

# ASYMPTOTIC SOLUTION OF THE THIN SHELL EQUATIONS CONTAINING A TURNING POINT

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# Contents

|                                                                          |           |
|--------------------------------------------------------------------------|-----------|
| <b>List of Figures</b>                                                   | <b>3</b>  |
| <b>List of Tables</b>                                                    | <b>5</b>  |
| <b>Declaration</b>                                                       | <b>9</b>  |
| <b>1 Construction of a shell theory</b>                                  | <b>15</b> |
| 1.1 Preliminaries . . . . .                                              | 15        |
| 1.2 Early shell structures . . . . .                                     | 16        |
| 1.3 Modern shell applications . . . . .                                  | 17        |
| 1.4 Introduction . . . . .                                               | 17        |
| 1.5 Assumptions . . . . .                                                | 18        |
| 1.6 Distribution of extensional and flexural effects in shells . . . . . | 19        |
| 1.7 The Geometry of shells . . . . .                                     | 21        |
| 1.8 External Loadings . . . . .                                          | 23        |

|          |                                                    |           |
|----------|----------------------------------------------------|-----------|
| 1.9      | Internal Stresses . . . . .                        | 24        |
| 1.10     | Condition of Equilibrium . . . . .                 | 25        |
| 1.11     | Membrane Theory for Shells of Revolution . . . . . | 26        |
| 1.12     | Hooke's Law . . . . .                              | 27        |
| 1.13     | Deformation of an element of a shell . . . . .     | 28        |
| <b>2</b> | <b>Rayleigh Method</b>                             | <b>31</b> |
| <b>3</b> | <b>Asymptotic Methods</b>                          | <b>39</b> |
| 3.1      | Watson's Lemma . . . . .                           | 39        |
| 3.2      | Laplace's Method . . . . .                         | 41        |

# List of Figures

|     |                                                                           |    |
|-----|---------------------------------------------------------------------------|----|
| 1.1 | Types of shell structures: (a) synclastic (b) anticlastic (c) developable | 20 |
| 1.2 | shell geometry . . . . .                                                  | 21 |
| 1.3 | Shell of Revolution . . . . .                                             | 22 |
| 1.4 | Loading components from deadweight . . . . .                              | 23 |
| 1.5 | membrane forces . . . . .                                                 | 24 |
| 1.6 | transverse shear forces . . . . .                                         | 24 |
| 1.7 | Bending and Twisting Moments . . . . .                                    | 25 |
| 1.8 | Forces on a differential element . . . . .                                | 26 |
| 1.9 | Deformation of an element of a shell . . . . .                            | 28 |



## List of Tables

# Abstract

# Abstract

The subject matter of this report is the vibrating behavior of thin shells of revolution when the generating line has a point of inflection at  $s^*$ . At this point  $s^*$ , the curvature changes its sign. We develop from the deformation of a shell of revolution and obtain the natural frequency of vibration using Lord Rayleigh's method.

We make use of the law of conservation of energy which states that, at equilibrium, the total kinetic energy is equal to the total potential energy. We then equate the kinetic energy,  $J_\nu$ , (which is proportional to the square of the natural frequency,  $\omega$ ,) to the total potential energy,  $J_\kappa$ . To solve the integrals we make use of the Laplace's method and a programme from mathematica and then compare the two results.





# Declaration

No portion of the work in this thesis has been submitted for another degree or qualification of this or any other university or another institution of learning.



# ACKNOWLEDGEMENTS

This report is devoted to my family their patience, understanding and encouragement throughout its preparation. I am also indebted my supervisor Professor M. B. Petrov during my MSc studies for having introduced me to the study of shells, and for providing guidance from the beginning to the end of my MSc studies.



# DEDICATION

*To my mum and dad, Jayson and Partricia Mukungunugwa, brothers Victor,  
Lawrence and sister Audrey*



# Chapter 1

## Construction of a shell theory

### 1.1 Preliminaries

The applications of shell structures has been found in many fields of engineering, notably civil, mechanical and aeronautical. Considerable effort has been expended on the development of rigorous theories to describe the behavior of shells in the elastic range as realistically as possible. Such rigorous formulations have led to sets of differential equations for which analytical solutions are either yet to be found or those that are available are not very interesting from an engineering point of view.

The problems faced by engineers, physicists and applied mathematicians involve difficulties such as nonlinear governing equations and boundary conditions, variable coefficients and complex boundary shapes that preclude exact solutions. The development of numerical formulations such as finite element methods has seen the gradual cessation of attempts to find analytical solutions to rigorous formulation of shell behavior. The finite element method is a general numerical procedure that can be used to tackle any shell problem to any desired degree of accuracy. Initially, one only needs to understand the response of the basic



individual elements making up a shell structure (rather than the response of the whole structure), and then to assemble such elements by enforcing compatibility of displacements between the elements and their common nodes, and equilibrium of forces and moments at all nodes as well as constraint conditions at the boundary nodes of the structures. In this way, the overall response of the complete structure is predicted.

While such a solution approach is effective from the practical viewpoint of obtaining actual values of stresses and displacements in a structure, it is not very efficient for the effect of varying a single parameter. We need to obtain some insight into the character of the solutions and their dependence on certain parameters. Often one or more of the parameters become either very large or very small. These situations are difficult to treat by straight forward numerical procedure. We shall employ the methods of perturbation or asymptotic expansions in terms of a small or large parameter or coordinate. These are analytic methods that can provide an accurate approximation.

## 1.2 Early shell structures

Man-made shell structures have been in existence for many centuries. One of the earliest applications of the shell as a structural form is represented by the several domes that have been constructed for the purpose of providing for temples, cathedrals, monuments and other buildings. Notable historical examples include the Pantheon of ancient Rome, built around 2000 years ago; the Taj Mahal of India built in the seventeenth century just to mention a few.

## 1.3 Modern shell applications

The generally high strength-to-weight ratio of the shell form, combines with its inherent stiffness, has formed the basis of modern applications of shell structures. As may be seen in more detail in the historical account of Sechler, developments over the past 200 years in various sectors of human activity have opened up numerous fields of application for shells. Among these are thin concrete roofs, such as the hyperbolic paraboloidal roof of buildings like halls.

Apart from roofing, shells find applications in many other fields. In industry, boilers, pressure vessels and associated piping are further examples of shell structures in metal construction. Hollow members of large industrial steel structures such as offshore oil platforms, are another example of shell applications, as are bodies of transportation structures such as motor vehicles, ships, aircraft and spacecraft. The great diversity of shell applications is seen in the curved panels of world famous Thames Barrier in London, and the giant egg-shaped sewage digestors that are becoming a familiar sight on the landscape of countries such as Germany.

A shell may be defined as a relatively thin structural element, in which the material of the element is bound between two curved surfaces a relatively small distance apart. The behavior of a shell is usually modelled on the basis of its middle surface (alternatively referred to as midsurface), which is the locus of interior points equidistant from the two bounding surfaces of the shell.

## 1.4 Introduction

The second half of the twentieth century has been, in many ways, a fruitful period in the development of applied mechanics. Among the many problems that would come into the purview of mechanics would be that of the dynamic response of elastic structural systems, and elements of those systems, to dynamic

loading. An important element in a structural system is an element which has one dimension, called thickness, much smaller than other dimensions. Shells (also Beams and Plates) form a class of such elements. The property of thickness being considerably smaller than any other length justifies the construction of two dimensional mathematical models which are simpler but more limited than the corresponding three-dimensional models. Consequently, the mathematical models of shells can recognise only those features in the input that are averaged over distances comparable to thickness.

A thin shell is a body that is bounded primarily by two closely spaced curves. Shell theory may be viewed as a two-dimensional subset of elasticity which is valid for certain structures. The solution to three-dimensional elasticity problem is tough. In elasticity theory there are three basic sets of equations- equilibrium, kinematic (strain-displacements) and constitutive (Hookes law). We shall begin our development of shell theory by stating the basic assumptions.

## 1.5 Assumptions

There are four principal assumptions that are used to establish thin shell theory. these assumptions first adopted by Kirchhoff for the plates and later extended to shells by Love are sometimes referred to as the Love-Kirchhoff assumptions. We state them as follows-

1. One dimension is considerably smaller than the other two; so we can speak of a thin shell.

-As a measure of thinness, we say that the ratio of a shell thickness,  $h$ , to one of the radii of curvatures,  $R_i$  is small i.e  $\frac{h}{R_i} \ll 1$ .

2. The shell deflections are assumed to be small.

-This implies that the products of deformation quantities occurring in the relevant governing equations theory may be neglected, ensuring that the system is described by a set of geometrically linear equations. Also this assumption allows

us to refer all equilibrium and kinematic equations to the original undeformed reference state of the shell.

3. We take the stress in the direction normal to the thin dimension to be negligible.

-This assumption is justified because the shell has already been assumed to be thin. However, the assumption is not valid in the vicinity of concentrated transverse loads, even if the shell is thin.

4. We assume that a line originally normal to the shell reference surface remains unstrained.

-This implies that the direct strain in the direction normal to the middle surface and due to transverse shear forces are all zero.

## 1.6 Distribution of extensional and flexural effects in shells

For shells, the relative proportions of extensional and flexural effects at a given point depend on several factors such as the type of shell surface (synclastic, anticlastic, or developable), support conditions, loading configuration, and the proximity of edges and certain discontinuities.

Synclastic surfaces are those with positive *Gaussian* curvature. Gaussian curvature is defined as the product of the two principal curvatures (the maximum and minimum) at a given point of a curved surface, such curvatures occurring in two planes called the principal planes, which are perpendicular to each other. As is clear from Fig.1.1(a), for a synclastic surface, the centres of maximum and minimum curvature at a given point lie on the same side of the surface, so that the surface, when suitably oriented in space, can hold liquid around the point.

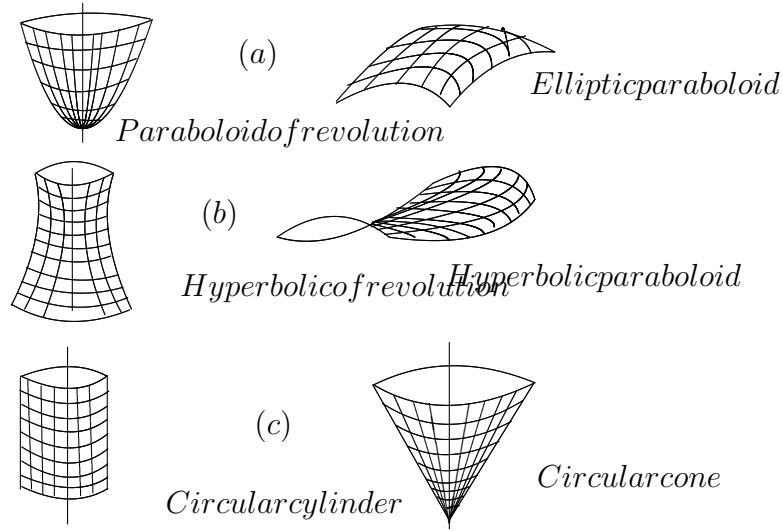


Figure 1.1: Types of shell structures: (a) synclastic (b) anticlastic (c) developable

Anticlastic surfaces possess negative Gaussian curvature, the centres of maximum and minimum curvatures at any given point on the surface lying on the opposite sides of the surface, so that the surface is 'saddle'-shaped and therefore incapable of holding liquid around an arbitrary point no matter how the surface is oriented in space (Fig 1.1(b)). Developable surfaces, as the name applies, are those that can be flattened into a plane surface, either directly or after making a single line cut in the surface (Fig.1.1(c)); they are characterised by zero Gaussian curvature and, as such, cannot hold liquid around an arbitrary point on the surface (the liquid would simply run off along the direction of zero curvature!)[2].

For synclastic shells, in particular, if both the shell geometry (i.e. shell thickness, midsurface slope in any arbitrary direction, and principal radii of curvature) and surface loads are varying smoothly (i.e. exhibiting no discontinuities in the variation of the shell geometrical parameters and loading components over the surface of the shell, nor in their first derivative with respect to arc length along a given direction), then extensional effects generally predominate in the interior regions of the shell, while in the edge zones bordering the supports, extensional and flexural effects usually become equally significant for most practical constructions of supports. Such localization of bending effects to the boundary zone of the shell

is also noted, but to a reduced extent, in developable shells; in anticlastic shells, bending emanating from the edges of the shell may propagate along certain directions deeply into the interior of the shell, a behaviour not unlike the bending in beams and plates.

## 1.7 The Geometry of shells

Our discussion is restricted to surfaces of revolution. A surface of revolution is obtained by rotation of a plane curve about an axis lying in the plane of the curve. This curve is called the meridian and its plane is the meridian plane. The intersection of the surface with the plane perpendicular to the axis of rotation are parallel circles and are called parallels. If we rotate the above curve about

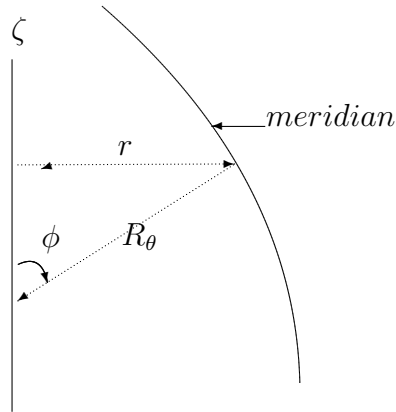


Figure 1.2: shell geometry

the axis of rotation  $\zeta$ , we obtain the shell of revolution that is shown below in Fig1.2. The directions  $x$ ,  $y$  and  $z$  are perpendicular to each other. For such a shell the lines of principal curvatures are its meridians and parallels.

Fig1.2 shows the following nomenclature

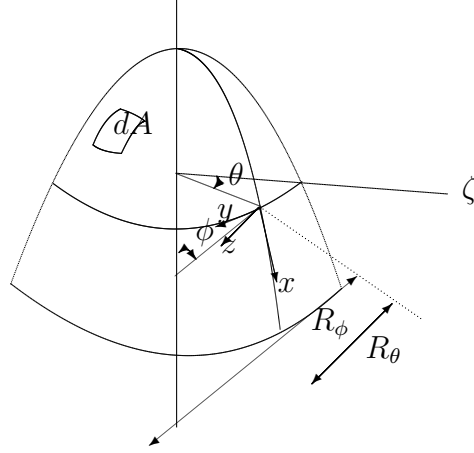


Figure 1.3: Shell of Revolution

$\phi$  is the angle between the axis of the shell and the shell normal at the point under consideration on the middle surface of the shell.

$\theta$  is the angle between  $r$  and the defined line  $\xi$

$R_\phi$  is the radius of curvature of the meridian.

$R_\theta$  is the length of the normal between any point on the middle surface and the axis of rotation.

$r$  is the radius of curvature of the parallel

## 1.8 External Loadings

The external loads consists of body forces that act on the element and surface forces that act on the upper and lower surfaces of the shell element. All loadings under consideration at any point on the shell can be resolved into three components in the  $x$ ,  $y$  and  $z$  directions. The  $x$ -direction is parallel to the tangent to the meridian. The  $y$ -direction is parallel to the tangent to the parallel circles. The  $z$ -direction is normal to the surface of the shell. The dead weight  $p$  (weight per unit area) for a shell of revolution can be resolved into load per unit area in the  $x$ ,  $y$  and  $z$  directions, respectively, in the following manner as shown by fig.1.3

$$p_x = p \sin \phi \quad p_y = 0 \quad p_z = \cos \phi$$

The external forces are resisted by internal forces, or stresses which are in equilibrium with the external forces. The internal forces consists of membrane forces, transverse shears, bending moments and twisting moments.

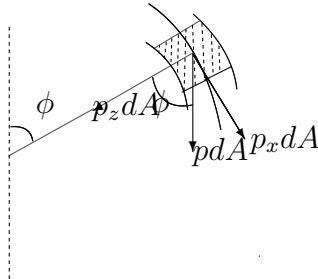


Figure 1.4: Loading components from deadweight



## 1.9 Internal Stresses

The external forces are resisted by internal forces, or stresses which are in equilibrium with the external loads. It is convenient to investigate the stresses along a meridian and parallel, which are defined by the angles  $\phi$  and  $\theta$ . 1. The membrane forces ( $N_\theta, N_\phi, N_{\phi\theta}, N_{\theta\phi}$ ), which act in the plane of the surface of the shell, are shown in Fig1.4

$N_\phi, N_\theta$  = normal inplane forces per unit length.

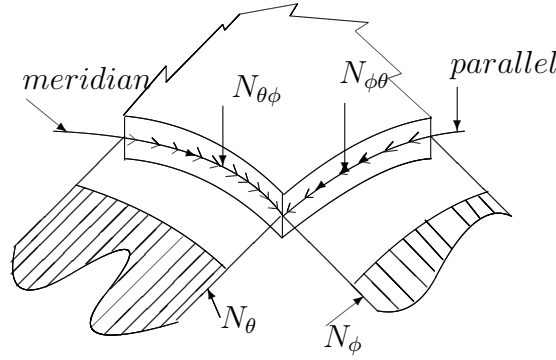


Figure 1.5: membrane forces

$N_{\theta\phi}, N_{\phi\theta}$  = inplane shear forces per unit length.

2. The transverse shear forces per unit length  $Q_\phi$  and  $Q_\theta$  are shown in Fig1.5.

$Q_\phi, Q_\theta$  = transverse shear forces per unit length.

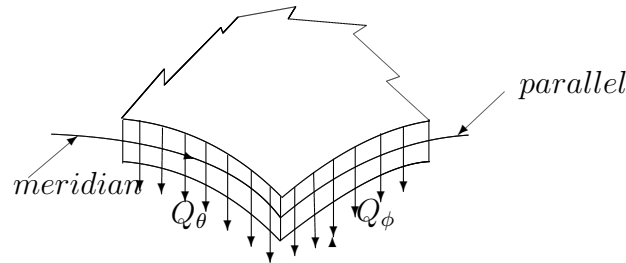


Figure 1.6: transverse shear forces

3. The bending moments  $M_\phi$  and  $M_\theta$  per unit length and twisting moments  $M_{\phi\theta}$  and  $M_{\theta\phi}$  per unit length are shown in Fig1.6.

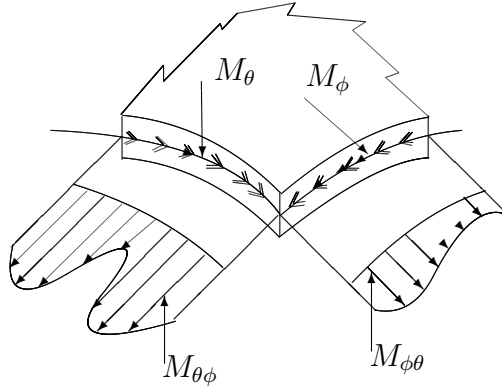


Figure 1.7: Bending and Twisting Moments

## 1.10 Condition of Equilibrium

We now determine the conditions for equilibrium of the shell element under external loads. We shall derive the equations arising by virtue of the demands of equilibrium and the compatibility of deformations by considering an individual differential shell element. These equations are relations between differential quantities or between differential changes in the internal forces and therefore are called differential equations. If the differential element is imagined separated from the loaded shell, it is stressed by ten internal components which must be in equilibrium with the external loads. These components are

$$N_\phi, N_\theta, N_{\theta\phi}, N_{\phi\theta}, Q_\phi, Q_\theta, M_\phi, M_\theta, M_{\phi\theta}, M_{\theta\phi}$$

To determine these components, there are only six known equilibrium equations-

$$\begin{aligned} \sum F_x &= 0 & \sum M_x &= 0 \\ \sum F_y &= 0 & \sum M_y &= 0 \\ \sum F_z &= 0 & \sum M_z &= 0 \end{aligned}$$

where  $\sum F_i$  is the sum of the forces in the  $i$ -direction ( $i = x, y, z$ ) and  $\sum M_i$  is the sum of the moments about the  $i$  axis.

## 1.11 Membrane Theory for Shells of Revolution

Let us assume that each member of the shell element is stressed only axially. End moments and shears are zero, and the shell is analysed as an internally statically determinate structure. Thus as a result of this assumption we have

$$Q_\phi = Q_\theta = M_\phi = M_\theta = M_{\phi\theta} = M_{\theta\phi} = 0$$

Consequently only four unknowns remain, the membrane forces:

$$N_\phi, N_\theta, N_{\theta\phi}, N_{\phi\theta}.$$

If a shell theory includes only membrane forces in the analysis, it is called a membrane theory. The components of the external loading are designated by X, Y and Z, which act in the  $x, y, z$  directions respectively and are in units of force. The forces shown are on one end only. On the opposite ends the forces will be

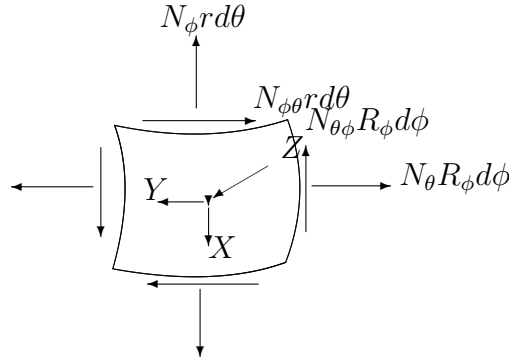


Figure 1.8: Forces on a differential element

differentially changed

- a)  $N_\phi r d\theta$  with change of  $\phi$  becomes  $N_\phi r d\theta + \frac{\partial}{\partial \phi}(N_\phi r d\theta) d\phi$
- b)  $N_\theta R_\phi d\phi$  with change of  $\theta$  becomes  $N_\theta R_\phi d\phi + \frac{\partial}{\partial \theta}(N_\theta R_\phi d\phi) d\theta$
- c)  $N_{\theta\phi} R_\phi d\phi$  with change of  $\theta$  becomes  $N_{\theta\phi} R_\phi d\phi + \frac{\partial}{\partial \theta}(N_{\theta\phi} R_\phi d\phi) d\theta$
- d)  $N_{\phi\theta} r d\theta$  with change of  $\phi$  becomes  $N_{\phi\theta} r d\theta + \frac{\partial}{\partial \phi}(N_{\phi\theta} r d\theta) d\phi$

The loading components are

- e)  $X = p_x r d\theta R_\phi d\phi,$

f)  $Y = p_y r d\theta R_\phi d\phi$ ,

g)  $Z = p_z r d\theta R_\phi d\phi$ .

The forces acting on the differential element must be in static equilibrium. Since there are no forces which would produce moments about the  $x$  and  $y$  axis we have  $\sum M_x = 0$  and  $\sum M_y = 0$ .

We now consider the moments about the  $z$  - *axis* which are given by

$$\sum M_z = N_{\phi\theta} r d\theta R_\phi d\phi - N_{\theta\phi} R_\phi d\phi r d\theta = 0.$$

Where the values of higher order terms are neglected. This leads to the relation

$$N_{\phi\theta} = N_{\theta\phi}.$$

If we continue with this analysis considering  $\sum F_x = 0$ ,  $\sum F_y = 0$  and  $\sum F_z = 0$ , we arrive at the following system of equations, which consists of two differential equations and one algebraic equation,

$$\frac{\partial}{\partial \phi}(N_\phi R_\theta \sin \phi) + \frac{\partial N_{\phi\theta}}{\partial \theta} R_\theta - N_\theta R_\phi \cos \theta + p_x R_\phi R_\theta \sin \phi = 0, \quad (1.1)$$

$$\frac{\partial N_\theta}{\partial \theta} R_\phi + \frac{\partial}{\partial \phi}(N_{\phi\theta} R_\phi \sin \phi) + N_{\phi\theta} R_\phi \cos \phi + p_y R_\phi R_\theta \sin \phi = 0, \quad (1.2)$$

$$N_\phi R_\theta + N_\theta R_\phi + p_z R_\phi R_\theta = 0. \quad (1.3)$$

The equations (1.1), (1.2), and (1.3) are used to evaluate the membrane forces  $N_\theta$ ,  $N_\phi$ , and  $N_{\phi\theta}$  for any loading condition.

## 1.12 Hooke's Law

Hooke's law relates the strains,  $\epsilon_{\phi_z}, \epsilon_{\theta_z}$ , with the corresponding stresses,  $\sigma_\phi$  and  $\sigma_\theta$

$$\epsilon_{\phi_z} = \frac{1}{E}(\sigma_\phi - \nu \sigma_\theta) \quad \epsilon_{\theta_z} = \frac{1}{E}(\sigma_\theta - \nu \sigma_\phi) \quad (1.4)$$

or

$$\sigma_\phi = \frac{E}{1 - \nu^2}(\epsilon_{\phi_z} + \nu\epsilon_{\theta_z}), \quad (1.5)$$

$$\sigma_\theta = \frac{E}{1 - \nu^2}(\epsilon_{\theta_z} + \nu\epsilon_{\phi_z}), \quad (1.6)$$

where  $E$  is Young's modulus of elasticity and  $\nu$  is Poisson's ratio for the shell material. The influence of the stress and strain in the  $z$ -direction is small and will not be considered in this report.

### 1.13 Deformation of an element of a shell

Let ABCD(Fig1.8) represent an infinitely small element cut out from a shell by two pairs of adjacent planes normal to the middle surface of the shell and containing its principal curvatures. Denote by  $R_x$  and  $R_y$  the radii of principal curvatures in the  $xz$  and  $yz$  planes respectively. We shall take the coordinate axes  $x$  and  $y$  tangent at  $O$  to the lines of principal curvatures and the  $z$ -axis normal to the middle surface. The thickness of the shell which we will assume to be constant, we denote by  $h \ll 1$ .

Let us begin with a simple case in which, during bending, the lateral faces of

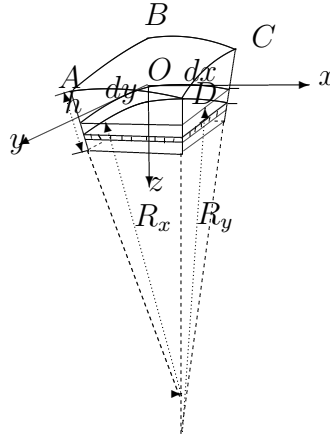


Figure 1.9: Deformation of an element of a shell

the element ABCD rotate only with respect to their lines of intersection with the middle surface. Let  $R'_x$  and  $R'_y$  be the values of the radii of curvature after deformation. Then the unit elongations of a thin lamina at a distance  $z$  from the middle surface are,

$$\epsilon_x = -\frac{z}{1-\frac{z}{R_x}}\left(\frac{1}{R'_x} - \frac{1}{R_x}\right) \quad \epsilon_y = -\frac{z}{1-\frac{z}{R_y}}\left(\frac{1}{R'_y} - \frac{1}{R_y}\right).$$

If the corresponding unit elongations of the middle surface in the  $x$ - and  $y$ -directions are denoted by  $\epsilon_1$  and  $\epsilon_2$  respectively. Also by the assumption stated earlier,  $h$  is small in comparison with the radii of curvature such that  $\frac{z}{R_x}$  and  $\frac{z}{R_y}$  can be neglected in comparison with the unit. We shall neglect also the effect of the elongations  $\epsilon_1$  and  $\epsilon_2$  on the curvatures. Thus

$$\epsilon_x = \epsilon_1 - z\left(\frac{1}{R'_x} - \frac{1}{R_x}\right) = \epsilon_1 - \kappa_1 z \quad (1.7)$$

$$\epsilon_y = \epsilon_2 - z\left(\frac{1}{R'_y} - \frac{1}{R_y}\right) = \epsilon_2 - \kappa_2 z \quad (1.8)$$

where  $\kappa_1$  and  $\kappa_2$  are the changes of curvature.

Assuming that there are no normal stresses and using the expressions above we obtain the the expressions for the components of stress given below

$$\sigma_x = \frac{E}{1-\nu^2}[\epsilon_1 + \nu\epsilon_2 - z(\kappa_1 + \nu\kappa_2)]$$

$$\sigma_y = \frac{E}{1-\nu^2}[\epsilon_2 + \nu\epsilon_1 - z(\kappa_2 + \nu\kappa_1)].$$

The resultant forces which act on the middle surface of the shell and the bending moments per unit length are given by

$$N_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x dz = \frac{Eh}{1-\nu^2}(\epsilon_1 + \nu\epsilon_2)$$

$$N_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y dz = \frac{Eh}{1-\nu^2}(\epsilon_2 + \nu\epsilon_1)$$

and also

$$M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_x dz = -D(\kappa_1 + \nu\kappa_2)$$

$$M_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} z \sigma_y dz = -D(\kappa_2 + \nu \kappa_1)$$

where  $D = \frac{Eh^3}{1-\nu^2}$  is the flexural rigidity.

Considering the element of the shell, as in fig1.8, and neglecting the strain energy due to shearing forces  $Q_x$  and  $Q_y$ , we find that the strain energy of the element is equal to the work done on the element by the bending moment  $M_x dy$  and  $M_y dx$  and the twisting moments  $M_{xy} dy$  and  $M_{yx} dx$ . Since we neglect the effect of vertical shearing forces on the curvature of the deflection surface, the strain energy due to bending moments will be represented by the expression

$$V = \frac{1}{2} D \int \int \left[ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + 2\nu \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] dx dy$$

derived in the theory of plates for the case pure bending. The strain energy of a deformed shell consists of two parts: (1) the strain energy due to bending and (2) the energy due to stretching of the middle surface. The maximum potential energy for the shell element is thus given by

$$J_{max} = \frac{D}{2} \int \int \left[ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu) \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] dx dy \quad (1.9)$$

The energy due to the stretching of the middle surface is given by

$$J_\epsilon = \frac{D}{2} \int \int [(\epsilon_1 + \epsilon_2)^2 - 2(1-\nu)(\epsilon_1 \epsilon_2 - \epsilon_{12})] dA. \quad (1.10)$$

Since the elongations,  $\epsilon_1$  and  $\epsilon_2$  on the curvatures has been neglected, the energy  $J_\epsilon$  shall be assumed to be zero.

## Chapter 2

# Rayleigh Method

The inextensional vibrations of thin shells were first studied by Lord Rayleigh in the second half of the twentieth century, and since that time his procedure has been used to estimate natural frequencies for various shell shapes. In considering the deformation of a thin shell the most important question which presents itself is whether the middle surface does, or does not, undergo extension. The common mathematical models for thin shells admit a strain energy expression that consists of two parts: extensional (stretching or membrane) and flexural (or bending). In general, the inextensional energy is produced by the extensional and shear strains in the middle surface, and it is proportional to the thickness. But by Love-Kirchoff's assumption 3 mentioned earlier, these strains are assumed to be small and therefore negligible. Flexural energy is produced by the changes in curvature and torsion of the middle surface, and it is also proportional to thickness.

Lord Rayleigh proposed the hypothesis that the low frequency modes must be flexural and that for a sufficiently thin shell, the contribution of extensional energy to the total strain energy must be negligible. Following his hypothesis, Lord Rayleigh assumed that during vibration the displacement field is such that the two extensional and the shearing strain of the middle surface are zero. He then



went on and obtained the three displacement components which he called inextensional displacements. These displacements shall be denoted by  $w$ ,  $u$ , and  $v$ .  $w$  is in the direction parallel to the tangent of the meridian,  $u$  is in the direction parallel to tangent of the parallel circles and  $v$  is in the direction normal to the meridian. The displacements  $u$  and  $v$  shall be assumed to be small as compared to  $w$ .

Lord Rayleigh used the inextensional displacements to calculate approximate natural frequencies from the identification of the strain and kinetic energies (Rayleigh Principle). Since the kinetic energy is proportional to thickness and to the square of the frequency, and flexural energy is proportional to the cube of thickness, Lord Rayleigh concluded that the natural frequency is proportional to thickness.

Now, let us assume that the shell is vibrating freely. By the law of conservation of energy, at equilibrium the total kinetic energy (which we shall denote as  $T = \omega^2 J_\nu$ ) is equal to the total potential energy (denoted by  $U = J_\kappa$ ). Thus we have that

$$T = \omega^2 J_\nu = J_\kappa = U \quad (A)$$

where the expressions for the kinetic and potential energies are given by

$$J_\nu = \int \int \omega^2 [w^2 + v^2 + u^2] dA, \quad (2.1)$$

and after substituting into (1.9) the changes of curvatures  $\kappa_1$ ,  $\kappa_2$  and  $\kappa_{12}$  we obtain

$$J_\kappa = \frac{Eh^3}{3(1-\nu^2)} \int \int [(\kappa_1 + \kappa_2)^2 - 2(1-\nu)(\kappa_1\kappa_2 - \kappa_{12})] dA, \quad (2.2)$$

where  $E$  is the young modulus,  $h$  is the thickness of the shell,  $\nu$  is poison's ratio and the changes of curvatures are given by

$$\kappa_1 = -w'' + (k_1 u)'$$

$$\kappa_2 = B' B^{-1} (k_1 u - w') + m B^{-1} (m B^{-1} w + k_2 v),$$

$$\kappa_{12} = m B^{-1} w' + m B' B^{-2} w - m B^{-1} k_1 u + B k_2 (B^{-1} v)'$$

Since  $E$ ,  $h$  and  $\nu$  do not play an important role in the asymptotic solution, to be

discussed in the next chapter, we have put the multiplier  $\frac{Eh^3}{3(1-\nu^2)} = 1$ . This also simplifies our expressions. Again we note that  $u$  and  $v$  are small compared to  $w$  i.e  $w \gg u$  and  $w \gg v$ . The above equations are therefore reduces to

$$J_\nu = \int \int \omega^2 w^2 dA,$$

$$\kappa_1 = -w'',$$

$$\kappa_2 = -B'B^{-1}w' + m^2B^{-2}w,$$

$$\kappa_{12} = mB^{-1}w' + mB'B^{-2}w.$$

The displacement  $w(s, m)$  shall be represented in the form of the combination of the Airy function,  $U(\eta)$  and its derivative  $\frac{dU(\eta)}{d\eta}$  as follows,

$$w(s, m) = w^{(1)}(s, m)U(\eta) + w^{(2)}(s, m)\frac{dU}{d\eta}, \quad (2.3)$$

where  $U(\eta)$  is a solution of of the Airy equation,

$$\frac{d^2U}{d\eta^2} + \eta U = 0, \quad \eta = m^{\frac{2}{3}}\zeta(s). \quad (2.4)$$

$\zeta(s)$  is a rapidly changing function which we shall derive shortly.  $w^{(k)}$  are asymptotic series of the form,

$$w^{(k)} = m^{\gamma_k(s)} \sum_{l=0}^{\infty} m^{-2l} w^{(k,l)}(s), \quad k = 1, 2. \quad (2.5)$$

We are mainly interested in the leading term of our asymptotic series thus we put  $l = 0$  so that

$$w^{(1)} = m^{\gamma_1(s)} w^{(1,0)}, \quad \text{and} \quad w^{(2)} = m^{\gamma_2(s)} w^{(2,0)}, \quad (2.6)$$

where

$$w^{(1,0)} = -(k_2 B)^{-1} v^{(1,0)}, \quad w^{(2,0)} = -(k_2 B)^{-1} v^{(2,0)} \quad (2.7)$$

$$v^{(1,0)} = (\zeta' B^{-1} k_2^{-1})^{-\frac{1}{2}}, \quad v^{(2,0)} = (|\zeta| \zeta' B^{-1} k_2^{-1})^{-\frac{1}{2}}. \quad (2.8)$$

Now substituting (2.8) into (2.7) we obtain

$$w^{(1,0)} = -(k_2 B)^{-1} (\zeta' B^{-1} k_2^{-1})^{-\frac{1}{2}} \quad (2.9)$$

and

$$w^{(2,0)} = -(k_2 B)^{-1} (|\zeta| \zeta' B^{-1} k_2^{-1})^{-\frac{1}{2}}. \quad (2.10)$$

Again substituting equations (2.9) and (2.10) into (2.3) we have the full expansion of  $w(s, m)$ , i.e

$$w(s, m) = -(k_2 B)^{-\frac{1}{2}} \left( m^{\gamma_1(s)} \zeta' U(\eta) + m^{\gamma_2(s)} |\zeta| \zeta' \frac{dU}{d\eta} \right). \quad (2.11)$$

We pause for a moment and express  $\zeta$  and  $U(\eta)$  wholly in terms of  $s$ . In general, the generating line of our shell of revolution has the form  $z = a - b \text{sinc}(s - s^*)$ . For this report we shall take  $a = 2$ ,  $b = c = 1$  and  $s^* = 0$ , where  $s^*$  is a turning point. By a turning point, we mean the point at which the curvature changes its sign from positive to negative. Thus the generating line of our shell of revolution is therefore given by [6],

$$z = 2 - \text{sinc}s. \quad (2.12)$$

Firstly, we are going to evaluate the expression for  $\zeta$  which is given by

$$\begin{aligned} \zeta(s) &= \left( \frac{3}{2} \int_{s^*}^s z ds \right)^{\frac{2}{3}} \\ &= \left( \frac{3}{2} \int_0^s (2 - \text{sinc}s) \right)^{\frac{2}{3}} ds \\ &= \left( \frac{3}{2} (2s + \text{coss} - 1) \right)^{\frac{2}{3}}. \end{aligned}$$

Since  $\text{coss} \approx 1$  for all  $s \ll 1$ ,  $\zeta(s)$  is therefore given by

$$\zeta(s) \cong (3s)^{\frac{2}{3}}. \quad (2.13)$$

The equation

$$\frac{d^2 U}{d\eta^2} + \eta U = 0$$

possess two linearly independent solutions  $U_1$  and  $U_2$ , and their asymptotic behavior is given by the following formulas

$$U_1 = \frac{1}{2}\pi^{\frac{-1}{2}}(\eta)^{-\frac{1}{4}}e^{-\delta},$$

and

$$U_2 = \frac{1}{2}\pi^{\frac{-1}{2}}(\eta)^{-\frac{1}{4}}e^{\delta}.$$

For the Airy function in (2.11), we shall define it to be  $U(\eta) := U_1(\eta)$ , and drop  $U_2$  so that we can speak of a bounded solution which will not misbehave when one parameter becomes very large.

We now write  $\eta$  and  $\delta$  as functions of  $s$  alone i.e

$$\eta = m^{\frac{2}{3}}\zeta(s) = (3ms)^{\frac{2}{3}} \quad (2.14)$$

and

$$\begin{aligned} \delta &= m \int_{s^*}^s |z| ds \\ &= m \int_0^s |2 - \sin s| ds \\ &= 2ms. \end{aligned}$$

We are now in a position to write the Airy function  $U(\eta)$  and its derivative  $\frac{dU}{d\eta}$  as functions of  $s$  alone. The double integral in expressions (2.1) and (2.2) will be converted to a single integral, where this single integral is evaluated with respect to  $s$ . Thus we have

$$U(\eta) = \frac{1}{2}\pi^{-\frac{1}{2}}(3ms)^{-\frac{1}{6}}e^{-2ms} \quad (2.15)$$

and

$$\frac{dU}{d\eta} = \frac{1}{8}\pi^{-\frac{1}{2}}(3ms)^{-\frac{5}{6}}e^{-2ms}. \quad (2.16)$$

Substituting these equations into the expression for  $w(s, m)$  we obtain

$$w(s, m) = -(k_2 B \pi)^{-\frac{1}{2}} [m^{\gamma_1 - \frac{1}{6}} + \frac{1}{4} m^{\gamma_2 - \frac{5}{6}}] (3s)^{\frac{-1}{2}} e^{-2ms}. \quad (2.17)$$

For simplicity of the expression we put  $K' = -(k_2 B \pi)^{-\frac{1}{2}} [m^{\gamma_1 - \frac{1}{6}} + \frac{1}{4} m^{\gamma_2 - \frac{5}{6}}]$  so that

$$w(s, m) = K'(3s)^{-\frac{1}{2}} e^{-2ms}, \quad (2.18)$$

$$w' = -mK'[2(3s)^{-\frac{1}{2}} + O(m^{-1})]e^{-2ms} \quad (2.19)$$

and

$$w'' = m^2 K'[4(3s)^{-\frac{1}{2}} + O(m^{-1})]e^{-2ms}, \quad (2.20)$$

where all the derivatives are with respect to  $s$ .

We then express the curvatures  $\kappa_1$ ,  $\kappa_2$  and  $\kappa_{12}$  in terms of  $s$  and try to compute the two integrals in (2.1) and (2.2). Now if we substitute

$$\kappa_1^2 = 16m^2 K'^2 (3s)^{-1} e^{-4ms}, \quad (2.21)$$

$$\kappa_2^2 = [B' B^{-1} K'^2 (1 + m^2 B^{-2})]^2 (3s)^{-1} e^{-4ms}, \quad (2.22)$$

$$\kappa_{12}^2 = [m B' B^{-2} K' - 2m^2 B^{-1} K']^2 (3s)^{-1} e^{-4ms}, \quad (2.23)$$

$$\kappa_1 \kappa_2 = 4m B' B^{-1} K'^2 (1 + m^2 B^{-2}) (3s)^{-1} e^{-4ms}. \quad (2.24)$$

From expression (A), we have that the natural frequencies of vibration are expressed as a quotient of the kinetic energy and potential energy as follows:

$$\omega^2 = \frac{J_\kappa}{J_\nu} \quad (2.25)$$

where  $J_\kappa$  and  $J_\nu$  are given by (2.1) and (2.2) respectively.

Let us first note, from expression (2.18) and (2.21) to (2.24), that there is a common integrand,  $s^{-1} e^{-4ms}$ , that will appear in both the numerator and denominator of (2.25). What seems to be the logical thing to do is to divide the numerator and denominator by this value and conclude that these natural frequencies depend on some constant function of  $m$ , the number of waves in the circumferential direction, and the Poisson's ratio  $\nu$ . But, this is only possible if

the value of the integral is known to be finite. If it is infinite or zero then the expression for the natural frequencies is indeterminate. We shall develop the method of Laplace for evaluating integrals of real variable and use these methods or theorems to evaluate our integral.



# Chapter 3

## Asymptotic Methods

### 3.1 Watson's Lemma

Consider an integral of the form

$$I(\alpha) = \int_0^\infty q(t)e^{-\alpha t} dt. \quad (3.1)$$

A direct way of computing (3.1) is to substitute the Maclaurin's expansion for  $q(t)$

$$q(t) = q(0) + tq'(0) + t^2q''(0) + \cdots ,$$

and integrating term by term. This formal process suggest a natural extension: can a similar asymptotic result be constructed by termwise integration in cases when the expansion of  $q(t)$  near  $t = 0$  is in terms of non-integer powers of  $t$ ?

Watson (1918) supplied an answer which state that it is immaterial whether the expansion of  $q(t)$  ascends merely in regularly spaced powers, or whether the series converges or is merely asymptotic. The general result is illustrated sufficiently well by the following theorem [5] which is probably the most frequently used result for deriving asymptotic expansions.



**Theorem 1** *Let  $q(t)$  be a function of the real variable  $t$  such that*

$$q(t) = \sum_{n=0}^{\infty} a_n t^{n+\beta} = t^{\beta}(a_0 + a_1 t + a_2 t^2 + \cdots) \quad \beta > -1$$

*where  $q(t)$  is regular on  $[0, b]$  and near  $t = 0$ ,  $a_0 > 0$ . If*

$$I(\alpha) = \int_0^b q(t) e^{-\alpha t} dt \quad (3.2)$$

*then*

$$I(\alpha) \approx \sum_{n=0}^{\infty} \frac{\Gamma(n + \beta + 1)}{\alpha^{n+\beta+1}} a_n. \quad (3.3)$$

**Proof:**

We want to show that  $\forall N \in \mathbb{N}$ ,

$$I(\alpha) \approx \sum_{n=0}^N \frac{\Gamma(n + \beta + 1)}{\alpha^{n+\beta+1}} a_n + O\left(\frac{1}{\alpha^{\beta+N+2}}\right).$$

We first note that

$$\int_0^b = \int_0^{\epsilon} + \int_{\epsilon}^b.$$

Thus if

$$I(\alpha) = \int_0^b F(t, \alpha) dt, \quad F(t, \alpha) = q(t) e^{-\alpha t},$$

we have that

$$I(\alpha) = \int_0^{\epsilon} F(t, \alpha) dt + \int_{\epsilon}^b F(t, \alpha) dt.$$

$q(t)$  is regular on the interval  $[\epsilon, b]$  and hence bounded. Thus we have

$$|F(t, \alpha)| \leq M < \infty, \quad \text{on} \quad [\epsilon, b]$$

$$\begin{aligned} \left| \int_{\epsilon}^b F(t, \alpha) dt \right| &\leq \int_{\epsilon}^b |M e^{-\alpha t}| dt \\ &= M \int_{\epsilon}^b e^{-\alpha t} dt \\ &= M \left( \frac{1}{\alpha} e^{-\alpha \epsilon} - \frac{1}{\alpha} e^{-\alpha b} \right) \end{aligned}$$

which are exponentially small terms.

Put  $q(t) = t^\beta (\sum_{n=0}^N a_n t^n + O(t^{N+1}))$ , this implies that

$$I(\alpha) \sim \sum_{n=0}^{\infty} a_n \int_0^{\epsilon} t^{\beta+n} e^{-\alpha t} dt + \text{exponentially small terms}$$

We put  $\alpha t = u$ , hence  $dt = \frac{du}{\alpha}$ . Thus

$$\begin{aligned} a_n \int_0^{\epsilon} t^{\beta+n} e^{-\alpha t} dt &= a_n \int_0^{\epsilon} \frac{u^{\beta+n}}{\alpha^{\beta+n+1}} e^{-u} du \\ &\sim \frac{a_n}{\alpha^{\beta+n+1}} \int_0^{\infty} u^{\beta+n} e^{-u} du \\ &= \frac{a_n}{\alpha^{\beta+n+1}} \Gamma(\beta + n + 1). \end{aligned}$$

Hence

$$I(\alpha) \approx \sum_{n=0}^{\infty} \frac{a_n \Gamma(\beta + n + 1)}{\alpha^{\beta+n+1}}. \quad (3.4)$$

In the notion of the integrand of expression (3.1),  $q(s) = s^{-1}$ , for which if we compare its series expansion which only consists of  $a_0$ , we notice that  $\beta = -1$  which contradicts Watson's lemma which requires that  $\beta > -1$ . We now develop a generalisation of the integral (3.1), which was originated by Laplace.

## 3.2 Laplace's Method

Consider the integral

$$I(\alpha) = \int_a^b e^{-\alpha p(t)} q(t) dt, \quad (3.5)$$

in which  $a$ ,  $b$ ,  $p(t)$  and  $q(t)$  are independent of the positive parameter  $\alpha$ . Either  $a$  or  $b$  or both may be infinite. Laplace (1820) originated the following powerful method for approximating  $I(\alpha)$ . The peak value of the factor  $e^{-\alpha p(t)}$  occurs at a point  $t = t_0$ , say, at which  $p(t)$  is a minimum. When  $\alpha$  is large, this peak is very sharp, and the graph of the integrand suggests that the overwhelming

contribution to the integral comes from the neighborhood of  $t_0$ . Accordingly, we replace  $p(t)$  and  $q(t)$  by the leading terms in series expansions in ascending powers of  $(t - t_0)$ , and then extend the integration limits to  $\infty$  or  $-\infty$ . The resulting integral is explicitly evaluable and yields the required approximation. Suppose, that  $t_0 = a$ ,  $p(a) > 0$  and  $q(a) \neq 0$ . Then Laplace's procedure is expressed by

$$\begin{aligned} I(\alpha) &= \int_a^b e^{-\alpha(p(a)+(t-a)p'(a))} q(a) dt \\ &= q(a) e^{-\alpha p(a)} \int_a^b e^{-\alpha(t-a)p'(a)} dt \\ &\approx \frac{q(a) e^{-\alpha p(a)}}{\alpha p'(a)} (1 - e^{-\alpha(b-a)p'(a)}). \end{aligned}$$

If we define

$$I(m) = \int_0^s s^{-1} e^{-4ms} ds$$

with  $m \equiv \alpha$ ,  $a = 0$ ,  $p(s) = 4s$  and  $q(s) = s^{-1}$ , we find that  $q(0)$  does not exist. We proceed with Laplace's methods and try to develop a general procedure for evaluating  $I(m)$ .

Without any loss of generality, let us assume that  $a$  is finite and the minimum of  $p(t)$  occurs at  $t = a$ . With precisely formulated conditions on  $p(t)$  and  $q(t)$  it was proved in Olver that the Laplace approximation is asymptotic to the integral as  $\alpha \rightarrow \infty$ .

We suppose that the limits  $a$  and  $b$  are independent of the large parameter  $\alpha$ ,  $a$  being finite and  $b(> a)$  finite or infinite. The functions  $p(t)$  and  $q(t)$  are independent of  $\alpha$ ,  $p(t)$  being real and  $q(t)$  either real or complex[5, 10]. In addition to these assumptions

- (i)  $p(t) > p(a)$  when  $t \in (a, b)$
- (ii)  $p'(t)$  and  $q(t)$  are continuous in a neighborhood of  $a$ , except possibly at  $a$ .
- (iii) as  $t \rightarrow a$  from above  $p(t) - p(a) \sim P(t - a)^\mu$ ,  $q(t) \sim Q(t - a)^\lambda$  and we require that the first of these relations be differentiable. Here  $P, \mu$  and  $\lambda$  are constants

and  $Q$  is either real or complex constant.

(iv)  $I(\alpha) = \int_a^b e^{-\alpha p(t)} q(t) dt$  converges absolutely throughout its range for all sufficiently large  $\alpha$ .

**Theorem 2** *With all the conditions listed above*

$$I(\alpha) \sim \frac{Q}{\mu} \Gamma\left(\frac{\lambda}{\mu}\right) \frac{e^{-\alpha p(a)}}{(P\alpha)^{\frac{\lambda}{\mu}}} \quad (3.6)$$

**Proof:**

From conditions (ii) and (iii), a number  $k$  can be found which is close to  $a$  to ensure that in  $(a, k]$ ,  $p'(t)$  is continuous and positive and  $q(t)$  is continuous. Since  $p(t)$  is increasing in  $(a, k)$ , we define

$$v := p(t) - p(a)$$

and take it as the new integration variable in this interval. Now that  $v$  and  $t$  are continuous functions of each other

$$e^{\alpha p(a)} \int_a^k e^{-\alpha p(t)} q(t) dt = \int_0^\rho e^{-\alpha v} f(v) dv, \quad (3.7)$$

where

$$\rho = p(k) - p(a), \quad f(v) = q(t) \frac{dt}{dv} = \frac{q(t)}{p'(t)}$$

We can clearly see that  $\rho$  is finite and positive, and  $f(v)$  is continuous when  $v \in (0, \rho]$ . Since  $v \sim P(t - a)^\mu$  as  $t \rightarrow a$ , we have that

$$t - a \sim \left(\frac{v}{P}\right)^{\frac{1}{\mu}} \quad (v \rightarrow 0^+)$$

and hence after differentiation we have

$$f(v) \sim \frac{Q v^{\frac{\lambda}{\mu}-1}}{\mu P^{\frac{\lambda}{\mu}}} \quad (v \rightarrow 0^+).$$

As a result of this relation we rearrange the integral (3.7) in the form

$$\int_0^\rho e^{-\alpha v} f(v) dv = \frac{Q}{\mu P^{\frac{\lambda}{\mu}}} \left\{ \int_0^\infty e^{-\alpha v} v^{\frac{\lambda}{\mu}-1} dv - \varepsilon_1(\alpha) \right\} + \varepsilon_2(\alpha), \quad (3.8)$$

where

$$\varepsilon_1(\alpha) = \int_{\rho}^{\infty} e^{-\alpha v} v^{\frac{\lambda}{\mu}-1} dv, \quad \varepsilon_2(\alpha) = \int_0^{\rho} e^{-\alpha v} \left( f(v) - \frac{Q v^{\frac{\lambda}{\mu}-1}}{\mu P^{\frac{\lambda}{\mu}}} \right) dv$$

the first term on the right hand side of equation (3.8) is evaluable by the use of Euler's integral and immediately yields the required approximation (3.6).

Secondly, given an arbitrary positive number  $\epsilon$ , we make  $\rho$  small enough (by choosing  $k$  sufficiently close to  $a$ ) to ensure that

$$\left| f(v) - \frac{Q v^{\frac{\lambda}{\mu}-1}}{\mu P^{\frac{\lambda}{\mu}}} \right| < \epsilon \frac{Q v^{\frac{\lambda}{\mu}-1}}{\mu P^{\frac{\lambda}{\mu}}}. \quad (3.9)$$

Then by the use of Euler's integral we have

$$|\varepsilon_2(\alpha)| < \epsilon \frac{Q}{\mu} \Gamma\left(\frac{\lambda}{\mu}\right) \frac{1}{(P\alpha)^{\frac{\lambda}{\mu}}}. \quad (3.10)$$

Thirdly, in the notation of the incomplete Gamma function we have

$$\varepsilon_1(\alpha) = \frac{1}{\alpha^{\frac{\lambda}{\mu}}} \Gamma\left(\frac{\lambda}{\mu}, \rho\alpha\right) = O\left(\frac{e^{-\rho\alpha}}{\alpha}\right) \quad (3.11)$$

for large  $\alpha$ .

The proof of Theorem 2 is completed by making  $\alpha$  large enough to guarantee that the right hand side of (3.11) is bounded and small.

Theorem 2 confirms the prediction of this section that in a wide range of circumstances the asymptotic form of the integral (3.5) for large  $\alpha$  depends solely on the behaviour of the integrand near the minimum of  $p(t)$ .

Now substituting (2.21) to (2.24) into (2.2) we have the expression

$$J_{\kappa} = \int_0^s \left[ \frac{16}{3} m^2 K'^2 - (B^{-1} B' K' (1 - m^2 B^{-2}))^2 + 8\nu m B' B^{-1} K'^2 \right. \\ \left. (1 + m^2 B^{-2}) + 2(1 - \nu)(m B' B^{-2} K' - 2m^2 B' K')^2 \right] s^{-1} e^{-4ms} ds. \quad (3.12)$$

Putting

$$T = \frac{16}{3} m^2 K'^2 - (B^{-1} B' K' (1 - m^2 B^{-2}))^2 + 8\nu m B' B^{-1} K'^2 (1 + m^2 B^{-2}) + 2(1 - \nu)(m B' B^{-2} K' - 2m^2 B' K')^2,$$

we have

$$J_{\kappa} = T \int_0^s s^{-1} e^{-4ms} ds. \quad (3.13)$$

Again if we substitute (2.18) into (2.3) we have

$$J_\nu = \int_0^s \omega^2 s^{-1} e^{-4ms} ds. \quad (3.14)$$

Let us denote by  $I(m)$ , the integral

$$I(m) = \int_0^s s^{-1} e^{-4ms} ds, \quad (3.15)$$

and use the results of theorem 2 to evaluate this integral. We first take the integral to  $\infty$  by making use of the fact that

$$\int_0^s = \int_0^\infty - \int_s^\infty, \quad (3.16)$$

where the last term on the right hand side of (2.13) are exponentially small terms.

Thus (2.12) becomes

$$I(m) \sim \int_0^\infty s^{-1} e^{-4ms} ds. \quad (3.17)$$

In the notion of theorem 2,  $q(s) = s^{-1}$  and  $p(s) = 4s$ . We note that the minimum of  $p(s)$  in the integration range occurs at  $s = 0$ , thus we put  $a = 0$ . From these choices of  $p(s)$  and  $q(s)$  and theorem 2 we have that

$P = 4$ ,  $\mu = 1$ ,  $\lambda = 0$   $Q = 1$  and  $p(0) = 0$ .

Hence

$$I(m) \sim \frac{Q}{\mu} \Gamma\left(\frac{\lambda}{\mu}\right) \frac{e^{-\alpha p(a)}}{(P\alpha)^{\frac{\lambda}{\mu}}} = \Gamma(0). \quad (3.18)$$

The *Gamma function* originated as a solution of an interpolation problem for the factorial function, where we have

$$\Gamma(n+1) = n!. \quad (3.19)$$

From the notion of 3.19,  $n = -1$ , we have that the right hand side of (3.19) will be indeterminant. Thus Laplace's procedure collapses. Again from Euler's integral of second order,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

This requires that  $\text{Re} z > 0$  and in our case  $\text{Re} z = 0$  which also collapse.



# Conclusion

The two theorems discussed in this work have all collapsed or have failed to provide a solution to our problem. I will conclude that further investigation on the evaluation of this integral will be appropriate. I will live it in the indeterminate state and wish to develop my problem as my PHD starting point.





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