(1,1)}}{ds}\Big) U + m^{\frac{2}{3}}\xi'\Big(m^{\gamma_1(x)} x^{(1,0)} + m^{\gamma_1(x)-2} x^{(1,1)}\Big) \frac{dU}{d\eta} \\ &+ \Big(m^{\gamma_2(x)} \frac{dx^{(2,0)}}{ds} + m^{\gamma_2(x)-2} \frac{dx^{(2,1)}}{ds}\Big) \frac{dU}{d\eta} + \Big(m^{\gamma_2(x)} x^{(2,0)} + m^{\gamma_2(x)-2} x^{(2,1)}\Big) \frac{d^2U}{d\eta^2}\frac{d\eta}{ds} \\ &= \Big(m^{\gamma_1(x)} \frac{dx^{(1,0)}}{ds} + m^{\gamma_1(x)-2} \frac{dx^{(1,1)}}{ds}\Big) U + \Big(\xi' m^{\gamma_1(x)+\frac{2}{3}} x^{(1,0)} + \xi' m^{\gamma_1(x)-\frac{4}{3}} x^{(1,1)}\Big) \frac{dU}{d\eta} \\ &+ \Big(m^{\gamma_2(x)} \frac{dx^{(2,0)}}{ds} + m^{\gamma_2(x)-2} \frac{dx^{(2,1)}}{ds}\Big) \frac{dU}{d\eta} + -\xi\xi' m^{\frac{4}{3}}\Big(m^{\gamma_2(x)} x^{(2,0)} + m^{\gamma_2(x)-2} x^{(2,1)}\Big) U \\ &= \Big(m^{\gamma_1(x)} \frac{dx^{(1,0)}}{ds} + m^{\gamma_1(x)-2} \frac{dx^{(1,1)}}{ds}\Big) U + \Big(\xi' m^{\gamma_1(x)+\frac{2}{3}} x^{(1,0)} + \xi' m^{\gamma_1(x)-\frac{4}{3}} x^{(1,1)}\Big) \frac{dU}{d\eta} \\ &+ \Big(m^{\gamma_2(x)} \frac{dx^{(2,0)}}{ds} + m^{\gamma_2(x)-2} x^{(2,1)}\Big) \frac{dU}{d\eta} - \Big(\xi\xi' m^{\gamma_2(x)+\frac{4}{3}} x^{(2,0)} + \xi\xi' m^{\gamma_2(x)-\frac{2}{3}} x^{(2,1)}\Big) U. \end{aligned} \tag{3.36}$$

For the equilibrium equation

$$\frac{dT_1}{ds} + \frac{B'}{B}(T_1 - T_2) + \frac{m}{B} S = 0, \tag{3.37}$$

substituting (3.35) and (3.36) into (3.37) and rearranging terms

$$
\begin{aligned}
0 = &\Big(m^{\gamma_1(T_1)}\frac{dT_1^{(1,0)}}{ds} + m^{\gamma_1(T_1)-2}\frac{dT_1^{(1,1)}}{ds}\Big)U + \Big(\xi' m^{\gamma_1(T_1)+\frac{2}{3}}T_1^{(1,0)} + \xi' m^{\gamma_1(T_1)-\frac{4}{3}}T_1^{(1,1)}\Big)\frac{dU}{d\eta}\\
&+ \Big(\xi\xi' m^{\gamma_2(T_1)+\frac{4}{3}}T_1^{(2,0)} + \xi\xi' m^{\gamma_2(T_1)-\frac{2}{3}}T_1^{(2,1)}\Big)U\\
&+ \frac{B'}{B}\Big[\Big(m^{\gamma_1(T_1)}T_1^{(1,0)} + m^{\gamma_1(T_1)-2}T_1^{(1,1)}\Big)U + \Big(m^{\gamma_2(T_1)}T_1^{(2,0)} + m^{\gamma_2(T_1)-2}T_1^{(2,1)}\Big)\frac{dU}{d\eta}\Big]\\
&- \frac{B'}{B}\Big[\Big(m^{\gamma_1(T_2)}T_2^{(1,0)} + m^{\gamma_1(T_2)-2}T_2^{(1,1)}\Big)U + \Big(m^{\gamma_2(T_2)}T_2^{(2,0)} + m^{\gamma_2(T_2)-2}T_2^{(2,1)}\Big)\frac{dU}{d\eta}\Big]\\
&+ \frac{m}{B}\Big[\Big(m^{\gamma_1(S)}S^{(1,0)} + m^{\gamma_1(S)-2}S^{(1,1)}\Big)U + \Big(m^{\gamma_2(S)}S^{(2,0)} + m^{\gamma_2(S)-2}S^{(2,1)}\frac{dU}{d\eta}\Big],
\end{aligned}
\tag{3.38}
$$

equating coefficients of U,

$$
m^{\gamma_1(T_1)}\frac{dT_1^{(1,0)}}{ds} - \xi\xi' m^{\gamma_2(T_1)-\frac{2}{3}}T_1^{(2,1)} + \frac{B'}{B}\Big(m^{\gamma_1(T_1)}T_1^{(1,0)} - m^{\gamma_1(T_2)}T_2^{(1,0)}\Big) + \frac{1}{B}m^{\gamma_1(S)-1}S^{(1,1)} = 0,
$$

substituting (3.31), we have

$$
\frac{1}{B}S^{(1,1)}m^{\alpha_2-\frac{2}{3}} = \xi\xi' m^{\alpha_2-\frac{2}{3}}T_1^{(2,1)} - m^{\alpha_1}\left[\frac{dT_1^{(1,0)}}{ds} + \frac{B'}{B}\Big(T_1^{(1,0)} - T_2^{(1,0)}\Big)\right],
$$

dividing by $m^{\alpha_2-\frac{2}{3}}$ and cross-multiplying by B, we have

$$
S^{(1,1)} = B\xi\xi' T_1^{(2,1)} - m^{\alpha_1-\alpha_2+\frac{2}{3}}\left[\frac{dT_1^{(1,0)}}{ds} + \frac{B'}{B}\Big(T_1^{(1,0)} - T_2^{(1,0)}\Big)\right],
$$

hence

$$
S^{(1,1)} = B\xi\xi' T_1^{(2,1)} - m^{\delta_2}\left[\frac{dT_1^{(1,0)}}{ds} + \frac{B'}{B}\Big(T_1^{(1,0)} - T_2^{(1,0)}\Big)\right]. \tag{3.39}
$$

Equating coefficients of $\frac{dU}{d\eta}$

$$
\begin{aligned}
0 = &\ \xi' m^{\gamma_1(T_1)+\frac{2}{3}}T_1^{(1,0)} + \xi' m^{\gamma_1(T_1)-\frac{4}{3}}T_1^{(1,1)} + m^{\gamma_2(T_1)}\frac{dT_1^{(2,0)}}{ds} + m^{\gamma_2(T_1)-2}\frac{dT_1^{(2,1)}}{ds}\\
&+ \frac{B'}{B}\left[m^{\gamma_2(T_1)}T_1^{(2,0)} + m^{\gamma_2(T_1)-2}T_1^{(2,1)} - m^{\gamma_2(T_2)}T_2^{(2,0)} + m^{\gamma_2(T_2)-2}T_2^{(2,1)}\right]\\
&+ \frac{m}{B}\Big(m^{\gamma_2(S)}S^{(2,0)} + m^{\gamma_2(S)-2}S^{(2,1)}\Big),
\end{aligned}
$$

leading to

$$
m^{\gamma_2(T_1)}\frac{dT_1^{(2,0)}}{ds} + \frac{B'}{B}\left[m^{\gamma_2(T_1)}T_1^{(2,0)} - m^{\gamma_2(T_2)}T_2^{(2,0)}\right] + \frac{1}{B}m^{\gamma_2(S)-1}S^{(2,1)} + \xi' m^{\gamma_1(T_1)-\frac{4}{3}}T_1^{(1,1)} = 0,
$$

substituting (3.31) we have

$$m^{\alpha_2}\frac{dT_1^{(2,0)}}{ds}+\frac{B'}{B}\left[m^{\alpha_2}T_1^{(2,0)}-m^{\alpha_2}T_2^{(2,0)}\right]+\frac{1}{B}m^{\alpha_1-\frac{4}{3}}S^{(2,1)}+\xi' m^{\alpha_1-\frac{4}{3}}T_1^{(1,1)}=0,$$

making $S^{(2,1)}$ subject of formula, dividing by $m^{\alpha_1-\frac{4}{3}}$ and cross-multipying by B we get

$$S^{(2,1)}=-B\xi' T_1^{(1,1)}-m^{\delta_1}\left[B\frac{dT_1^{(2,0)}}{ds}+B'\Big(T_1^{(2,0)}-T_2^{(2,0)}\Big)\right]. \tag{3.40}$$

For

$$\frac{dS}{ds}+2\frac{B'}{B}S-\frac{m}{B}T_2=0, \tag{3.41}$$

equating coefficients of U,we get

$$T_2^{(1,1)}=-\xi\xi' BS^{(2,1)}+m^{\delta_1}\left[B\frac{dS^{(1,0)}}{ds}+2B'S^{(1,0)}\right], \tag{3.42}$$

and for $\dfrac{dU}{d\eta}$

$$T_2^{(2,1)}=\xi' BS^{(1,1)}+m^{\delta_2}\left[B\frac{dS^{(2,0)}}{ds}+2B'S^{(2,0)}\right]. \tag{3.43}$$

From

$$\frac{T_1}{R_1}+\frac{T_2}{R_2}=0, \tag{3.44}$$

we have after equating coefficients of U

$$\frac{T_1^{(1,1)}}{R_1}+\frac{T_2^{(1,1)}}{R_2}=0, \tag{3.45}$$

and from the coefficients of $\dfrac{dU}{d\eta}$

$$\frac{T_1^{(2,1)}}{R_1}+\frac{T_2^{(2,1)}}{R_2}=0. \tag{3.46}$$

From the bending equations, we have for

$$-\frac{m}{B}u+B\frac{d}{ds}\left(\frac{v}{B}\right)=2(1+\nu)S, \tag{3.47}$$

equating the coefficients of U, we have

$$u^{(1,1)}=-\xi\xi' v^{(2,1)}+m^{\delta_1}\left[\frac{dv^{(1,0)}}{ds}-2B(1+\nu)S^{(1,0)}\right], \tag{3.48}$$

and that from the coefficients of $\dfrac{dU}{d\eta}$

$$u^{(2,1)} = B\xi' v^{(1,1)} + m^{\delta_2}\left[\frac{dv^{(2,0)}}{ds} - B'v^{(2,0)} - 2B(1+\nu)S^{(2,0)}\right]. \tag{3.49}$$

For

$$\frac{du}{ds} - \frac{w}{R_1} = T_1 - \nu T_2, \tag{3.50}$$

the coefficients of U, gives

$$w^{(1,1)} = \frac{R_1}{B}v^{(1,1)} + m^{\delta_2}\left[\frac{B'}{B}R_2 u^{(1,0)} - R_2\Big(T_2^{(1,0)} - \nu T_1^{(1,0)}\Big)\right], \tag{3.51}$$

and the coefficients from $\dfrac{dU}{d\eta}$, gives

$$w^{(2,1)} = \frac{R_2}{B}v^{(2,1)} + m^{\delta_1}\left[\frac{B'}{B}u^{(2,0)} - R_2\Big(T_2^{(2,0)} - \nu T_1^{(2,0)}\Big)\right], \tag{3.52}$$

where

$$\delta_1 = \alpha_2 - \alpha_1 + \frac{4}{3}, \qquad \delta_2 = \alpha_1 - \alpha_2 + \frac{2}{3}, \tag{3.53}$$

when $\delta_1 = \delta_2 = 0$, we have

$$\begin{aligned} \alpha_1 &= \alpha_2 + \frac{4}{3}, \\ \alpha_2 &= \alpha_1 + \frac{2}{3}, \end{aligned}$$

and therefore values of δ_i, for $i = 1, 2$ are as follows

$$\begin{aligned} \text{when}\ \ \alpha_1 = 0,\quad \alpha_2 = -\frac{4}{3} &\implies \delta_1 = 0, \quad \delta_2 = 2, \\ \alpha_2 = 0,\quad \alpha_2 = -\frac{2}{3} &\implies \delta_2 = 0, \quad \delta_1 = 2. \end{aligned}$$

The consistency conditions for the next approximation give the leading terms for the solutions, for example, the leading terms of the asymptotic expansions for the membrane equations for a stress, T_1 are as follows

$$T_1 = m^{\gamma_1(T_1)}\Big[T_1^{(1,0)} + \underline{O}(m^{-2})\Big]U + m^{\gamma_2(T_1)}\Big[T_1^{(2,0)} + \underline{O}(m^{-2})\Big]\frac{dU}{d\eta},$$

but

$$\gamma_1(T_1) = \gamma_1(T_2) = \alpha_1,$$

$$\gamma_2(T_1) = \gamma_2(T_2) = \alpha_2,$$

and if $\alpha_1 = 0, \alpha_2 = -\frac{4}{3}$, then

$$\begin{aligned} T_1 &= m^{\alpha_1}\left[T_1^{(1,0)} + \underline{O}(m^{-2})\right]U + m^{\alpha_2)}\left[T_1^{(2,0)} + \underline{O}(m^{-2})\right]\frac{dU}{d\eta}, \\ &= \left[T_1^{(1,0)} + \underline{O}(m^{-2})\right]U + m^{-\frac{4}{3}}\left[T_1^{(2,0)} + \underline{O}(m^{-2})\right]\frac{dU}{d\eta}, \end{aligned}$$

and if $\alpha_2 = 0,\ \alpha_1 = -\frac{2}{3}$, then

$$T_1 = m^{-\frac{2}{3}}\left[T_1^{(1,0)} + \underline{O}(m^{-2})\right]U + \left[T_1^{(2,0)} + \underline{O}(m^{-2})\right]\frac{dU}{d\eta},$$

and the others have the following

$$\begin{aligned} -\frac{1}{B}u^{(1,0)} - m^{\frac{1}{3}}\xi\xi' v^{(2,0)} &= 0, \\ u^{(1,0)} &= -m^{\frac{1}{3}}\xi\xi' v^{(2,0)}, \\ \text{from}\quad \frac{1}{R_2}w^{(2,0)} &= \frac{m}{B}v^{(2,0)}, \\ w^{(2,0)} &= \frac{m}{B}R_2 v^{(2,0)}, \\ m^{\frac{1}{3}}\xi' T_1^{(1,0)} + \frac{1}{B}S^{(2,0)} &= 0, \\ S^{(2,0)} &= -m^{\frac{1}{3}}\xi' B T_1^{(1,0)}, \\ \text{from}\quad \frac{1}{R_1}T_1^{(1,0)} + \frac{1}{R_2}T_2^{(1,0)} &= 0, \\ T_2^{(1,0)} &= -\frac{R_2}{R_1}T_1^{(1,0)}, \end{aligned}$$

together the leading terms are as follows, where, for example, $u^{(1,0)}$ is replaced by $u^{(1)}$,

$$\begin{aligned} T_1^{(1)} &= -\frac{2EhrR_1^2}{(R_1-R_2)^2}\left(\frac{B^3}{\xi' R_2}\right)^{\frac{1}{2}}\frac{dy}{ds}, & S^{(2)} &= -\frac{m^{\frac{1}{3}}}{\xi B}T_1^{(1)}, \\ T_2^{(1)} &= -\frac{R_2}{R_1}T_1^{(1)}, & u^{(1)} &= -m^{\frac{1}{3}}\xi\xi' B v^{(2)}, \\ v^{(2)} &= \frac{m^{-\frac{1}{3}}}{r}\left(\xi'\frac{B}{R_2}\right)^{\frac{1}{2}}y, & w^{(2)} &= \frac{mR_2}{B}v^{(2)}, \end{aligned} \tag{3.54}$$

and for the other two solutions, the leading terms are, for example

$$u = m^{\gamma_1(v)}\left[v^{(1,0)} + \underline{O}(m^{-2})\right]U + m^{\gamma_2(v)}\left[v^{(2,0)} + \underline{O}(m^{-2})\right]\frac{dU}{d\eta},$$

and if $\alpha_1 = 0, \alpha_2 = -\frac{4}{3}$,

$$\begin{aligned} u &= m^{\alpha_2+\frac{1}{3}}\left[v^{(1,0)} + \underline{O}(m^{-2})\right]U + m^{\alpha_1-\frac{1}{3}}\left[v^{(2,0)} + \underline{O}(m^{-2})\right]\frac{dU}{d\eta}, \\ &= m^{-1}\left[v^{(1,0)} + \underline{O}(m^{-2})\right]U + m^{-\frac{1}{3}}\left[v^{(2,0)} + \underline{O}(m^{-2})\right]\frac{dU}{d\eta}, \end{aligned}$$

and in the case $\alpha_2 = 0,\ \alpha_1 = -\frac{2}{3}$,

$$\begin{aligned} u &= m^{\alpha_2+\frac{1}{3}}[v^{(1,0)} + \underline{O}(m^{-2})]U + m^{\alpha_1-\frac{1}{3}}\Big[v^{(2,0)} + \underline{O}(m^{-2})\Big]\frac{dU}{d\eta}, \\ &= m^{\frac{1}{3}}[v^{(1,0)} + \underline{O}(m^{-2})]U + m^{-1}\Big[v^{(2,0)} + \underline{O}(m^{-2})\Big]\frac{dU}{d\eta}, \end{aligned}$$

and the remaining terms have the following

$$\begin{aligned} \frac{1}{B}v^{(1,0)} &= \frac{m}{R_2}w^{(1,0)}, \\ w^{(1,0)} &= \frac{mR_2}{B}v^{(1,0)}, \\ \text{from} \quad \frac{1}{B}u^{(2,0)} &= m^{-\frac{1}{3}}\xi' v^{(1,0)}, \\ u^{(2,0)} &= m^{-\frac{1}{3}}\xi' B v^{(1,0)}, \\ \text{from} \quad \frac{1}{R_1}T_1^{(2,0)} + \frac{1}{R_2}T_2^{(2,0)} &= 0, \\ T_2^{(2,0)} &= -\frac{R_2}{R_1}T_1^{(2,0)}, \end{aligned}$$

together the leading terms are

$$\begin{aligned} T_1^{(2)} &= \frac{2EhR_1^2}{(R_1-R_2)^2}\Big(\frac{B^3\xi'}{R_2}\Big)^{\frac{1}{2}}\frac{dy}{ds}, & S^{(1)} &= m^{\frac{1}{3}}B\xi\xi' T_1^{(2)}, \\ T_2^{(2)} &= -\frac{R_2}{R_1}T_1^{(2)}, & u^{(2)} &= m^{-\frac{1}{3}}B\xi' v^{(1)}, \\ v^{(1)} &= m^{\frac{1}{3}}\Big(\frac{B}{\xi' R_2}\Big)^{\frac{1}{2}}y, & w^{(1)} &= \frac{mR_2}{B}v^{(1)}, \end{aligned} \tag{3.55}$$

where y satisfies the differential equation

$$\frac{d}{ds}\Big(rg\frac{dy}{ds}\Big) + \frac{f}{r}y = 0, \tag{3.56}$$

and where $g(s) = \dfrac{2R_1^2B^4}{R_2(R_1-R_2)^2}$, $\ f(s) = \dfrac{\lambda m^2 R_2}{2} - \dfrac{m^6h^2}{12(1-\nu^2)g}$ and $f, g \sim 1$.

The differential equation (3.56) is derived from rearranging the equation (3.43) to obtain an expression for $S^{(1,1)}$ and substitute into equation (3.39), we have

$$T_2^{(2,1)} - \xi\xi' B^2 T_1^{(2,1)} = B\frac{dS^{(2,0)}}{ds} + 2B'S^{(2,0)} - \xi' B^2\frac{dT_1^{(1,0)}}{ds} + B'\Big(T_1^{(1,0)} - T_2^{(1,0)}\Big),$$

solving this equation together with

$$\frac{T_1^{(2,1)}}{R_1} + \frac{T_2^{(2,1)}}{R_2} = \frac{m^3\lambda}{B^3}w^{(2,0)} - \frac{h^2 m}{12(1-\nu^2)}\Big(\frac{m^2}{B^2}\Big)^2 w^{(2,0)},$$

we get

$$\frac{d}{ds}\Big(B\xi' T_1^{(1,0)}\Big) + T_1^{(1,0)}\Big(\xi' B' \frac{R_2}{R_1} - 4B\xi''\Big) - \frac{m^3\lambda}{B^2} w^{(2,0)} - \frac{h^2 m}{12(1-\nu^2)}\Big(\frac{m^2}{B^2}\Big)^2 w^{(2,0)} = 0,$$

which gives the differential equation (3.56).

In (3.54) and (3.55) the omitted terms are sufficiently small, for example, in (3.54)

$$T_1^{(1)} \sim m^{-\frac{4}{3}}, \quad S^{(2)} \sim m^{-1}, \quad u^{(1)} \sim m^{-\frac{1}{3}}, \quad v^{(2)} \sim 1 \quad \text{and} \quad w^{(2)} \sim m^{-\frac{2}{3}},$$

and in (3.55)

$$T_1^{(2)} \sim m^{-\frac{2}{3}}, \quad S^{(1)} \sim m^{-\frac{1}{3}}, \quad u^{(2)} \sim m^{-1}, \quad v^{(1)} \sim 1 \quad \text{and} \quad w^{(1)} \sim m^{-\frac{4}{3}}.$$

3.2 Solution to the Differential Equation (3.56)

For the general second order differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0,$$

a point $x = x_0$ is an ordinary point of the differential equation, if we can find two linearly independent solutions in the form of a power series centred at x_0,

$$y = \sum_{n=0}^{\infty} C_n (x - x_0)^n,$$

and is called a regular singular point if both $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are analytic at x_0 and if not then it is an irregular singular point [10].

If $x = x_0$ is a regular singular point, by method of Frobenius, then there exists at least one solution of the form

$$\begin{aligned} y &= (x - x_0)^p \sum_{n=0}^{\infty} C_n (x - x_0)^n, \\ &= \sum_{n=0}^{\infty} C_n (x - x_0)^{n+p}, \end{aligned}$$

where p is a constant to be determined [10].

The differential equation (3.56) in expanded form is

$$rg\frac{d^2y}{ds^2} + \frac{d(rg)}{ds}\frac{dy}{ds} + \frac{f}{r}y = 0,$$

and assuming the solution to be in the form

$$y = \sum_{n=0}^{\infty} C_n(s - s_*)^{n+p},$$

after differentiating, gives the relations

$$\begin{aligned}
\frac{dy}{ds} &= \sum_{n=0}^{\infty} (n+p)C_n(s - s_*)^{n+p-1}, \\
\frac{d^2y}{ds^2} &= \sum_{n=0}^{\infty} (n+p)(n+p-1)C_n(s - s_*)^{n+p-2},
\end{aligned}$$

and substituting into the differential equation, gives

$$\begin{aligned}
0 = (s - s_*)^{\frac{1}{2}} &\sum_{n=0}^{\infty} (n+p)(n+p-1)C_n(s - s_*)^{n+p-2} \\
&+ \frac{1}{2}(s - s_*)^{-\frac{1}{2}} \sum_{n=0}^{\infty} (n+p)C_n(s - s_*)^{n+p-1} + (s - s_*)^{-\frac{1}{2}} \sum_{n=0}^{\infty} C_n(s - s_*)^{n+p},
\end{aligned}$$

$$\begin{aligned}
0 = &\sum_{n=0}^{\infty} (n+p)(n+p-1)C_n(s - s_*)^{n+p-\frac{3}{2}} \\
&+ \frac{1}{2}\sum_{n=0}^{\infty} (n+p)C_n(s - s_*)^{n+p-\frac{3}{2}} + \sum_{n=0}^{\infty} C_n(s - s_*)^{n+p-\frac{1}{2}},
\end{aligned}$$

collecting terms, we have

$$\sum_{n=0}^{\infty} (n+p)(n+p-\frac{1}{2})C_n(s - s_*)^{n+p-\frac{3}{2}} + \sum_{n=0}^{\infty} C_n(s - s_*)^{n+p-\frac{1}{2}} = 0,$$

for $n = 0$,

$$p\Big(p - \frac{1}{2}\Big)C_0(s - s_*)^{p-\frac{3}{2}} + \sum_{n=1}^{\infty} (n+p)(n+p-\frac{1}{2})C_n(s - s_*)^{n+p-\frac{3}{2}} + \sum_{n=0}^{\infty} C_n(s - s_*)^{n+p-\frac{1}{2}} = 0,$$

factoring out $(s - s_*)^p$,

$$(s-s_*)^p\left[p\Big(p-\frac{1}{2}\Big)C_0(s-s_*)^{-\frac{3}{2}} + \sum_{n=1}^{\infty} (n+p)(n+p-\frac{1}{2})C_n(s - s_*)^{n-\frac{3}{2}} + \sum_{n=0}^{\infty} C_n(s - s_*)^{n-\frac{1}{2}}\right] = 0,$$

letting $k = n - 1$,

$$(s-s_*)^p\left[p\Big(p-\frac{1}{2}\Big)C_0(s-s_*)^{-\frac{3}{2}} + \sum_{k=0}^{\infty} (k+1+p)(k+p+\frac{1}{2})C_{k+1}(s - s_*)^{k-\frac{1}{2}} + \sum_{k=0}^{\infty} C_k(s - s_*)^{k-\frac{1}{2}}\right] = 0,$$

collecting like terms

$$(s-s_*)^p\left[p\Big(p-\frac{1}{2}\Big)C_0(s-s_*)^{-\frac{3}{2}}+\sum_{k=0}^{\infty}\Big((k+p+1)(k+p+\frac{1}{2})C_{k+1}+C_k\Big)(s-s_*)^{k-\frac{1}{2}}\right]=0,$$

but

$$p\Big(p-\frac{1}{2}\Big)C_0=0,\quad (k+p+1)(k+p+\frac{1}{2})C_{k+1}+C_k=0,$$

from

$$p\Big(p-\frac{1}{2}\Big)C_0=0,\implies p\Big(p-\frac{1}{2}\Big)=0,\quad \text{for}\ \ C_0\neq 0,$$

then, the two values of p are

$$p_1=0\ \ \text{and}\quad p_2=\frac{1}{2},$$

and therefore, the two linearly independent solutions of (3.56) have the expansions

$$\text{for}\ \ p_1=0,\quad y_1=a_0+a_1(s-s_*)+\cdots,$$

and

$$\begin{aligned}\text{for}\quad p_2=\frac{1}{2},\quad y_2&=b_0(s-s_*)^{\frac{1}{2}}+b_1(s-s_*)^{\frac{3}{2}}+\cdots,\\ y_2&=(s-s_*)^{\frac{1}{2}}\big(b_0+b_1(s-s_*)+\cdots\big).\end{aligned}$$

To get regular functions $x^{(k)}$ in (3.54) we should put $y=y_2$ and in (3.55) $y=y_1$. Since for $s>s_*$, $R_1<0$ and $R_1(s_*)=\infty$, then from the equation

$$r^2=-\frac{R_2}{B^2R_1},\tag{3.57}$$

in the neighbourhood of the point s_*, we have by Taylor's expansion

$$B(s)=B(s_*)+\frac{dB}{ds}\Big|_{s_*}(s-s_*)+\cdots,\tag{3.58}$$

$$R_2(s)=R_2(s_*)+\frac{dR_2}{ds}\Big|_{s_*}(s-s_*)+\cdots,\tag{3.59}$$

$$\frac{1}{R_1(s)}=\frac{1}{R_1(s_*)}+\Big(\frac{dR_1}{ds}\Big)^{-1}\Big|_{s_*}(s-s_*)+\cdots,\tag{3.60}$$

noting that $R_1(s_*)=\infty$, then $\dfrac{1}{R_1(s_*)}\sim 0$.

Thus from (3.57), we have

$$r^2(s)=-\frac{R_2(s_*)+\frac{dR_2}{ds}\Big|_{s_*}(s-s_*)+\cdots}{\Big[B(s_*)+\frac{dB}{ds}\Big|_{s_*}(s-s_*)+\cdots\Big]^2}\left[\frac{dR_1}{ds}\Big|_{s_*}(s-s_*)+\cdots\right],$$

expanding the denominator

$$\begin{aligned}
[B^2]^{-1} &= B^{-2}\left[1+2\frac{1}{B}\frac{dB}{ds}\Big|_{s_*}(s-s_*)+\frac{1}{B^2}\left(\frac{dB}{ds}\right)^2+\cdots\right],\\
&= \frac{1}{B^2(s_*)}-2B\frac{dB}{ds}\Big|_{s_*}(s-s_*)+\cdots,
\end{aligned}$$

therefore simplifying, gives

$$r^2(s)=\left(\frac{dR_1}{ds}\right)\frac{R_2}{B^2}\Big|_{s_*}(s-s_*)+\underline{O}(s-s_*),$$

and taking square roots, we have that

$$r\sim(s-s_*)^{\frac{1}{2}}.$$

Substituting $f, g\sim 1$ and $r\sim(s-s_*)^{\frac{1}{2}}$ into the differential equation (3.56), we have

$$\begin{aligned}
(s-s_*)^{\frac{1}{2}}\frac{d^2y}{ds^2}+\frac{1}{2}(s-s_*)^{-\frac{1}{2}}\frac{dy}{ds}+\frac{y}{(s-s_*)^{\frac{1}{2}}}&=0,\\
\frac{d^2y}{ds^2}+\frac{1}{2(s-s_*)}\frac{dy}{ds}+\frac{y}{(s-s_*)}&=0,
\end{aligned}$$

which shows that (3.56) has at $s=s_*$ a regular singular point.

3.2.1 Airy Solutions

Let us take two solutions of equation (3.18)

$$U_1(\eta)=Ai(-\eta)\quad,\quad U_2(\eta)=Bi(-\eta), \tag{3.61}$$

then the functions (3.54) and (3.55) give four solutions of (3.17). Consider their linear combination as

$$x=\sum_{k=1}^{4}C_kx_k(s,m), \tag{3.62}$$

where C_k are arbitrary constants, x_1 and x_2 are constructed in accordance with formulae (3.54) and (3.55), for $U=U_1$ and x_3 and x_4 for $U=U_2$.

The leading terms of asymptotic solution of (3.61) have the following terms

$$\begin{aligned}
U_1(\eta)&=\pi^{-\frac{1}{2}}\eta^{-\frac{1}{4}}\cos\varphi, & U_2(\eta)&=\pi^{-\frac{1}{2}}\eta^{-\frac{1}{4}}\sin\varphi,(\eta\to\infty),\\
U_1(\eta)&=\frac{1}{2}\pi^{-\frac{1}{2}}(-\eta)^{-\frac{1}{4}}e^{-\delta}, & U_2(\eta)&=\pi^{-\frac{1}{2}}(-\eta)^{-\frac{1}{4}}e^{\delta},(\eta\to\infty),
\end{aligned} \tag{3.63}$$

where $\varphi = \frac{2}{3}\eta^{\frac{3}{2}} - \dfrac{\pi}{4}$ and $\delta = \frac{2}{3}(-\eta)^{\frac{3}{2}}$.

From (3.63) it follows that all solutions, x_k, oscillate for $s > s_*$ $(\xi > 0, \eta > 0)$ which is a typical situation for the asymmetric vibrations (and loss of stability) of the shells of negative Gaussian curvature. For $s < s_*$, solutions, x_1 and x_2 exponentially decrease with increase of $s_* - s$, and x_3 and x_4 increasing.

Four solutions from (3.62), for $s > s_*$ we get

$$\begin{aligned}
T_1 &\approx V_1[(-C_1^0\cos\varphi - C_3^0\sin\varphi)y_2' + (-C_2^0\sin\varphi + C_4^0\cos\varphi)y_1'],\\
S &\approx V_2[(-C_1^0\sin\varphi + C_3^0\cos\varphi)y_2' + (C_2^0\cos\varphi + C_4^0\sin\varphi)y_1'],\\
u &\approx V_3[(-C_1^0\cos\varphi - C_3^0\sin\varphi)y_2 + (-C_2^0\sin\varphi + C_4^0\cos\varphi)y_1],\\
v &\approx V_4[(-C_1^0\sin\varphi + C_3^0\cos\varphi)y_2 + (C_2^0\cos\varphi + C_4^0\sin\varphi)y_1],
\end{aligned} \tag{3.64}$$

and from (3.62), for $s < s_*$ the four solution are

$$\begin{aligned}
T_1 &\approx -V_1\Big(\frac{1}{2}C_1^0e^{-\delta} + C_3^0e^{\delta}\Big)y_2' + |V_1|\Big(\frac{1}{2}C_2^0e^{-\delta} - C_4^0e^{\delta}\Big)y_1',\\
S &\approx V_2\Big(\frac{1}{2}C_1^0e^{-\delta} - C_3^0e^{\delta}\Big)y_2' - |V_2|\Big(\frac{1}{2}C_2^0e^{-\delta} + C_4^0e^{\delta}\Big)y_1',\\
u &\approx -V_3\Big(\frac{1}{2}C_1^0e^{-\delta} + C_3^0e^{\delta}\Big)y_2 + |V_3|\Big(\frac{1}{2}C_2^0e^{-\delta} - C_4^0e^{\delta}\Big)y_1,\\
v &\approx V_4\Big(\frac{1}{2}C_1^0e^{-\delta} - C_3^0e^{\delta}\Big)y_2 - |V_4|\Big(\frac{1}{2}C_2^0e^{-\delta} + C_4^0e^{\delta}\Big)y_1,
\end{aligned} \tag{3.65}$$

where

$$\begin{aligned}
V_1 &= \frac{B}{|r|}V_2, \qquad V_2 = \frac{2R_1^2|r|}{(R_1-R_2)^2}\Big(\frac{B|r|}{\pi R_2}\Big)^{\frac{1}{2}}(\mathrm{sign}(-R_1))^{\frac{1}{2}}, \qquad \delta = m\int_s^{s_*}|r|\,ds,\\
V_3 &= \frac{|r|}{B}V_4, \qquad V_4 = \Big(\frac{B|r|}{\pi R_2}\Big)^{\frac{1}{2}}(\mathrm{sign}(-R_1))^{\frac{1}{2}}, \qquad \varphi = m\int_{s_*}^{s} r\,ds - \frac{\pi}{4},
\end{aligned}$$

and

$$C_k^0 = C_k m^{\frac{(-1)k}{6}}.$$

From $\delta = -\frac{2}{3}(-\eta)^{\frac{3}{2}}$ and substituting $\eta = m^{\frac{2}{3}}\xi$, we have

$$\begin{aligned}
\delta &= -\frac{2}{3}\left(\left(-m^{\frac{2}{3}}\xi\right)^{\frac{3}{2}}\right),\\
&= -\frac{2}{3}\left(-m\frac{3}{2}\int_s^{s_*}|r|\,ds\right),\\
&= m\int_s^{s_*}|r|\,ds,
\end{aligned}$$

and from $\varphi = \frac{2}{3}\eta^{\frac{3}{2}} - \dfrac{\pi}{4}$, we have

$$\begin{aligned}
\varphi &= \frac{2}{3}\left(m^{\frac{2}{3}}\xi\right)^{\frac{3}{2}} - \frac{\pi}{4}, \\
&= \frac{2}{3}m\xi^{\frac{3}{2}} - \frac{\pi}{4}, \\
&= \frac{2}{3}\cdot\frac{3}{2}m\left(\int_{s_*}^{s} r\,ds\right)^{\frac{2}{3}\cdot\frac{3}{2}} - \frac{\pi}{4}, \\
&= m\int_{s_*}^{s} r\,ds - \frac{\pi}{4}.
\end{aligned}$$

Coefficients V_k for $s > s_*$ are positive and for $s < s_*$, pure imaginary, but the products $V_k y_2$ in (3.65) are real.

To find T_2 and w use the approximate formulae

$$T_2 = -\frac{R_2}{R_1}T_1, \quad w = \frac{mR_2}{B}v. \tag{3.66}$$

3.3 Boundary Value Problem

Using the constructed integrals for solution of boundary value problems, restricting ourselves with tangential boundary conditions. Also suppose that edges $s = s_1$ and $s = s_2$ for $s_1 < s_* < s_2$ are sufficiently far from $s = s_*$.

Let on $s = s_1$, there be at least one tangential fix (constant)(fixed either u or v or both) for excluding bending at the positive part of the shell.

Approximate solutions may be obtained if the linear combination of solutions decrease near $s = s_1$.

Let

$$x = C_1 x_1 + C_2 x_2. \tag{3.67}$$

The error done has order $\exp(-2m\int_{s_1}^{s_*}|r|\,ds)$.

We give the following equations for frequency for different boundary conditions of $s = s_2$,

for $u = v = 0$, equation (3.64) gives

$$-C_1^0\cos\phi y_2 - C_2^0\sin\varphi y_1 = 0, \tag{3.68}$$

$$-C_1^0\sin\varphi y_2 + C_2^0\cos\varphi y_1 = 0, \tag{3.69}$$

multiplying (3.68) by $\cos\varphi$ and (3.69) by $\sin\varphi$ and adding the two we have

$$-C_1^0 y_2 = 0, \quad \Rightarrow y_2 = 0, \tag{3.70}$$

multiplying (3.68) by $\sin\varphi$ and (3.69) by $\cos\varphi$ and adding the two we have

$$2C_2^0 y_1 = 0, \quad \Rightarrow y_1 = 0, \tag{3.71}$$

together

$$y_1 y_2 = 0. \tag{3.72}$$

For $u = S = 0$, equation (3.64) gives

$$-C_1^0 \sin\varphi y_2' + C_2^0 \cos\varphi y_1' = 0, \tag{3.73}$$
$$-C_1^0 \cos\varphi y_2 - C_2^0 \sin\varphi y_1 = 0, \tag{3.74}$$

multiplying (3.73) by $\cos\varphi y_2$ and (3.74) by $\sin\varphi y_2'$ and adding the two we have

$$\cos^2\varphi y_1' y_2 + \sin^2\varphi y_1 y_2' = 0. \tag{3.75}$$

For $v = T_1 = 0$ equation (3.64) gives

$$-C_1^0 \sin\varphi y_2 + C_2^0 \cos\varphi y_1 = 0, \tag{3.76}$$
$$-C_1^0 \cos\varphi y_2' - C_2^0 \sin\varphi y_1' = 0, \tag{3.77}$$

multiplying (3.76) by $\cos\varphi y_2'$ and (3.77) by $\sin\varphi y_2$ and adding the two we have

$$\sin^2\varphi y_1' y_2 + \cos^2\varphi y_1 y_2' = 0. \tag{3.78}$$

For $T_1 = S = 0$, equation (3.64) gives

$$-C_1^0 \cos\varphi y_2' - C_2^0 \sin\varphi y_1' = 0, \tag{3.79}$$
$$-C_1^0 \sin\varphi y_2' + C_2^0 \sin\varphi y_1' = 0, \tag{3.80}$$

multiplying (3.79) by $\sin\varphi$ and (3.80) by $\cos\varphi$ and adding the two we have

$$2C_2^0 y_1' = 0, \quad \Rightarrow y_1' = 0, \tag{3.81}$$

and multiplying (3.79) by $\cos\varphi$ and (3.80) by $\sin\varphi$ and adding the two we have

$$-2C_1^0 y_2' = 0, \quad \Rightarrow y_2' = 0, \tag{3.82}$$

together

$$y_1' y_2' = 0, \tag{3.83}$$

grouping them together gives

$$y_1 y_2 = 0 \quad \text{for} \;\; u = v = 0, \tag{3.84a}$$

$$y_1 y_2' \sin^2 \varphi + y_2 y_1' \cos^2 \varphi = 0 \quad \text{for} \;\; u = S = 0, \tag{3.84b}$$

$$y_1 y_2' \cos^2 \varphi + y_2 y_1' \sin^2 \varphi = 0 \quad \text{for} \;\; v = T_1 = 0, \tag{3.84c}$$

$$y_1' y_2' = 0 \quad \text{for} \;\; T_1 = S = 0, \tag{3.84d}$$

where functions y_k,y_k' and φ are evaluated for $s = s_2$. Equations (3.84) do not depend on boundary conditions at $s = s_1$.

3.3.1 Variational Approach

When $m \sim h^{-\frac{1}{3}}$ is fixed then from equations (3.84) we can find eigenvalues $\lambda \sim h^{\frac{2}{3}}$. The smallest frequency may be found while changing m. In case (a) for the smallest frequency we have estimate (3.14). In case (b) and (c) there is a possibility for further decrease of frequency when decreasing m, if the shell has the so called eigensizes (the same as for shells of negative curvature). For example in (c), our task is to find the smallest $\lambda > 0$ for which there exists a solution for (3.11) and (3.12) satisfying boundary conditions and also we should find an m which corresponds to this λ.

To find approximate solutions we shall substitute functions (3.64) into tangential boundary conditions (3.84). Consider different variations of boundary conditions

1. if on both $s = s_1$ and $s = s_2$ are (3.84a), form equation for different λ is

$$y_1(s_2) = 0,$$

2. for the case $s = s_1$ in (3.84a) and $s = s_2$, either in (3.84b) or (3.84c), we have

$$y_1'(s_2) = 0,$$

3. if both $s = s_1$ and $s = s_2$ then for the same (3.84b) and (3.84c) is

$$y_1' y_2 \sin^2 \varphi + y_1 y_2' \cos^2 \varphi = 0, \quad \text{on} \;\; s = s_2, \tag{3.85}$$

or if different

$$y_1' y_2 \cos^2 \varphi + y_1 y_2' \sin^2 \varphi = 0, \quad \text{on} \;\; s = s_2. \tag{3.86}$$

We have the differential equation

$$\frac{d}{ds}\Big(rg(s)\frac{dy}{ds}\Big) - \frac{m^6h^2}{12(1-\nu^2)}\frac{y(s)}{g(s)} + \frac{\lambda m^4 f(s)}{r}y(s) = 0, \tag{3.87}$$

considering these variations in detail, taking into account that

$$y_1(s_1) = 0, \quad y_1'(s_1) = 1, \quad y_2(s_1) = 1, \quad y_2'(s_1) = 0, \tag{3.88}$$

we see that the first and second variations i.e. 1 and 2 lead to a typical Sturm-Liouville problem : Find λ for which there exists solution of equation (3.87) satisfying conditions

$$y(s_1) = y(s_2) = 0, \quad \text{in the first variation,}$$

and

$$y(s_1) = y'(s_2) = 0, \quad \text{in the second variation.} \tag{3.89}$$

One should find the smallest λ for different m. Consider a variational method for simple formulas,

hence $\lambda = \lambda_*$ follows from

$$\lambda_* = \min_{m,y}\left(\frac{I_1 + \frac{m^6h^2}{12(1-\nu^2)}I_2}{m^4 I_3}\right), \tag{3.90}$$

where

$$I_1 = \int_{s_*}^{s_2} rg(s)\Big(\frac{dy}{ds}\Big)^2 ds, \quad I_2 = \int_{s_*}^{s_2} \frac{y^2}{rg(s)}\, ds, \quad I_3 = \int_{s_*}^{s_2} \frac{f(s)y^2}{r}\, ds.$$

The Wronskian of (3.87) is

$$y_1'(s)y_2(s) - y_1(s)y_2'(s) = \frac{g(s_1)}{g(s)}, \tag{3.91}$$

then conditions (3.85) and (3.86) may be rewritten in the form

$$y_1(s_2)y_2'(s_2) = -\frac{g(s_1)}{g(s_2)}\sin^2\varphi, \tag{3.92}$$

$$y_1(s_2)y_2'(s_2) = -\frac{g(s_1)}{g(s_2)}\cos^2\varphi, \tag{3.93}$$

where

$$\varphi = \varphi(s_2) = m\int_{s_*}^{s_2}\Big|\frac{R_2}{B^2R_1}\Big|^{\frac{1}{2}} ds. \tag{3.94}$$

Let us consider (3.93) corresponding to the fourth variant of boundary conditions.

3.3.2 Eigensizes

If the boundary conditions at s_1 and s_2 are such that there are constraints only on one of the two tangential displacements (u or v) the so called eigensizes exist. These are sizes of a shell which are such that for a given boundary condition, for example, $u(s_2) = S(s_2) = 0$ and a particular m the frequency of vibration decreases significantly due to non-trivial bending.

To find the sizes (eigensizes) of the shell and m such that $\cos\varphi = 0$, we then get the Sturm-Liouville problem containing differential equation (3.87) and boundary conditions

$$y'(s_1) = y'(s_2) = 0. \tag{3.95}$$

For general m, eigenvalue $\lambda = \lambda(m)$ may be found from (3.90) where y satisfies the conditions (3.95).

For approximate evaluation of $\lambda(m)$ put $y(s) \equiv 1$. Then

$$\lambda(m) \approx \frac{m^2h^2}{12(1-\nu^2)} \int_{s_*}^{s_2} \frac{ds}{rg(s)} \left(\int_{s_*}^{s_2} \frac{f(s)}{r}\, ds \right)^{-1}. \tag{3.96}$$

To check whether (3.96) gives a critical load, we should consider other values of m. Let now $\cos\varphi \neq 0$, then $y_1(s)$ and $y_2(s)$ satisfy the integral equations

$$\begin{aligned} y_1(s) &= rg(s) \int_{s_*}^{s_2} \frac{dt}{g(t)} - \int_{s_*}^{s_2} \left(\int_{s_*}^{t} \frac{f(t_1)}{r(t_1)} y_1(t_1)\, dt_1 \right) \frac{dt}{g(t)}, \\ y_2(s) &= 1 - \int_{s_*}^{s_2} \left(\int_{s_*}^{t} \frac{f(t_1)}{r(t_1)} y_2(t_1)\, dt_1 \right) \frac{dt}{g(t)}. \end{aligned} \tag{3.97}$$

Exact solutions of (3.97) may be found by iterative methods. This method converges quickly if $\left|\dfrac{f}{r}\right| \ll 1$.

Let $\left|\dfrac{f}{r}\right| \ll 1$, then we get

$$y_1(s) = \int_{s_*}^{s_2} \frac{rg(s_1)}{g(t)}\, dt, \quad y_2(s) = 1 - \int_{s_*}^{s_2} \left(\int_{s_*}^{t} \frac{f(t_1)}{r(t_1)}\, dt_1 \right) \frac{dt}{g(t)}. \tag{3.98}$$

Substituting (3.98) into (3.93) we get

$$\lambda(m) = \frac{\cos^2\varphi}{m^2 I_1 I_2} + \frac{m^4h^2 I_1}{12(1-\nu^2) I_2}, \tag{3.99}$$

where
$$I_1 = \int_{s_*}^{s_2} \frac{ds}{rg}, \qquad I_2 = \int_{s_*}^{s_2} \frac{R_2}{2r}\, ds.$$

Eigensizes corresponding to $\cos\varphi = 0$ and the approximate formula (3.99) are valid only if
$$\cos^2\varphi \ll 1, \qquad \frac{1}{12} m^6 h^2 \ll 1.$$

Taking into account that as m decreases the influence of two still not taken into account facts : boundary conditions at $s = s_2$ and the type of boundary conditions on $s = s_1$. In case (d) as m decreases there is always reduction of frequency and these facts are getting essential.

Conclusions

In the region $s > s_*(\xi > 0, \eta > 0)$ both the solutions oscillate. This is a typical situation for the asymmetric vibrations (and loss of stability) of the shells of negative Gaussian curvature. In the remaining region where the curvature is positive, both Airy functions, and the unknown displacements and stresses exponentially increase or decrease. The low frequency vibrations take place when the leading role belongs to the bending solutions.

Chapter 4

Numerical Vs Asymptotic

As an example consider a shell whose middle surface is got as a result of rotation of the curve

$$z = a - b \sin c(x - x_*), \quad x_1 \leq x \leq x_2,$$

about the OX axis. At $x = x_*$ there is change in sign of curvature. So we have, observing that

$$\begin{aligned} ds^2 &= dx^2 + dz^2, \\ ds^2 &= dx^2 + (b^2c^2 \cos^2[c(x - x_*)]dx^2, \\ &= [1 + b^2c^2 \cos^2[c(x - x_*)]]dx^2, \\ ds &= \Phi(x)dx, \end{aligned}$$

where $\Phi = \sqrt{1 + b^2c^2 \cos^2[c(x - x_*)]}$. Therefore

$$s(x) = \int_{s_*}^{s} ds = \int_{x_*}^{x} \Phi(x)\, dx = \int_{x_*}^{x} \Phi(t)\, dt, \quad \Phi(t) = \left(1 + b^2c^2 \cos^2[c(t - x_*)]\right)^{\frac{1}{2}}. \tag{4.1}$$

Let on both sides

$$u = v = w = M_1 = 0, \quad x = x_1, \quad x_2. \tag{4.2}$$

For the formal approximation of the smallest eigenvalue, we use the first of equations (3.84a).

To construct y_1 and y_2 solutions of equation (3.56) take the following initial conditions

$$\begin{aligned} y_1(x_* + \varepsilon) &= 1, \quad \left.\frac{dy_1}{dx}\right|_{x_*+\varepsilon} = -\left.\frac{2B^2\Phi^2 f}{R_2 g \frac{dR_1^{-1}}{dx}}\right|_{x=x_*}, \\ y_2(x_* + \varepsilon) &= \sqrt{\varepsilon}, \quad \left.\frac{dy_2}{dx}\right|_{x_*+\varepsilon} = \frac{1}{2\sqrt{\varepsilon}}. \end{aligned} \tag{4.3}$$

Put

$$\varepsilon = 10^{-4}, \quad a = 2, \quad b = c = 1, \quad x_* = 0, \quad x_2 = -x_1 = \frac{\pi}{2}, \quad \nu = 0.3.$$

As a result of the solution by Runge-Kutta fourth order of problem (3.56) and (4.3) as $h = 0.003$ and $h = 0.01$ it happens that the smallest eigenvalue is when $y_2(x_2, \lambda) = 0$.

Denote the found λ as λ_{ac} from the variational approach and solve the equations (3.11), (3.12) and (4.2) numerically, denoting the smallest eigenvalue found as λ_{np}.

For the different number of waves m, the values of the frequency parameter λ are presented in the following tables for different values of h.

m	5	6	7	8	9	10
λ_{ac}	0.0437	0.0431	0.0307	0.0306	0.0328	0.0347
λ_{np}	0.0408	0.0336	0.0311	0.0306	0.0330	0.0351

Table 4.1: when $h = 0.01$

m	9	10	11	12	13	14
λ_{ac}	0.0153	0.0139	0.0135	0.0136	0.0141	0.0148
λ_{np}	0.0156	0.0144	0.0138	0.0138	0.0142	0.0152

Table 4.2: when $h = 0.003$

From these tables there is good agreement.

4.1 Graphs

In the region $s > s_*(\xi > 0, \eta > 0)$ the solutions oscillate. In the remaining region, both Airy functions, and the unknown displacements and stresses exponentially increase or decrease as shown by the graphs below.

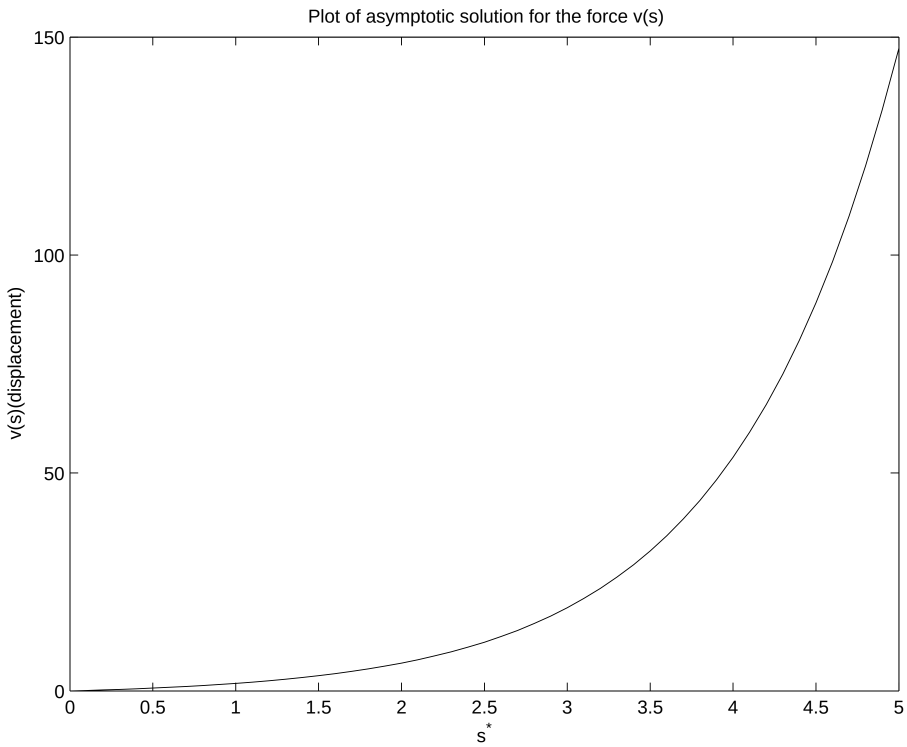


Figure 4.1: Asymptotic solution for the force $v(s)$

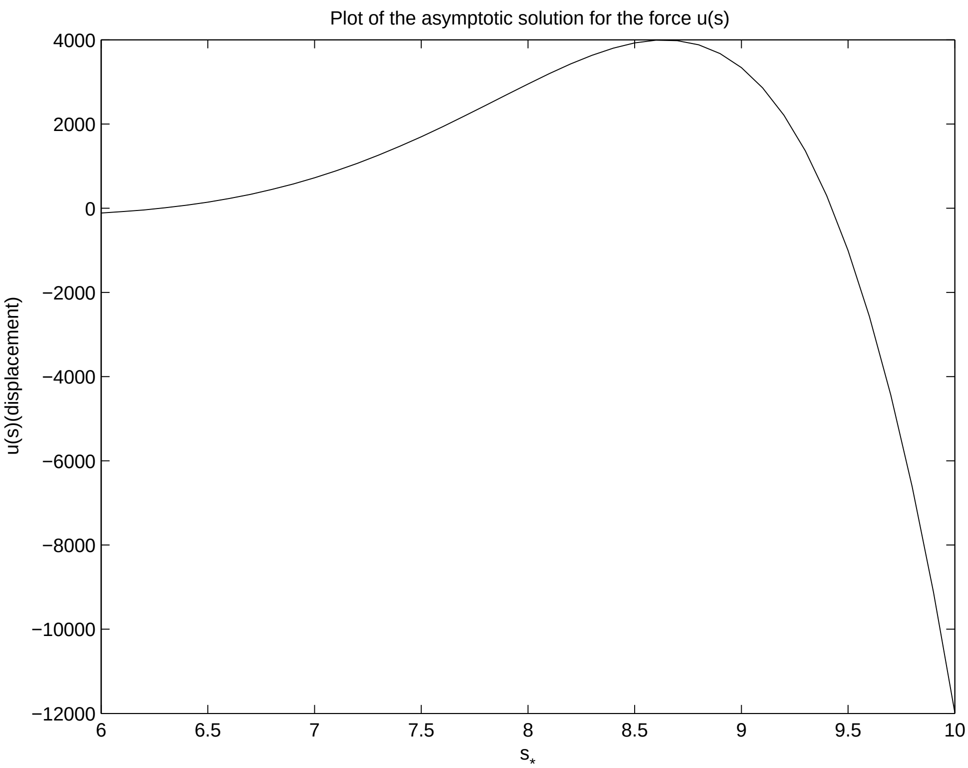


Figure 4.2: Asymptotic solution for the force $u(s)$

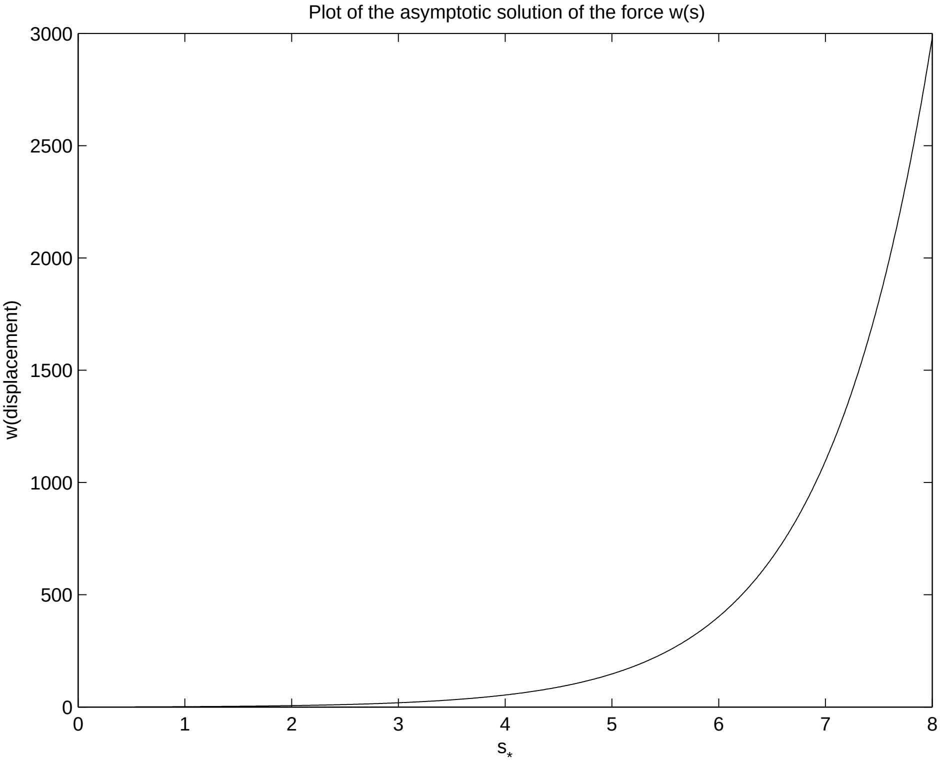


Figure 4.3: Asymptotic solution for the force $w(s)$

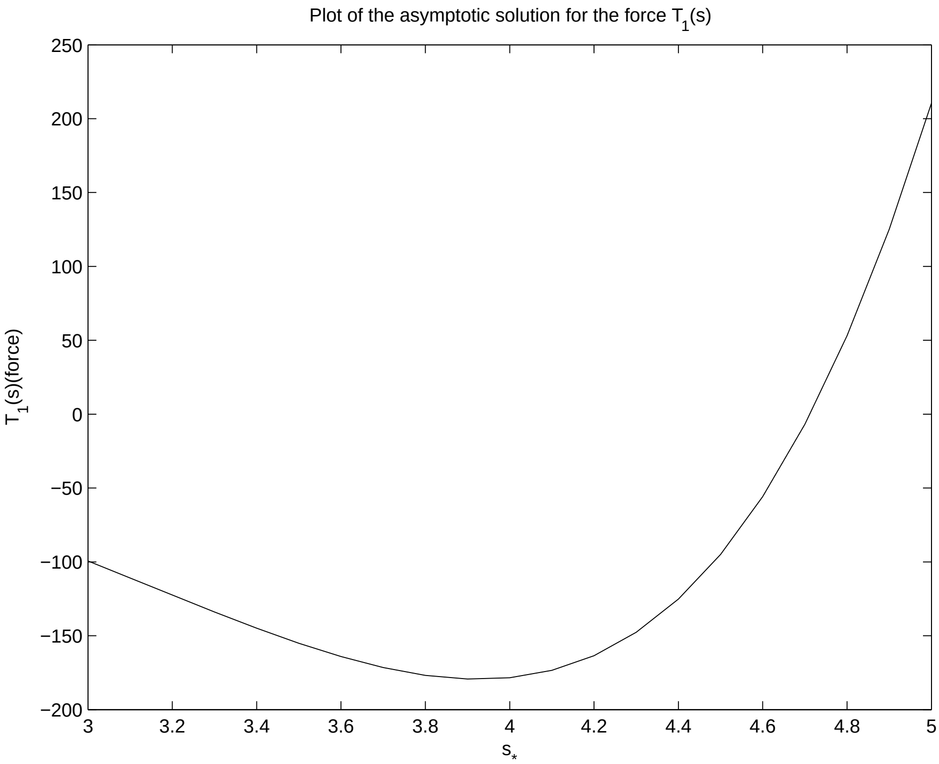


Figure 4.4: Asymptotic solution for the force $T_1(s)$

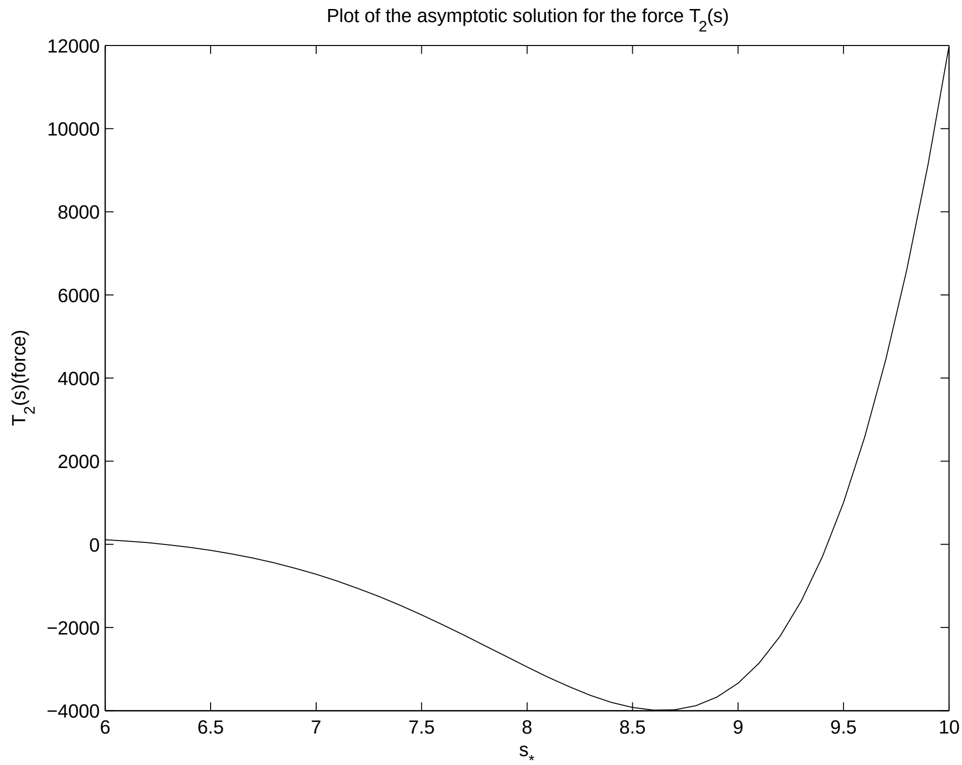


Figure 4.5: Asymptotic solution for the force $T_2(s)$

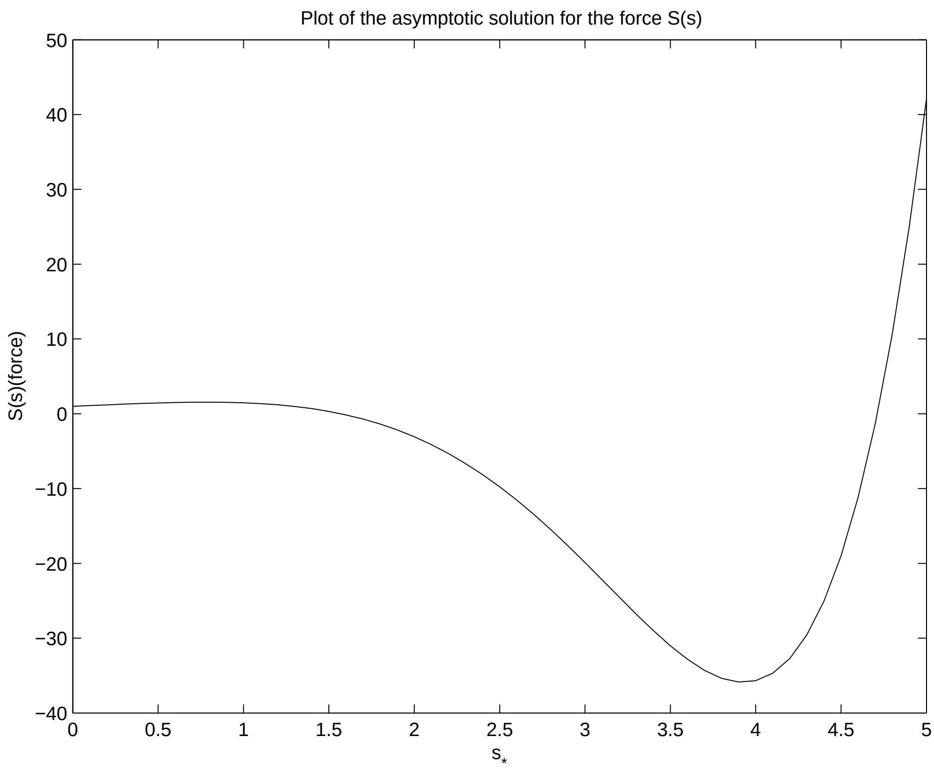


Figure 4.6: Asymptotic solution for the force $S(s)$

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