SINGLE PERIOD INVENTORY MODELS FOR PRODUCTS WHOSE DEMAND FOLLOWS COMPOUND DISTRIBUTIONS

BY

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DECLARATION

This dissertation is submitted for assessment of the Integrated Independent Project module of the degree of Master of Science in Mathematical Modelling at the University of Zimbabwe.

I certify that this is my own work.

Signed:

Date:

Abstract

Up to this point, the Single Period Inventory models developed have focussed on a product whose demand distribution in a single period is some simple classical distribution like the normal distribution, log-normal distribution, etc. These models are silent on the fact that the number of customers visiting a retail outlet to demand the product is a random variable with some probability distribution, typically the Poisson distribution (for this project). In this thesis, a model that assumes that the number of customers visiting a retail outlet during a single period is Poisson distributed is developed; each customer's demand for the product in turn is assumed to be a random variable X that follows some known distribution $\mathcal{D}(\underline{\theta})$, where $\underline{\theta}$ is the parametric vector for the distribution.

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Dedication

This project is dedicated to Nontando, Anold, Nolleen and Absalom (Jnr). I thank you for the love and support you gave me throughout the years.

Contents

1 Introduction **1**
1.1 Background of the Study 1
1.2 Motivation of the Study 4
1.3 Statement of the Problem 5
1.4 Aims of the Study 6
1.5 Layout of the Dissertation 7

2 Literature Review **8**
2.1 Review of Inventory Models 8
2.1.1 The Simple Economic Order Quantity Model 13
2.1.2 The Economic Production Quantity Model 15
2.1.3 Single-period Inventory Models 17
2.2 Multi-period Inventory Models 20
2.2.1 Multi-period Inventory Model with Backordering . . . 22
2.3 Sensitivity Analysis 24

3 Proposed Single-period Inventory Models **25**
3.1 Single-period Inventory Models 25
3.2 Customer Arrival Pattern 27
3.3 Single Period Demand Distribution 27
3.4 Confidence Interval for λ 29
3.4.1 Estimation of μ and σ^2 30

4 Methodology **31**
4.1 Simulation 31
4.2 Data for the project 32
4.3 Optimum Order Quantity, Q 33
4.4 Total Expected Cost, TEC 33

4.5 Sensitivity Analysis 34

5 Results **35**
5.1 Customer Arrivals 35
5.2 Customer Transactions 35
5.3 Optimum Order Quantity, Q^* 36
5.4 Total Expected Cost 37
5.5 Classical Single-period Inventory Model 38
5.6 Sensitivity Analysis 39
5.6.1 Variation in p_s 39
5.6.2 Variation in p_r 40
5.6.3 Variation in c 40
5.6.4 Variation in h_e 40

6 Summary, Conclusions and Recommendations **42**
6.1 Summary 42
6.2 Conclusions 43
6.2.1 Current developments 44
6.3 Recommendations 44

7 Appendices **48**
7.1 Appendix A 48
7.2 Appendix B 52

List of Tables

5.1 Total Expected Cost values 37
5.2 Variation of p_s 40
5.3 Variation of p_r 40
5.4 Variation of c 40
5.5 Variation of h_e 41

List of Figures

2.1 Variation of on-hand inventory with time 14
2.2 Production and Inventory phase 16
2.3 Variation of Inventory level with Time 21

Chapter 1

Introduction

1.1 Background of the Study

People use mathematical techniques or tools to solve some of their problems. These techniques, known as Quantitative Techniques, have been successfully applied in solving different types of problems in private and public sectors such as banks, government, health care, education, manufacturing and transport. Quantitative techniques help in establishing a satisfactory level of operational efficiency. The successful use of quantitative techniques result in solutions that are timely, accurate, flexible, economical, reliable and easy to understand.

The Quantitative Analysis approach starts with data which is manipulated or processed into information that is valuable to people making decisions. This has been easier and simpler with the advent of computers.

One area of quantitative methods that is fast evolving is a subject called Operations Research - which is broadly speaking, a subject where Mathematical and Statistical tools are harnessed to optimise business operations. Render [12], Taha [13] and Lapin [6] confirm this. In this subject, there is an area called Inventory Analysis and Control. In this area, many models aimed at determining inventory policies that result in the minimisation of inventory costs have been developed.

An inventory policy, by and large, is uniquely determined by two characteristics:

- when to order (i.e. to what level will have stocks been depleted when a requisition for fresh supplies is made), and
- what quantity to order each time an order is placed.

By varying these two quantities appropriately, huge cost savings can be made by a retailing establishment.

Definition 1.1 *The level to which inventory stock will have been depleted when a requisition for fresh supplies is made is called the* **re-order point**, r.

Definition 1.2 *The fixed quantity involved everytime an order is made is called the* **order quantity**, *Q.*

The two quntities defining an inventory policy, r and Q are in turn determined by the:

1. type of commodity concerned (whether a perishable or non-perishable).

2. demand for the product per given unit of time.

3. various inventory cost components such as the cost of placing orders, the cost involved in holding inventory stock over time, the procurement costs, the cost associated with shortages when customers come looking for the commodity when it is out of stock.

Definition 1.3 *The time duration between two successive orders for a commodity is called an* **inventory cycle** *or an* **inventory period**.

There are two classes of inventory models:

• **multi-period inventory models** - models that deal with commodities that can be held in stock for a number of inventory periods; this applies to commodities with extended shelf-lives, and

• **single-period inventory models** - models which deal with commodities whose shelf-lives are limited.

In the case of single-period inventory models, the shelf life of the commodity coincides with the inventory period. It is assumed that the commodity concerned is stocked at the beginning of the cycle. Should stocks run out mid-way through the cycle, then a shortage will occur in that cycle and if on the other hand there are left-over stocks at the end of the cycle, the excess stocks are disposed off appropriately. When we have such left-overs, the business will either incur an additional cost of disposal or will get something back if someone is willing to buy the excess stocks at a discounted price.

Remark 1.1 *The guiding principle behind determining an optimal inventory policy is the minimisation of inventory cost per unit time.*

1.2 Motivation of the Study

On average, any organisation that is in the retail business has about 40% of its invested capital as inventory thats according to Render [12]. Hence there is need to effectively manage inventory and reduce operational costs within inventory. Inventory management represents a vast segment of total economic activity such that a slight modification in an inventory policy can produce huge gains or losses in cost savings. The need to maximise the gains or/and minimise the cost influences organisations to embark on inventory policies with models that are *optimal*.

After doing the course-work for my MSc degree programme, I felt that I should do my thesis in something that would potentially benefit the business world - and hence this topic.

1.3 Statement of the Problem

One short-coming with single-period inventory models developed to date is that they do not adequately address the conformance of demand data to a distribution. Mainly, they are silent on the dependence of the overall demand on the number of customers that come to demand the inventory item during a given inventory cycle.

In this research project, it is assumed that the number of customers N who turn up to demand a given commodity is a random variable, typically Poisson distributed. Each customer demand is assumed to be a random variable X which follows some distribution $\mathcal{D}(\underline{\theta})$, where $\underline{\theta}$ is the associated parametric vector. Consequently, the demand for the commodity during a single period will be a compound distribution. The project seeks to determine the associated optimum inventory policy or more specifically, the economic order quantity and other important parameters.

1.4 Aims of the Study

In this research project, I hope to:

1. Investigate the nature of overall demand for an item per single period when the number of customers queuing for service is Poisson distributed and each customer demand for the product follows some known classical distribution (Normal distribution).

2. Determine the optimal inventory policy, i.e. an inventory policy that specifies the Economic Order Quantity, Q^* which results in the minimisation of the total expected inventory cost per single-period inventory cycle.

3. Carry out a sensitivity analysis to see how p_s, p_r, h_e, and c directly affect the optimal quantity, Q^*.

1.5 Layout of the Dissertation

Chapter One presents the introdutory material for this research thesis. Chapter Two presents the Literature Review while Chapter Three presents the derivation of the proposed model. In Chapter Four, the reader will find the methodology while Chapter Five presents the results obtained from some simulated data. Chapter Six contains the summary, conclusions and recommendations.

Chapter 2

Literature Review

2.1 Review of Inventory Models

Inventory policies are concerned with maintaining stocks of goods that ensures a smooth operation of a production system or a business activity usually at the minimum possible cost. An inventory policy is mainly characterised by:

- how much to order; and
- when to order?

The main assumption of inventory policies is that each time replenishment of stock takes place, a fixed quantity Q is ordered; this fixed quantity is called the **order quantity.**

Definition 2.1 *The optimal amount of stock to be ordered when an order is* *placed is called the* **Economic Order Quantity**.

Taha [13] said the times to order depends on the inventory system or policy.

Definition 2.2 *The level to which on-hand inventory will have dropped when a requisition for fresh supplies is triggered is called the* **re-order point**.

Inventory cost components mainly include:

- ordering costs
- procurement costs
- holding costs
- shortage costs.

Lapin [6] defines the above costs as follows:

Definition 2.3 Ordering Costs - *Ordering costs represent all expenses incurred in ordering.*

Remark 2.1 *Normally, the cost of placing a single order is not dependent on the order size.*

Ordering costs may arise due to:

i) Inspection and return of poor quality products

ii) Clerical or labour costs of processing orders

iii) Transport costs

iv) Handling costs

Definition 2.4 Holding or carrying costs - *Holding costs are expenses incurred during storage of items.*

Holding costs include physical costs such as operation of warehouse facilities, insurance costs, property taxes, labour, storage costs, theft, rent, depreciation, overheads (e.g. heating, lighting, and security), opportunity costs i.e. funds tied up in stock which would have been profitably invested elsewhere, obsolescence costs i.e. costs incurred when stock is leftover at the end of a cycle.

Holding costs depend largely on the quantity of items stored and length of storage.

Definition 2.5 Shortage costs - *Shortage costs occur when there is demand for items that are not currently in stock.*

The smallest shortage cost per each item short is that per unit profit that would have been earned had the commodity been in stock. Shortage costs may also increase due to the potential loss of customer goodwill that may be expected to increase in proportion to the length of delay in ordering fresh supplies.

There are two classes of inventory models:

- single-period inventory models, and
- multi-period inventory models.

Single-period inventory models are suitable for commodities with limited shelf-lives, i.e. perishables. At the end of the shelf-life (or single period), any left-over quantities are disposed of at a cost or alternatively, sold at a discounted price. Whether or not there will be a shortage or surplus at the end of a cycle, depends on the quantity demanded during the cycle and this demand is a random variable. The total demand over an inventory cycle is an accumulation of the individual demand values of the customers that may pitch up during the inventory cycle.

An additional complication is that the number of customers that will come asking for the commodity during a cycle may turn out to be a random variable as well.

Multi-period inventory models mainly deal with commodities that have an extended shelf-life. Items for such commodities can be stocked over several time periods and can be reordered with considerable flexibility.

Remark 2.2 *This research work deals mainly with single-period inventory models.*

The mathematical models derived will use the notation given below:

C_o - Fixed costs of placing an order

C_s - Fixed costs for setting a production run

C_h - Annual holding cost of the item in the inventory

C - Unit cost of procuring or producing an item

D - Annual demand of an item

Q_o - Economic order quantity for the model.

To have an in-depth understanding of the models developed in this thesis, a brief treatise of the elementary models is given first followed by more complicated models and then the models that I develop come later on.

2.1.1 The Simple Economic Order Quantity Model

The *Simple Economic Order Quantity* inventory model is the simplest type of inventory model where demand is assumed to be uniform over time and periodic replenishment is done when stocks just run dry so that no shortages are experienced. The model applies to organizations or service providers that buy items for resell or use as raw materials in their production lines or service. The assumptions of this model are:

1. Demand is known in advance and occurs at a uniform rate of D units per annum.
2. No stock shortages occur.
3. Holding costs vary with order size.
4. Instantaneous replenishment is expected as soon as stock level is zero.

An illustration of the variation of inventory level over time is shown in Figure 2.1.1.

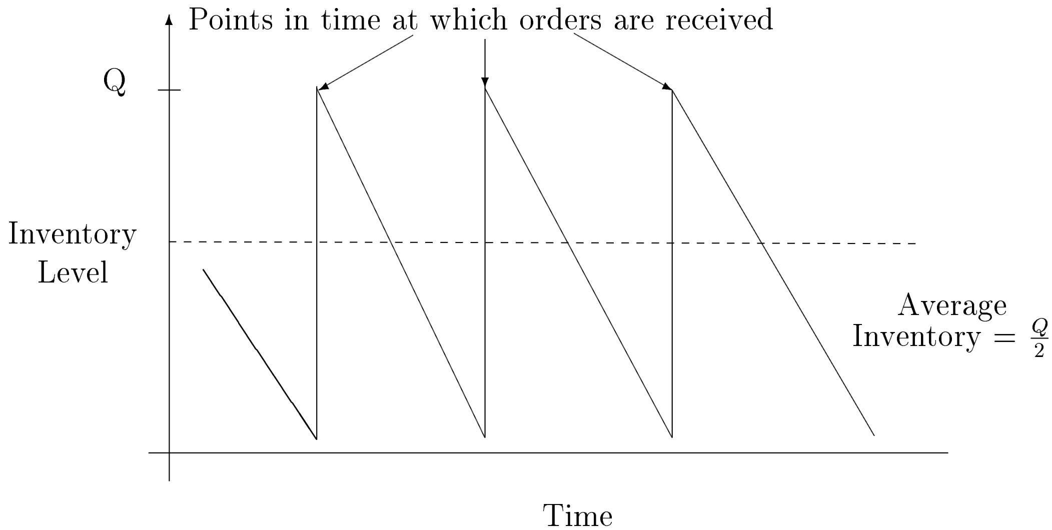


Figure 2.1: Variation of on-hand inventory with time

The Total Annual Cost, TAC(Q), of doing business in inventory depends on the order quantity, Q and comprise of three components, which are ordering, procurement and holding costs.

The Total Annual Inventory Cost, TAC(Q) is given by:

$$TAC(Q) = \frac{DC_o}{Q} + DC + \frac{QC_h}{2} \tag{2.1}$$

The Economic Order Quantity, Q_o obtained by minimising TAC(Q) in 2.1 is given by:

$$Q_o = \sqrt{\frac{2dC_o}{C_h}} \tag{2.2}$$

2.1.2 The Economic Production Quantity Model

The Economic Production Quantity model applies to the case where manufacturing and demand for a commodity can take place concurrently. The assumptions of this model are:

1. Demand for the commodity is uniform over time and the annual demand rate, D is known in advance.

2. The annual production rate (B units per annum) exceeds annual demand rate D thereby necessitating periodic shut downs in production.

3. In every production run, exactly Q units (called the production quantity) are produced.

The maximum inventory built-up is given by:

$$Y_{max} = \frac{Q(B - D)}{B} \tag{2.3}$$

The variation of the level of on-hand inventory with time is illustrated in the Figure below.

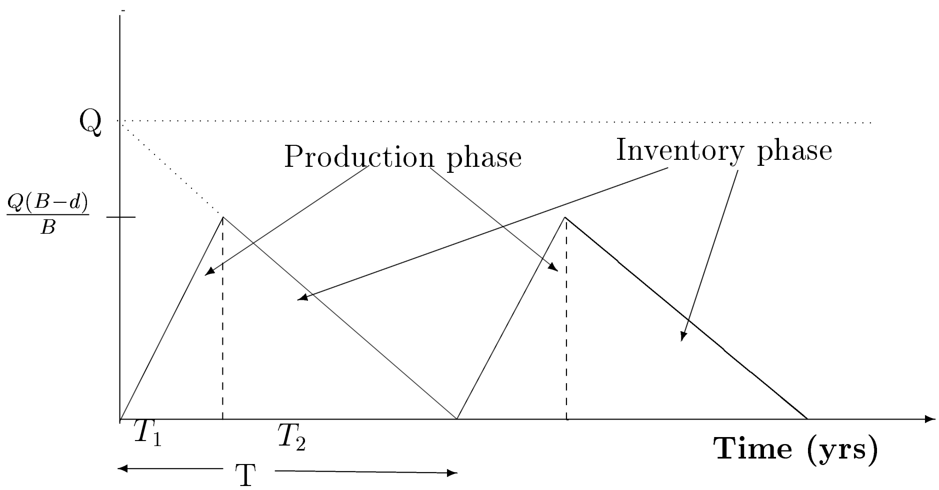


Figure 2.2: Production and Inventory phase

The Total Annual Inventory Cost, $TAC(Q)$ for this model is:

$$TAC(Q) = \frac{DC_s}{Q} + C_h \frac{Q(B-D)}{2B} + DC \tag{2.4}$$

Definition 2.6 *The Economic Production Quantity is the quantity that minimises total inventory costs per unit of time.*

The Economic Production Quantity, Q_o is given by:

$$Q_o = \sqrt{\frac{2BDC_s}{(B-D)C_h}} \tag{2.5}$$

2.1.3 Single-period Inventory Models

As mentioned ealier, Single-period inventory models apply when demand for a product is uncertain and the product has a limited shelf-life. Render, B et al [12] said single-period inventory models involve items whose shelf-life is limited or perishable products that may not be held in inventory to satisfy demand in another period. Examples of such commodities include include:

- periodicals such as daily newspaper and magazine whose effective life time is just one day or one month,
- fresh bread sold at a bakery or restaurant has a lifetime of a day,
- produce, i.e. fresh vegetables and fruits, to be sold in a grocery store,
- a certain brand of flowers sold by a florist which withers away within 3 days, say.

In developing a single-period inventory model, one notes the following:

- the objective is to minimize the expected inventory costs.
- there is no inventory stocks brought foward from the preceding inventory period - fresh stock is ordered at the beginning of the cycle.
- the demand in a given inventory cycle is a random variable with some known distribution $\mathcal{D}(\underline{\theta})$, where $\underline{\theta}$ is the associated parametric vector

for the distribution.

Remark 2.3 *The decision to be made concerns the order quantity or production quantity at the beginning of the inventory cycle or period.*

The following notation is employed:

c = Unit procurement cost

h_e = Unit holding cost at the end of an inventory cycle

Q = Quantity of the commodity to be stocked at the beginning of each cycle

D = a random variable representing the demand per cycle

p_r = selling price per unit

p_s = Penalty incurred for each item short

Remark 2.4 *At the beginning of each inventory cycle, it makes business sense for the entrepreneur stocks a fixed quantity, Q which is less than or equal to the maximum possible demand and greater than the minimum possible demand level per cycle i.e. if the minimum demand level is known to be d_{min} and the maximum demand level is known to be d_{max}, then $d_{min} \leq Q \leq d_{max}$.*

When demand per cycle for the commodity exceeds the quantity ordered at the beginning of the cycle, Q, a shortage for the commodity will occur and consequently, the entrepreneur will incur a shortage cost that is proportionate to the quantity short; on the other hand, if demand is less than the quantity ordered, an excess at the end of the cycle will be recorded and the entrepreneur will incur a holding cost. Demand, D can be a discrete or continuous random variable. Hillier, F [5] gives the result for Total Expected Cost per inventory cycle that applies when demand D is a continuos random variable with probability density $f_D(.)$. The total expected cost $TEC(Q)$:

$$TEC(Q) = c\mu + (h+c)\int_0^Q (Q-x)f_D(x)dx + (p_r+p_s-c)\int_Q^\infty (x-Q)f_D(x)dx \tag{2.6}$$

and the Economic Order Quantity, Q_o is such that:

$$Pr(D \leq Q_o) = \frac{p_r+p_s-c}{p_r+p_s+h} \tag{2.7}$$

If demand is a discrete random variable with probability mass function $f_D(x) = Pr(D = x)$ then,

$$TEC(Q) = c\mu + (h+c)\sum_{x<Q}(Q-x)P(D=x) + (p_r+p_s-c)\sum_{x>Q}(x-Q)P(D=x) \tag{2.8}$$

where $\mu = E[D]$.

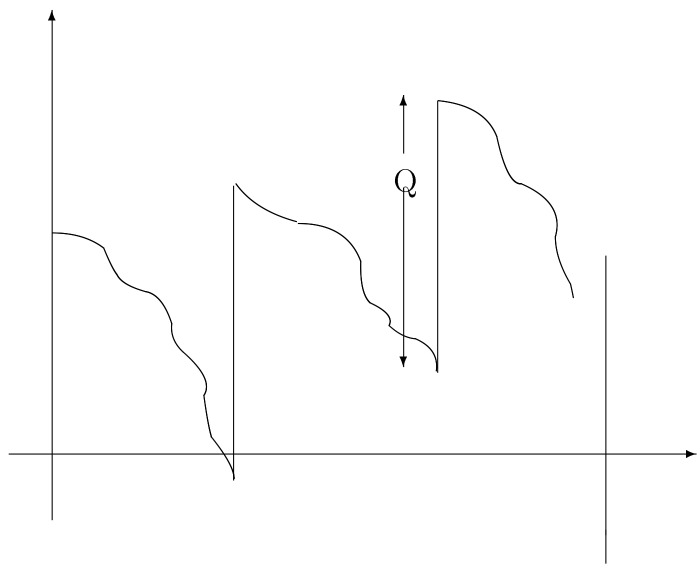
Remark 2.5 *Lapin, L. L. [6] says that the Economic Order Quantity formulae are obtained using marginal analysis whereby Q_o is value of Q that* *satisfies the two conditions that* $TEC(Q-1) \geq TEC(Q)$ *and* $TEC(Q+1) \geq TEC(Q)$. Q_o *is the smallest possible demand level such that:*

$$P(D \leq Q_o) \geq \frac{p_r + p_s - c}{p_r + p_s + h} \tag{2.9}$$

2.2 Multi-period Inventory Models

Multi-period inventory policy are applicable to commodities that may be stored over several time periods and that may be re-ordered with flexibility. In these models, there are two random variables: *demand* and *lead-time*. Render et al [12] state that multi-period inventory models fall within two categories: *continuous* and *periodic review models*.

A continuous review model assumes that a continuous monitorship of inventory position takes place so that decisions regarding replenishment are based on the current inventory level. This category of models give rise to (r, Q) policies where an order of size Q is placed whenever the current inventory position falls to or just below the reorder point, r. The typical variation of inventory with time is illustrated in the Figure below.


Q

For multi-period inventory models, should a shortage occur, the entrepreneur has the option to backorder goods to satisfy demand or may simply treat the shortage as a *lost sales case*.

Back-ordering occurs for items demanded in the preceding cycle that were out of stock. Any demand occuring during this time will be filled when outstanding orders arrive. A shortage penalty applies that includes the added cost of processing plus any cost assigned to the loss of customer goodwill.

2.2.1 Multi-period Inventory Model with Backordering

This model is appropriate whenever items are backordered. Each inventory cycle begins when an order arrives. Stock is then depleted until it falls below the reorder point, r and a new order is placed. Continued depletion occurs during the lead time until the order arrives. Any items short are then provided for in the incoming order. Letting μ be the expected lead time demand, the expected level of inventory just before the order arrives is $r-\mu$ so that the subsequent cycle begins with an expected inventory of $(Q+r-\mu)$ items where μ is the mean lead-time demand. The expected level of inventory per cycle, $E[Y]$:

$$E[Y] = \frac{1}{2}\left((Q + r - \mu) + r - \mu\right) = \frac{Q}{2} + r - \mu \tag{2.11}$$

so that

$$\text{The expected annual holding cost} = C_h \left(\frac{Q}{2} + r - \mu\right) \tag{2.12}$$

Items are backordered when lead-time demand exceeds the reorder point, r, hence the expected shortage quantity per inventory cycle, $B(r)$ is determined as:

$$B(r) = \sum_{d>r} (d - r) Pr_L(D = d) \tag{2.13}$$

where $Pr_L(D = d)$ is the probability of d units being demanded during the lead-time.

The expected annual shortage cost is determined by multiplying $B(r)$ by the expected number of cycles per year and penalty, p_s for each item short i.e.

$$\text{Expected annual shortage cost} = p_s \times \frac{A}{Q} \times B(r) \tag{2.14}$$

Using the above cost components, the expected total annual inventory cost is:

$$TEC(r, Q) = \left(\frac{A}{Q}\right) C_o + C_h \left(\frac{Q}{2} + r - \mu\right) + p_s \frac{A}{Q} B(r) \tag{2.15}$$

The economic order quantity, Q^* and economic re-order point, r^* are arrived at iteratively using the following algorithm:

Algorithm for finding Q^* and r^*

1. Begin by calculating $Q_o = \sqrt{\frac{2AC_o}{C_h}}$

2. For a given order quantity, Q the optimal reorder point r^* is the smallest quantity having a cumulative lead-time demand probability satisfying:

$$P_L(D \leq r^*) \geq 1 - \frac{hcQ}{p_s D} \tag{2.16}$$

3. For a given reorder point, r the optimal order quantity Q^* is given by

$$Q^* = \sqrt{\frac{2D(k + p_s B(r))}{hc}} \tag{2.17}$$

2.3 Sensitivity Analysis

Definition 2.7 Sensitivity Analysis - *This is an analysis of parameter changes and the impact on their optimal solution*

This is an analysis of the impact of cost elements on the TEC and EOQ. Sensivity Analysis tells how sensitive the optimal solution is to the variations in the value assumed for the parameter constant. The elements or variables that affect the TEC or EOQ are the penalty short, p_s, cost price, c, holding cost, h_e, and the unit selling price, p_r. Varying a variable and observing the changes in TEC or EOQ whilst the other variables are kept constant does an analysis of how that variable affects TEC or EOQ.

Chapter 3

Proposed Single-period Inventory Models

3.1 Single-period Inventory Models

As discussed earlier, at the beginning of each inventory cycle, the entrepreneur orders stock to be sold during the cycle. Current models assume that demand for the commodity concerned is a random variable following some known distribution $\mathcal{D}(.)$. During the cycle, if the demand for the commodity exceeds what has been ordered (i.e. if the inventory cycle demand is greater than the order quantity), a shortage will occur resulting in shortage costs being incurred. On the other hand, if demand is less than the order quantity, a surplus will be recorded at the end of the cycle resulting in *surplus costs or end of cycle inventory holding costs* being incurred. Too few items result in shortage costs which again increases inventory costs. The optimum order quantity will result in the minimisation of the total expected inventory costs per cycle and as such, it induces a compromise on shortage costs and holding costs.

As outlined in the Statement of the Problem section of Chapter 1, the demand probability distributions discussed in the inventory models developed to date are silent on the impact of the number of customers who turn up to demand the commodity. Of course, typically the number of customers who pitch up to demand the product during a given inventory cycle is a random variable with some distribution $\mathcal{D}_N(.)$. In this project, I assume that

1. the demand levels $X_1, \ldots, X_N$ of N customers who pitch up on a given day are independently and identically distributed random variables from a distribution $D_X(.)$.

2. if no customer pitches up, then demand is recorded as 0.

The resultant probability distribution of the overall demand per single period, thus, $\mathcal{D}(.)$ is a compound distribution derived from the distributions $\mathcal{D}_N(.)$ and $D_X(.)$.

Remark 3.1 *Two distributions are exhaustively discussed: N being Poisson distributed and $X \sim N(\mu, \sigma^2)$.*

3.2 Customer Arrival Pattern

The models discussed assume that the number of customers arriving per single period inventory cycle, N is a Poisson distributed random variable with mean λ, so that:

$$P(N = n) = \frac{e^{-\lambda}\lambda^n}{n!} I_{[0,1,2\ldots]}(n) \tag{3.1}$$

3.3 Single Period Demand Distribution

When the $X_i's$ follow a normal distribution with parameters μ and σ^2 and N is the number of customers in a single period inventory model, the distribution of the overall demand S is derived as follows noting that

$$S = \begin{cases} \sum_{i=1}^{N} X_i, N = 1, 2, \ldots \\ \\ 0, N = 0 \end{cases} \tag{3.2}$$

$$\begin{aligned}
F_S(s) &= P(S \leq s) \\
&= P\left[(S \leq s \cap N = 0) \cup (S \leq s \cap N = 1) \cup (S \leq s \cap N = 2) \cup \ldots\right] \\
&= P\left[S \leq s \cap N = 0\right] + \sum_{n=1}^{\infty} P(S \leq s \cap N = n) \\
&= P\left[S \leq s \cap N = 0\right] + \sum_{n=1}^{\infty} P(S \leq s | N = n) \times P(N = n) \\
&= e^{-\lambda} \cdot I_{(0,\infty)}(s) + \sum_{n=1}^{\infty} P\left(\frac{S - n\mu}{\sigma\sqrt{n}} \leq \frac{s - n\mu}{\sigma\sqrt{n}}\right) \times P(N = n) \\
&= e^{-\lambda} \cdot I_{(0,\infty)}(s) + \sum_{n=1}^{\infty} \Phi\left(\frac{s - n\mu}{\sigma\sqrt{n}}\right) \times \left(\frac{e^{-\lambda}\lambda^n}{n!}\right)
\end{aligned}$$

$$\text{Thus,} \quad f_S(s) = \frac{d}{ds}F_S(s)$$
$$= \sum_{n=1}^{\infty} \frac{1}{\sigma\sqrt{n}}\Phi'(\frac{s-n\mu}{\sigma\sqrt{n}}) \times \frac{e^{-\lambda}\lambda^n}{n!}$$
$$= \sum_{n=1}^{\infty} \frac{1}{\sigma\sqrt{n}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{s-n\mu}{\sigma\sqrt{n}}\right)^2} \times \frac{e^{-\lambda}\lambda^n}{n!}$$

Lemma 3.1 $E[S] = \lambda\mu$ *and* $var(S) = \lambda(\mu^2 + \sigma^2)$.

Proof:

(1)
$$E[S] = \int_{-\infty}^{\infty} s f_S(s) ds$$
$$= \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{1}{\sigma\sqrt{n}}\frac{e^{-\lambda}\lambda^n}{n!}\int_{-\infty}^{\infty} se^{-\frac{1}{2}\left(\frac{s-n\mu}{\sigma\sqrt{n}}\right)^2}ds$$
$$= \sum_{n=1}^{\infty}\frac{e^{-\lambda}\lambda^n}{n!}(n\mu)$$
$$= \sum_{n=1}^{\infty}\frac{e^{-\lambda}\lambda^n\mu}{(n-1)!}$$
$$= \lambda\mu e^{-\lambda}\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(n-1)!}$$
$$= \lambda\mu.$$

and

(2)
$$E[S^2] = \int_{-\infty}^{\infty} s^2 f_S(s) ds$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{1}{\sigma\sqrt{n}} \frac{e^{-\lambda}\lambda^n}{n!} \int_{-\infty}^{\infty} s^2 e^{-\frac{1}{2}\left(\frac{s-n\mu}{\sigma\sqrt{n}}\right)^2} ds \\
&= \sum_{n=1}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} \left(n\sigma^2 + (n\mu)^2\right) \\
&= \lambda^2\mu^2 + \lambda\mu^2 + \lambda\sigma^2
\end{aligned}
$$

Thus, $var(S) = E[S^2] - E^2[S] = \lambda(\sigma^2 + \mu^2)$.

Proposition 3.1 *The optimum order quantity for this model, Q^* is a solution of the equation*

$$
e^{-\lambda} + e^{-\lambda} \sum_{n=1}^{\infty} \Phi\left(\frac{Q^* - n\mu}{\sigma\sqrt{n}}\right) \times \left(\frac{\lambda^n}{n!}\right) = \frac{p_r + p_s - c}{p_r + p_s + h_e} \qquad (3.3)
$$

3.4 Confidence Interval for λ

The point estimator for λ that is used is the Uniformly Minimum Variance Unbiased estimator, $\bar{N}$. If we invoke the Central Limit Theop15Tf245.6390Td[()]TJ/F1511.955Tf-

3.4.1 Estimation of μ and σ^2

The following unbiased estimators of μ and σ^2, respectively, are used in drawing comparisons:

$$\hat{\mu} = \overline{\overline{X}} = \frac{1}{\sum_{i=1}^{m} n_i} \sum_{i=1}^{m} \sum_{j=1}^{n_i} X_{ij}, i = 1, \ldots, m; \quad j = 1, \ldots, n_i \quad and \tag{3.4}$$

$$\widehat{\sigma^2} = S^2_{pooled} = \frac{\sum_{i=1}^{k} (n_i - 1) S_i^2}{\sum_{i=1}^{k} n_i - k} \tag{3.5}$$

Chapter 4

Methodology

4.1 Simulation

According to Morgan B. J, simulation is a procedure used to find numerical optimal solutions to problems through trial-and-error. This technique may be applied to any decision problem that involves uncertainty. In solving problems for this research, simulation has been used.

Simulation evaluates each alternative by generating a series of values for each random variable at the frequency indicated by the probability distributions. The resulting quantities are combined in accordance with an underlying mathematical model to provide a particular value. After a number of repititive calculations, a statistical pattern in the results can be seen.

4.2 Data for the project

The data used in this project was generated using S-PLUS, statistical package. In generating the data, the assumption taken was, the mean number of customers served per single period (per day for this project), λ is 50, and the number of customers served on the i^{th} day (for each of the 50 days) was generated in S-PLUS.

49	51	47	51	54
56	51	44	61	49
48	59	52	48	40
49	44	53	47	60
55	52	51	51	55
55	48	55	48	48
51	50	53	59	58
55	58	57	31	47
44	41	60	51	53
61	54	45	52	42

The table above shows the data generated, which indicates the number of customers served in a single day for the fifty days. From this data set, fifty samples, each of size $n_i, i = 1, 2, ...50$ were then generated. n_i is the number of customers that received service on the i^{th} day e.g n_1 is 49 which is the number of customers who received service on the first day.

Appendix B shows the data generated for fifty days and these were stored in 50 columns in a spreadsheet. This is the data that was used to test the model developed in Chapter 3.

4.3 Optimum Order Quantity, Q

To determine the Optimum Order Quantity, Q the first program in Appendix A gave the result. It is an iterative method that assumes an initial value of Q and then locates the optimal value Q^* depending on whether it is less than or more than an initial value.

4.4 Total Expected Cost, TEC

The Total Expected Cost, TEC was computed using the second program in Appendix A. In addition to the parameters defined in Chapter 2, p_s, p_r, h_e and c, other parameters were defined and used in the program and these are expected mean, expected shortage, expected surplus, demand and the optimal order quantity denoted by *xpmean, xpshortsm, xpsurplsm, d* and Q, respectively. The program also iterates to calculate expected surplus or shortage costs and then calculate the Total Expected Cost.

4.5 Sensitivity Analysis

A sensitivity analysis of the parameters p_r, p_s, h_e and c was done using the first S-PLUS program in Appendix A. The program iterates with different values of a chosen parameter whilst other parameters are constant and comes up with optimal values of Q^*. These values are tabulated and a conclusion made on how the parameter varies with Q^*.

Chapter 5

Results

5.1 Customer Arrivals

The simulated number of customers that qeue for service on each of the fifty days are presented in Appendix B. From the data, the first observation indicates that 49 customers were served on the first day.

5.2 Customer Transactions

The data was generated using the Normal distribution using the parameters $\mu = 20000$ and $\sigma = 4000$. Using these parameters, the mean and standard deviation were calculated and found to be $\hat{\mu} = 19931.092$ and $\hat{\sigma} = 3961.552$, respectively.

5.3 Optimum Order Quantity, Q^*

The following hypothesised cost components values were used basing on the suppositions that:

1. The cost in terms of lost goodwill as a result of a shortage of money to despatch to customers, $p_s = \$0.3$ i.e. the cost per dollar short is 30 cents.

2. Revenue generated from every dollar served to a customer, $p_r = \$0.05$, i.e. by serving a customer a dollar the bank earns 5 cents.

3. The operational costs of processing a withdrawal, $c = \$0.2$, i.e. for every dollar dispatched to a customer, the bank incurs a cost of 20 cents.

4. The cost of holding a dollar at the end of an inventory cycle, $h_e = \$0.1$. These holding costs arise due to money having to be safely kept in a safe overnight.

A model developed in Chapter 3 that determines the optimal order quantity, Q^* is:

$$e^{-\lambda} + e^{-\lambda} \sum_{n=1}^{\infty} \Phi\left(\frac{Q^* - n\mu}{\sigma\sqrt{n}}\right) \times \left(\frac{\lambda^n}{n!}\right) = \frac{p_r + p_s - c}{p_r + p_s + h_e}. \tag{5.1}$$

An S-PLUS program in Appendix A was used with the following parameters defined as λ as 51.06, μ as 19931.092 and σ as 3961.552. On running this first program, with the above parameters, the optimal order quantity, Q^* was found to be **\$48 669 078**.

5.4 Total Expected Cost

In determining the Total Expected Cost, TEC(Q^*), a computer program was written in S-PLUS. The model developed that determines the TEC is:

$$TEC(Q) = c\mu + (h+c)\int_0^Q (Q-D)f(D)dD + (p_r + p_s - c)\int_Q^\infty (D-Q)f(D)dD \tag{5.2}$$

This model was used to develop second program shown in Appendix A. On running the program with different values of Q, different values of TEC were obtained as shown in the table below.

Q	46990000	47500000	47750000	48250000	48500000
TEC	30755300	7575800	1913300	957050	3788300

Table 5.1: Total Expected Cost values

Values in the above table indicates that TEC decreases when the order quantities, Q are below the Optimal Order Quantity, Q^* of 48 669 078 and this is attributed by shortage costs. For quantities above the optimal order quantity (the last value in the table), TEC increases because of the storage costs.

5.5 Classical Single-period Inventory Model

The data in Appendix B was generated using the data in Section 5.3. Using these parameters, and the expression below, which is equation 2.7, and noting that data is Normally distributed:

$$P(d \leq Q^*) = \frac{p_r + p_s - c}{p_r + p_s + h} \tag{5.3}$$

To determine Q^*,

$$\begin{aligned} \frac{p_r + p_s - c}{p_r + p_s + h} &= \frac{0.05 + 0.3 - 0.2}{0.05 + 0.3 + 0.1} \\ \frac{p_r + p_s - c}{p_r + p_s + h} &= 0.333 \end{aligned} \tag{5.4}$$

Hence Q^*, is found as:

$$\begin{aligned} P(d \leq Q^*) &= 0.333 \\ P\left(\frac{d - n\mu_1\lambda}{\sqrt{n\lambda(\mu_1^2 + \sigma_1^2)}} < \frac{Q^* - (50)(19931.09)(51.06)}{\sqrt{(50)(51.06)(3961.552^2 + 19931.09^2)}}\right) &= 0.333 \end{aligned}$$

Where $n = 50$ days generated using the Poisson distribution, $\lambda\mu$ is the mean and $\lambda(\sigma^2 + \mu^2)$ is the variance for the demand distribution, S. Thus, Q^* will be given by

$$\begin{aligned} \frac{Q^* - 50884077.88}{1026762.752} &= \Phi^{-1}(0.333) \\ \frac{Q^* - 50884077.88}{1026762.752} &= -0.44 \\ Q^* &= 1026762.752(-0.44) + 50884077.88 \\ Q^* &= \mathbf{50432302.26} \end{aligned} \tag{5.5}$$

Remark 5.1 *One can argue that the value \$48 669 078 given by the model developed here is approximately the same as the value of \$50 432 302.26 given by the classical model. A difference of \$1 763 224.26 (which is approximately 3.5%) is not much.*

The Classical Single-period Inventory model gave us a value similar to the one got from the model developed indicating that we can confidently use the developed model with demand which is Normally distributed.

5.6 Sensitivity Analysis

This is an analysis of how the parameters used in the model varies with the optimal value, Q^*. Using the first program, one parameter was chosen and varied whilst others were kept constant and the changes in the value of Q^* observed. Tables below indicate the results obtained for the four parameters, p_s, p_r, c and h_e.

5.6.1 Variation in p_s

The table indicates that p_s, the loss of goodwill varies directly with Q^*, the optimal order quantity, i.e. an increase in p_s results in an increase in the value of Q^*.

Chapter 6

Summary, Conclusions and Recommendations

In this project, I presented an Inventory Model that addresses the dependence of overall demand of a commodity per cycle on:

i) The number of customers that queue up to receive the commodity per cycle and,

ii) The distribution of demand for the commodity by individual customers.

6.1 Summary

Models for overall demand for customers following Normal distribution was established and was tested with simulated data.

Using this data, computer programs were developed to calculate Economic

Order Quantity and Total Expected Cost using the proposed models developed.

A sensitive analysis on the parameters used to determine the optimal order quantity was done using a computer program.

6.2 Conclusions

The three aims for this projects were met since computer programs for calculating the optimal order quantity, Q^* and Total Expected Cost, TEC were developed and models for overall demand for compound distributions was also established. A sensitivity analysis of the parameters p_r, p_s, c and h_e was done.

The results found for the Optimum Order Quantity and Total Expected Cost from the computer program developed are very encouraging especially in the view of the fact that the Optimal Order Quantity value obtained from the model developed is close to that one found using the classical model. I can confidently recommend anyone wanting to use the computer programme developed in calculating optimum order quantity for data that follows a compound distribution (Poisson and Normal distributions) to use the one developed in this project.

From the sensitive analysis that was done, results indicated that p_s and p_r are directly proportional to the optimal order quantity whilst c and h_e are inversely proportional to the optimal order quantity.

6.2.1 Current developments

I am also convinced that a lot is being done in the area of Inventory Management since this area plays a crucial role in Organisations. Recent research tells me that the Just In Time (JIT) approach has been established to reduce to costs associated with inventory. This is an approach where a commodity or product is produced or obtained at the instant they are required. Also and approach on Material Requirement Planning (MRP) was developed to reduce inventory costs. This is where demand for components or sub-components depends on the demand for the final product, hence the dependence affect production schedules and controls inventory.

6.3 Recommendations

In this project, I explored and came up with a model for the case of Normal distribution being the underlying individual customer demand distribution.

I managed to develop computer programs for calculating the Optimal order quantity and Total expected cost for the developed model. I recommend that any one interested in this area can also look at other probability distributions like the:

i) Modified Beta distribution

ii) Lognormal distribution

iii) Pareto distribution

iv) Truncated Normal distribution

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Chapter 7

Appendices

7.1 Appendix A

A computer program to find Optimal Order Quantity

lambda_51.06

sigma_3961.552

mu_19931.092

pr_0.05

ps_0.3

c_0.2

he_0.1

differ_0.2

Q_lambda*mu*50

e_exp(-lambda)

rho_$((pr + ps - c)/(pr + ps + he))$

sigma_$sqrt(lambda * (mu^2 + sigma^2))$

mu_lambda*mu

```
while(differ > 0.001)
{
    for (i in 1:160)
    {
    sm1[i]_pnorm((Q-i*mu)/(sigma*sqrt(i)))*(e*(lambda^i/gamma(i+1)))
    }
    sm_sum(sm1)+e
    differ_(sm-rho)
    Q_Q-5000
}
Q
rho
sm
differ
```

A computer program to find Total Expected Cost

```
mu_20000
sigma_4000
lambda_51.06
xpmean_4000
Q_48000000
demand_47990000
mean1_mu * lambda
std1_sqrt(lambda * (mu^2 + sigma^2))
pr_0.3
ps_0.05
c_0.2
he_0.1
xpshortsm_0
xpsurplsm_0
surpl_Q-demand
while (surpl > 0)
{
    xpsurpl_(Q - demand) * (pnorm((demand - mean1)/std1))
    xpsurplsm_xpsurplsm+xpsurpl
    demand_demand+5000
    surpl_(Q-demand)
}
xpsurplsm
short_demand-Q
```

while ($short > 0$)

{

xpshort_$(demand - Q) * (pnorm((demand - mean1)/std1))$

xpshortsm_xpshortsm+xpshort

Q_Q+5000

short_(demand-Q)

}

xpshortsm

xpsurplc_$(he + c)*$xpsurplsm

xpshortc_$(ps + pr - c)*$xpshortsm

procc_$(c * xpmean)$

tec_procc+xpshortc+xpsurplc

tec

Q

7.2 Appendix B