Undergraduate student teachers' conceptualisations of mathematical proof

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Abstract

Students face serious challenges in learning mathematical proofs. Although many studies have been done with the aim of improving the learning of mathematical proof beyond mere regurgitation of memorised facts, very few studies have been based on students' actual proof attempts. Motivated by the need to develop an understanding of students' thinking grounded in their actual proof attempts the main research question put forward was: In what terms do Zimbabwean undergraduate student teachers think of mathematical proof? The goal was to explore students' schemes of argumentation and how students' thoughts around mathematical proof evolve.

A case study approach guided by the scientific realist philosophy was applied in the context of a teaching experiment that involved 10 undergraduate mathematics education students, 6 female and 4 male. The student teachers involved in the study had enrolled for the Bachelor of Education Degree in Mathematics. Three tools were used to elicit data: written proof tasks, reflective interviews and think aloud interview protocols. Directed and summative content analysis techniques in which theoretical constructs about proof learning such as the notion of technical handles and conceptual insights, syntactic and semantic modes of argumentation and ideas drawn from Harel and Sowder's taxonomy of proof schemes were applied to written responses, the transcription texts of audio and video recorded interview data for the purpose of inferring the kinds of proof schemes held by the students and how students' proof schemes emerge.

The study revealed that undergraduate student teachers conceptualise mathematical proof in terms of logical steps and procedures in which axioms and definitions are handled in order to validate mathematical propositions. The manner in which the axioms and definitions are handled has revealed the dominance of low cognitive level proof schemes such as the external conviction authoritative and symbolic proof schemes. Trajectories of proof schemes illuminated lateral shifts which indicate lack of growth in the student teachers' schemes of argumentation. Trajectories of proof schemes also revealed scenarios whereby proof scheme states regressed from higher to lower cognitive level proof scheme states during the transition from pre-university to undergraduate mathematics learning. The decrease in cognitive level just described was accompanied by loss of meaning construction as the student teachers engaged with the concept of proof at undergraduate level.

The kinds of proof schemes held by the undergraduate students and the proof scheme trajectories that emerged were manifested through students' proof behaviours which include: contradictory proof behaviour shown through use of axiomatic argumentation in tasks that require proof method by refutation and conversely through use of specific instantiations in deductive proof tasks, oscillatory switches in students' proof behaviour that revealed some instability and fragility in the higher level axiomatic proof scheme in which students were forced to slide down the proof scheme ladder to lower level empirical proof scheme.

While, some findings from this study are consistent with existing literature on mathematical proof, the instability and fragility in higher proof scheme states, the inconsistent formal rhetoric aspects, lateral and vertical downward shifts in students' proof scheme trajectories are new observations that might inform the learning of mathematical proof. Hence, the study has developed an explanatory theory that accounts for the kinds of proof schemes held by the student teachers as well how student teachers' proof schemes evolve. In this regard the study has uncovered the existence of unstable axiomatic proof scheme states that compel students to switch to lower level cognitive proof scheme states such as the external conviction symbolic and empirical-numeric proof schemes. The study has implications for teacher preparation in terms the need to foster and sustain high proof schemes states among student teachers. The process of conjecturing about mathematical statements has been identified by the study as a loose end for further studies.

Dedication

I dedicate this work to my dear wife Getrude, my three sons: Tinashe, Takudzwa, and Tinomudaishe and my daughters Tendai and Tanyaradzwa. To Getrude, I say you have always been a constant source of love and support in my accomplishments— the wind beneath my wings during my scholarly journey. To my daughters, Tendai and Tanyaradzwa I say gender should not be an impediment to your aspirations to reach the same academic level or even go beyond. To my three sons, I challenge you all to aim higher.

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Let me clarify that the people and institutions mentioned made insightful suggestions to improve the process and product of this study. I remain responsible for any errors of interpretation or omissions that might be noticed in this study. Opinions expressed in this thesis are those of the author and do not necessarily reflect the opinions of the people and institutions that I have mentioned.

Preface

Chapter 1 identifies the research problem of the study that concerns the student teachers' superficial understanding of mathematical proof. This study is an attempt to understand the student teachers' thinking around the notion of mathematical proof based on the students' own actual *voices*, that is, their own proof productions.

Chapters 2 and 3 survey literature on mathematical proof including the taxonomies of proof schemes available and notions embedded in proof construction processes such as the constructs of manipulating, getting a sense-of-articulating (MGA), conceptual insights and technical handles, the distinctions between syntactic and semantic approaches to proving. Features of the Realist process are discussed and contextualised in a way that led to the conceptual framework. Related studies on student teachers' learning of mathematical proof are examined. Ideas discussed in chapters 2 and 3 then influenced Chapter 4 on research methodology. Chapter 4 discusses how a collective instrumental case study was employed in the context of a teaching experiment to design research instruments and develop a Realist Analytic Framework on the basis of ideas drawn from the Conceptual Framework from Chapter 2.

The Realist Analytic Framework was then applied to qualitative textual data using directed and summative content analysis techniques draw meaning about the terms in which the undergraduate mathematics education students think of mathematical proof. The presentation and discussion of results in Chapters 5 and 6 led to an overall conclusion about the students' schemes of argumentation and how students' thoughts about mathematical proof evolve. An attempt to account for conclusions to sub-research questions and the overall conclusion led to the formulation of new observations made about the student teachers' thinking about mathematical proof which stated in Chapter 6. Recommendations from the study from theoretical, teaching, and methodological perspectives are discussed. Finally, suggested areas for further research are discussed and the thesis ends with my personal reflections of the entire scholarly journey.

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Chapter One
The Research Problem

1.1 Introduction

Fundamental ideas driving mathematics education currently are based on the overall view of long term human learning. Hence, the ideas of this thesis are grounded in the belief of life-long learning of mathematics concepts. In Zimbabwe the mathematics education community faces the challenge of improving students' ability to autonomously produce proofs of mathematical statements at all scholastic levels. Generally, students focus on reproducing proofs, yet mathematics learning requires, far more than simply working on exercises by doing desired computations and regurgitation of routine proofs. The emphasis on routine proofs was also a common feature in my personal experience with undergraduate mathematics where, in most cases, the major thrust was on rote memorisation of instructor's notes. Further, there was a tendency to disregard the meaning of logico-mathematical statements and lack of appreciation and understanding in our learning of mathematical proof. For example, I never really understood why and when to use proof by contradiction or proof by contrapositive. Yet we were able to regurgitate proofs involving use of the contrapositive of mathematical assertions such as: If $f: X \to Y$ is continuous and A is a compact subset of X then $f(A)$ is a compact subset of Y.

Some research studies have indicated that some students have been found to lack the intellectual curiosity to wonder why given mathematical propositions are true (Harel & Sowder, 1998, p. 236). The lack of intellectual curiosity stems probably from learners' view of the role of proof as a tool needed to confirm something that is intuitively obvious and already known to be true (Schoenfeld, 1985). An exemplification of this point is the proof of the theorem: *The square root of 2 is irrational.* Before the truth seeking activity (proving process), students are well aware of the fact that the square root of 2 is irrational. The source such of awareness among learners can be explained in terms of their met-befores, which precisely refer to the learners' previous experiences with irrational numbers (Tall, 2008, p. 6). In secondary school mathematics students would have looked at topics that involve the use of irrational numbers such as: Quadratic equations with inexact roots. Thus, the proposition: *The square root of 2 is irrational*, is perceived as something that is intuitively obvious in the sense suggested by Harel and Sowder (1998, p. 236), and, hence, the lack of intellectual curiosity in the proposition's validation. Such a viewpoint is a consequence of the manner in which mathematics is usually presented to students— as a finished product. Closely linked to this is the idea that students expect to be told the proof rather than take part in the proof construction process. Such students have often exhibited lack of concern and appreciation for meaning of mathematical proof, as their perception of it is that it is a collection of truths that requires no intrinsic justification for its origin (Harel & Sowder, 1998, p. 236). Other research studies have pointed to a lack of fundamental grasp of the distinction between empirical arguments and mathematical proof as well as a tendency to accept a few examples as evidence of truth of a mathematical proposition (CadawalladerOlsker, 2011, p. 44; Stylianides, 2011, p. 11). Empirical verifications are mathematical argumentations in which mathematical statements are quantitatively evaluated in one or more particular instances, that is, a mathematical assertion is purportedly validated by finding numerical values in a proper subset of all its possible cases (Harel & Sowder, 1998; Stylianides, 2011, p. 1; Weber & Mejia-Ramos, 2015). Such arguments should not be elevated to the status of a proof because a mathematical statement may hold for numerous cases but may fail to hold for at least one case of all cases in its domain. This idea embodies the notion of a counter example in mathematics.

Jones (1997) points out that even the most qualified pre-service teachers may not necessarily have the specific kind of subject matter knowledge for promoting students' proving abilities. Jahnke (2007) remarks "many school and university students and even teachers of mathematics have only superficial ideas on the nature of proof" (p. 80). Yet, the knowledge of teachers related to proof and proving directly influences their ways of teaching proof. Limited subject knowledge on proofs will allow misconceptions in many students regarding proofs to persist (Uǧurel, Morali, Yiǧit & Karahan, 2016, p. 205). Thus, if undergraduate mathematics education students do not master the proving activity adequately, they are less likely later to propose it to their own learners in a persuasive manner. Undergraduate student teachers should have a deep understanding of mathematical proof and proving. Hence, there is need to put stress on proof construction on the student teachers during their educational training period. Therefore studies to determine competences of teacher content knowledge regarding proving are crucial.

An essential and powerful component involved in learning mathematical proving is the use of language. Language enables learners to formulate and communicate conjectures, describe, refine and deduce relationships and focus on important ideas about mathematical proofs (Tall, 2008, p. 6). However, the use of language in this core area of mathematics has not been smooth. Research has revealed that the interpretability of logico-mathematical terms and notions impedes students' learning of mathematical proof (Lee & Smith, 2009, p. 21). For instance, students at all grade levels have been found to have difficulties in negating statements when proving conjectures by contradiction and that only the ablest students have been able to make successful attempts at such indirect proofs (Antonini, Presmeg, Marriotti, & Zaslavsky, 2011; Harel & Sowder, 1998 , p .136; Morselli, 2006, p.185).

During the proving activity one has to eliminate one's doubts about the truth/falsity a mathematical claim (ascertaining) and, in addition, he or she should convince (persuade) others about the truth value of the proposition. This is what is referred to as one's proof scheme. Thus, an individual's proof scheme "consists of what constitutes ascertaining and persuading for that person" (Harel & Sowder, 1998, p. 244). A mathematical proof's potential to promote argumentation skills and understanding accounts for the central place of proof in the learning of mathematics (Stylianides, 2011, p.1). Thus, learning to argue about mathematical ideas is fundamental to truly understanding mathematics. However, a number of studies have shown the absence of a central understanding of proof and proving in pre-service teachers' proof schemes (Durand-Guerrier, 2003; Stylianides, 2011, p. 2). Yet effective teachers need to understand the mathematics they teach at a deep level (Varghese, 2009, p. 2). Hence, there is need for more studies that explore students' mental constructs (understandings) around the notion of mathematical proof.

This study aims to contribute to studies that focus on terms in which students think about the notion of mathematical proof, henceforth referred to as ontological commitments. The guiding theoretical principle is based on the view of mathematics as a problem solving activity (Schoenfeld, 1985), drawing ideas from several taxonomies of proof schemes as well as the mathematical underpinnings of the notion of mathematical proof. A collective instrumental case study research design in which the concept of a proof scheme is the unit of analysis employs a realist process approach and key notions about mathematical learning such as proof event, technical handles and conceptual insights (Maxwell, 2004; Moore, 1994; Raman, 2003; Sandefur, Mason, Stylianides, & Watson, 2013) to investigate undergraduate student teachers' thinking processes with respect to mathematical proving. The intent is to get insights into the kinds of mental constructs around the concept of mathematical proof among undergraduate students in mathematics education.

1.2 Statement of the problem

The current study focuses on knowledge of content and students (KCS) which Lesseig (2016) describes as knowledge of students' typical conceptions or misconceptions of mathematics. Mathematical knowledge develops and matures as students engage with proofs. Therefore mathematical proofs play an active role in generating mathematical knowledge and promoting and fostering thinking among learners (Stylianides, Stylianides & Phillipou, 2007 cited in Doruk & Kaplan, 2015). However, research has established that undergraduate students have a fragile understanding of mathematical proof (Yang, 2010). Pertinent questions in light of students' weak command of the concept include: why are student teachers so inept at producing deductive arguments? Why does mathematical proof fail to permeate the undergraduate mathematics curriculum? The focus of the current study, is therefore, on how students' thinking around the concept of mathematical proof can be apprehended. For the purposes of avoiding ambiguity, before elaborating on the research problem I begin by defining some key terms embedded in the concept of mathematical proof: thinking, justification/argumentation, mathematical proof and proving.

First, a mathematical proof is a socially sanctioned written product that results from mathematicians' attempts to justify whether a given conjecture is true (Weber & Mejia-Ramos, 2011, p. 330). Second, mathematical proving is the process of searching for arguments used to convince a person (community) about the accuracy of mathematical assertion (Bieda, 2010). Third, a justification or argumentation is the process of constructing an explanation or evaluating evidence used to validate a mathematical claim (Bostic, 2016; Jonassen & Kim, 2010). Finally, I adopt Jonassen and Kim's (2010) definition of mathematical thinking as "a form of formulating and weighting the arguments for or against a course of action, a point of view, or a solution to a problem" (p. 40). Following next are comments on these definitions as well as efforts to put the research problem into perspective.

The term "socially sanctioned" captured in the definition of a mathematical proof conveys the meaning that while proof can promote some form of understanding as one tries to convince oneself and others that a statement is true (or not true) the product of the proving effort is evaluated on the basis of proofs produced by a research mathematician who plays the role of an arbiter (Stylianides, 2007 in Bostic, 2016). To develop understanding of mathematical assertions students should engage in justification rather than assuming truth of mathematical assertions. Justification is central to the learning of mathematical proof (Bieda, 2010). Justification is defined as an act of convincing someone that a statement is valid. There are two types of justification in mathematics. Balacheff (1988) coined pragmatic and conceptual justifications as two modes of justifications prevalently used by students. Pragmatic justifications are based on examples (particular instantiations) while conceptual justifications are based on abstract formulations of properties and of relationships among pertinent mathematical ideas to the proof one wishes to construct. Students' efforts to justify mathematical statements yield exploratory arguments that have certain characteristics. These characteristics give rise to the concept of a proof scheme that is considered in the next section.

Harel and Rabin (2010) suggests that the concept of a proof scheme is based on the idea of proving which is defined as the mental act that a person or a community employs to get rid of doubts about the truth of a mathematical assertion. Harel and Rabin define a mathematical community as consisting of a particular setting together with people involved in that setting along with their expectations of what constitutes a mathematical proof. Examples of mathematical communities include journal publishers, an examining body or informal settings such as a conversation between two mathematicians in the same area of mathematical specialisation. Hence, proving can be seen as the process of constructing a sequence of assertions that supports a mathematical claim or that leads to the rejection of the mathematical claim. A mathematical proof can in this sense be conceived as a product that results from mathematicians' attempts to establish the truth-value of a mathematical claim (Weber & Mejia-Ramos, 2011). Hence, proof construction compares quite considerably with problem solving that requires that a manifold of mathematical resources come to mind at the right time.

Following Harel and Rabin (2010) a mathematical proof can be described as a particular argument one produces to ascertain for oneself and convince others about the truth-value of a mathematical proposition. The argument(s) produced to support mathematical claims by an individual can reveal some characteristics. Persistent characteristics of the proofs produced by an individual constitute what is called a proof scheme (Harel & Rabin, 2010). In this sense, proving is not confined to axiomatic proving (Goethe & Friend, 2010) but encompasses students' enculturation into socio-mathematical norms of how correctness is determined in mathematics (Oflaz, Bulut, & Akcakin, 2016, p. 134; Yackel & Cobb, 1996). With this conception of mathematical proof we describe a proof scheme as a collection of persistent cognitive characteristics of the proofs one produces.

When removing one's doubts about the accuracy (or lack thereof) of a mathematical proposition, one engages in activities that involve manipulating mathematical objects in some specific ways. Accordingly, proving can be viewed as a path followed by an individual in the process of producing mathematical generalisations (Ersen, 2016). Activities of the path followed in validating or justifying mathematical propositions (i.e., proving) are determined by provers' knowledge structures. Therefore investigating students thinking during proving becomes crucial because proofs are at the heart of mathematics as they promote thinking. Proofs help us to justify why an assertion is true. Bell (1976) in Ersen (2016) refers to this sense of proof as illumination which is an explanation for why the proposition is true. The other sense of a mathematical proof according to Bell is verification which concerns establishing the accuracy of a proposition. Finally, Bell sees proof as serving the systematization role. In this sense, proof organizes mathematical results into a deductive system of axioms, major concepts and theorems. This also helps to show the logical structure of pertinent ideas by making deductive chains of reasoning clear to a prover (Ersen, 2016). Explicit chains of reasoning revealed during proof construction are crucial for this study that aims to identify student teachers' schemes of argumentation in proving. The aim of the study is to identify the thinking processes and the emergence of those thinking processes among undergraduate student teachers.

Similar to Bell (1976), Baki (2008) describes three phases of the path followed when proving. First, there is the accuracy phase where the truth-value of an assertion is ascertained. Second, it is then explained why it is accurate (similar to the illumination sense in Bell's three senses of a proof). Third, the proof is abstracted by examining to see its application in other contexts. So depending on the truth value of a mathematical proposition proving can either be seen as a search for deductive argument to support a true mathematical proposition or the search for a counter-example for the purpose of refuting a false conjecture. This process of establishing the truth of a proposition by deductive means or refuting a false conjecture by envisioning conditions that undermine a conjecture (i.e., counter-argumentation) makes mathematics distinct from other scientific disciplines (Stylianou, Blanton & Rotou, 2015, p. 1). The act of proving separates mathematics from other scientific disciplines because it involves abstraction, a process used to obtain the essence of a mathematical concept through the structural mode of thought as opposed to the concrete operational mode. The structural mode of thought omits dependence on real world objects such as numeric examples which help to get some sense of the essence of the mathematical relationships (Herlina & Batusangkar, 2015).

It has been noted that the notion of mathematical proof is central in mathematical thinking and hence it serves as a vehicle for learning mathematics (Dreyfus, 1990 in Stylianou, Blanton & Rotou, 2015; Wilensky-Jerde & Wilensky 2011). A mathematical proof can also be viewed as a communicative act made within the mathematical community which ensures correctness of a given conjecture by using analytic arguments, that is, through application of both deductive and inductive means (Goethe & Friend, 2010). According to Goethe and Friend a deductive argument that supports a mathematical statement begins with some axioms, definitions and previously proven theorems and proceeds using sanctioned rules of inference to lead to conclusion. An inductive argument on the other hand, is an argument that is construed to include tables, figures specific examples and other displays used to ensure correctness of a proposition by structural-intuitive means (Weber & Mejia-Ramos, 2011). A structural-intuitive argument is whereby a prover examines the mathematical proposition to determine whether it is a consequence of mental models a prover associates with the mathematical concepts embedded in the proposition. In other words, an inductive argument is dependent on use of particular instantiations (Alcock, 2010)

As a tool for mathematical learning, proof leads to mathematical understanding as the prover explains a theorem and the content it concerns (Hersh, 1993; Wilkerson-Jerde & Wilensky, 2011). Polya (1957) in Maya and Surmamo (2011) proposes four levels of mathematical understanding ability: mechanical, inductive, rational, and intuitive understanding. Polya describes mechanical understanding as one in which a person memorizes rules and procedures that are then implemented correctly. Inductive understanding of mathematical proof occurs when a prover verifies the accuracy of a statement by using mathematical objects, (specific examples, diagrams) drawn from a proper subset of the set of objects to which the statement pertains (Weber & Mejia-Ramos, 2011). The definition of inductive understanding given here is similar to the structural-intuitive warrant (Weber & Mejia-Ramos, 2011). Polya defines rational understanding as when an individual applies rules and procedures to establish correctness of a mathematical assertion meaningfully, that is, application of such rules and procedures is accompanied by justification (reason). Finally, intuitive understanding is used to denote scenarios in which a person demonstrates awareness of the truth of an assertion and has no doubts about its truth. In other words, an individual possessing intuitive understanding of a mathematical claim would have attained absolute conviction about the truth-value of a claim (Weber & Mejia-Ramos, 2015). In this sense, this ontological study of undergraduate student teachers' proof schemes sought to determine the forms of mathematical proof understanding from the students' proof attempts as a way of establishing the kinds of proof schemes held by the student teachers.

As noted earlier, the notion of mathematical proof is central to mathematical practice because it provides the accuracy (or lack therefore) of mathematical claims and it also provides the reason(s) why a statement is considered to be accurate or inaccurate. Thus while emphasis has been on checking the fact that a given proof is indeed true, the essential mathematical activity should be on constructing or finding a mathematical proof (Doruk & Kaplan, 2015). When proof is conceived in this manner, proof has the potential to play an important role in developing and shaping the mathematical thinking of student teachers. Hence, it can be seen that doing proofs is different from reading a proof.

Doing proofs is described as a cognitive act performed to eliminate doubts of an individual or community regarding the accuracy of a mathematical claim (Iskenderoglu & Baki, 2011). Thus doing proofs refers to those efforts intended to eliminate students' doubts and should be based on these students' own *voices*. However, in stark contrast, Cirillo and Herbst (2012) have reported that proof-oriented instructors rarely ask students to compose proofs of statements or tasks students have not seen before so that students do not engage in independent reasoning. Reading a mathematical proof refers to the act of examining an existing argument to check its validity or for the purpose of understanding the essence of the given argument (Selden & Selden, 2003; Weber & Mejia-Ramos, 2011). The focus of this study is on developing an understanding of students' mathematical thinking as they construct proofs as opposed to reading proofs.

Further, discussions and research efforts on mathematical proof and proving have focused on the *front* of mathematical proof, that is, proof has been conceived as presented in journals and textbooks. However, not much has been explored with respect to what constitutes the *back* of mathematics (Gowers, 2007). Metaphorically, the *back* of mathematics is used to refer to activities that take place in the workshop of a research mathematician where there is interplay between both syntactic and semantic approaches to proof making, which is similar to the analytic mode of proof production proposed by Goethe and Friend (2010). There is overwhelming evidence of students' difficulties with producing proofs. One way of overcoming these difficulties could be by investigating the cognitive processes and activities of students during proof construction. Such understanding could be developed comprehensively by allowing students to engage in activities of the *back* of mathematics. An elaboration on the metaphors *back* and *front* of mathematics is presented in the next section for the purposes of elucidating the focus of the study.

In the workshop of a research mathematician who writes proofs analytic proofs are a prevalent feature (Goethe & Friend, 2010). An analytic prover strives to reach mathematical conviction by using a mixture of both deductive and induction moves. In a deductive move, the prover proceeds from axioms and then utilizes logical rules to lead to a conclusion. Induction is construed in this context to refer to instantiations of mathematical ideas such as graphs, tables, figures and other structural-intuitive displays of ideas pertinent to the mathematical proof task or claim. In the *front* of mathematics, the prover employs the axiomatic approach, behaving in a conventional ritual fashion that demonstrates coherent reasoning that would in turn lead to a clear well- polished proof product (Azrou, 2015). Yet, student teachers' proof activities should reflect more features of the *back* as opposed to the *front* of mathematics. In other words, student teachers should test and experience proofs by themselves because it has been noted that students can follow a proof when explained by their instructors in class but would not be able to compose proofs themselves (Maya & Sumarmo, 2011, p. 232; Moore, 1994). Hence, because of the paucity of research into the typology of warrant types that characterise the *back* of mathematics at undergraduate level little is known particularly in our local Zimbabwean context about how students conceptualise mathematical proof based on their actual *voices*. Therefore, the study was an attempt to investigate student teachers' reasoning based on their actual proving efforts.

Rav (1999, p. 6) has argued that "proofs are at the heart of mathematics." Rav's argument has been supported by extensive research carried out by many mathematics educators on matters related to proving (Kindron & Dreyfus, 2014 p. 302). However, as earlier noted those research studies have tended to focus on students' ability to reflect and validate proofs supplied (e.g., CadawalladerOlsker, 2011; Harel & Sowder, 1998). Once again I reiterate that there has been a scarcity of research that addresses how students go about constructing proofs. This dearth in research based on students' actual proof construction efforts has seen many researchers advocating for more in-depth studies into students' conceptions of mathematical proof that are based on students' personal constructions. For instance, Mariotti (2006, p.198) has pointed out that "further investigation is needed into students' active production of proofs with particular emphasis on analysis of the cognitive processes involved in producing and proving conjectures" (p.198).

Further, Selden and Selden (2003) even suggest that "considerably more could be also done in examining the process of proof construction" (p. 2). Even in circumstances in which proof-oriented instructors try to involve students in proof production Selden and Selden have noted that not much time is devoted to helping students to learn how to construct proofs. Rather, emphasis is on producing fragments of proofs or original proofs presented as lecture notes in a neat fashion with little or no resemblance at all with the *back* of mathematics where an analytic approach is employed in constructing proofs. Hence, while it has been useful to generate knowledge about students' conceptions of mathematical proof through proof validations (e.g., Bleiler, Thompson, & Krajčevski, 2014; Pfeiffer, 2010; Weber & Mejia-Ramos, 2015), it is also crucial to gain insights about undergraduate student teachers conceptualisations of mathematical proof from their actions and behaviour as they engage in proof constructions.

As noted earlier, proof is essential for deep mathematics learning. It can be argued that mathematics students' understanding and ability to construct proofs is not only important for their own learning but it is also crucial that these future high school teachers are able to help learners learn how to construct proofs (Uĝurel, et al., 2016). Hence, in order for the student teachers to be able to promote proving abilities among their future students they need to be able to build a strong foundation of the proof concept. Although it is now documented that constructing a mathematical proof is a complex process that calls for a large expanse of knowledge and skills and is determined by the learning context (Pfeiffer, 2010), researchers (e.g., Balacheff, 1998, 2007; Beiler et al., 2013; Selden & Selden, 2003) have based their conclusions on arguments students find convincing (convincement issues) and validation of proofs supplied to the participants by researchers. This observation was also made by Imamoğlu and Toğrol (2015) who stated that over the past decades researchers have focused on proof validations which Selden and Selden (2003) define as readings and reflections on proofs to check their correctness. These proof validations are carried out on proof texts supplied by researchers. I argue that investigations into students' understanding of mathematical proving should be grounded in students' own efforts. Apart from personal observations available literature sources suggest the need for in-depth studies into students' conceptualisations of mathematical proof. For instance, Mejia-Ramos and Inglis (2009) have reported that there are a few empirical studies on how well students understand proofs.

I reiterate that most studies have focused on students' abilities to recognize correct mathematical proofs, that is, proof validations (e.g., Bleiler et al., 2014; Selden & Selden, 2003). Yet research on learning of mathematical proof and associated difficulties must be based on what students really do by themselves, rather than relying on students expressing their conviction levels on the validity of arguments supplied by researchers (Mariotti, 2006). Hence the current study responds to the dearth in studies into ways in which individual students think around the notion of proof. The study was thus designed to develop an understanding of undergraduate students' thoughts as they engaged with mathematical proof tasks.

Furthermore, while mathematics students have shown a preference for deductive proofs, a higher level proof scheme, research studies have revealed that these students were not able to compose proofs by themselves (e.g., Azrou, 2015; Moore, 1994 in Maya & Sumarmo, 2011, p. 232;

Styliamdes, 2011). When those students were asked to produce proofs of tasks that required use of formal deductive means they resorted to particular instantiations. This sort of proof behaviour revealed a discrepancy between what students produced as proofs and what they chose as closest to their preferences. It can be inferred that it is often easier to read a proof than to produce a proof. This provides further evidence about the limitation of relying on proof readings as basis for measuring students' competences at constructing proofs. Hence, I suggest that an understanding of the mental processes involved in proving and how proof schemes develop amongst undergraduate students merit close attention and one way of ascertaining students' proof competencies is by examining their proof productions.

One of the primary goals of mathematics instruction is for students to develop standards of proving and conceptions of mathematical proof that are held by research mathematicians (Weber & Mejia-Ramos, 2015). Hence, research on students' mathematical proof competences should involve measuring discrepancies between students and mathematicians' conception of justification and proving processes of mathematics statements. Therefore, this study was in response to the call to bring students proof experiences as close as possible to the practice of mathematicians. Precisely, the intent of the study was thus to determine students' thinking abilities on justification and proof by addressing the questions of how student teachers go about constructing proofs and how the mathematical object (proof scheme) evolves. In other words, what is the ontology of the proof scheme with respect to student teachers' conceptualisations of mathematical proof?

Careful analysis of critical elements of students' proof schemes and the individual's thinking and reasoning around the notion of mathematical proof is needed in current research efforts to raise students' mathematics proof competence levels. The goal of the current study is to contribute to efforts intended to transform the students' view of a mathematical proof as a special form of producing written work to a conception of proof as a vehicle for producing reliable explanations for the accuracy (or lack thereof) of mathematical propositions and hence a means of achieving understanding (Liu & Manouchnri, 2013).

Mathematical proof is an essential tool in learning mathematics. Understanding how student teachers conceptualise mathematical proof is an essential consideration for thinking about how to teach proof. It was therefore anticipated that the current research could for instance, account for impasses that characterize students proving efforts whereby students fail to construct a proof because simply they do not know what to do. Developing an understanding of the nature of students' conceptions of mathematical proof will in turn inform the process of identifying appropriate learning opportunities for students to engage in during learning.

Researchers (e.g., Iaanone & Inglis, 2011; Mariotti, 2006; Varghese, 2009) have recommended the necessity for more studies that illuminate processes students use when they engage in constructing proofs. Calls for in-depth studies that would uncover salient features of students thinking around the notion of mathematical proof created a research base for this study on: the ontology of proof schemes in undergraduate student teachers' conceptualisations of mathematical proof. Identifying critical elements of students' knowledge involved in proving will provide a clearer picture of student teachers' knowledge of situations for proving (Ball, Thames & Phelps, 2008). The term knowledge of situations for proving is defined as part of teachers' knowledge about proof involved in the mobilisation of proving opportunities for students (Ball et al., 2008).

The problem of students' understanding of mathematical proof has also illuminated itself in our local context at tertiary level. In Zimbabwe there is paucity in research on undergraduate students' understanding of mathematical proof, particularly studies with grounding in students' individual proof attempts. In the teaching and learning of proof-laden courses such as Real Analysis at undergraduate level the sequencing of instruction has followed the format "definition-theorem-proof." In addition, assessment modes, have focused on how students' comprehension of a given mathematical proof can be measured through efforts such as reproducing deductive arguments from lecture notes or modifying the proof slightly to prove an analogous statement e.g., lecture notes on the theorem: *The least upper bound of a subset of* $\mathbb{R}$ *that is bounded above is unique* can be modified slightly by the student to prove the analogous statement: *The greatest lower bound of a subset of* $\mathbb{R}$ *that is bounded below is unique.* It can be noted that these types of assessment only serve to provide a superficial understanding of mathematical proof because to accomplish the proof the student proceeds in a secure ritual manner by just modifying slightly lecture notes on the uniqueness of a least upper bound of a subset of real numbers that is bounded above (Mejia-Ramos, Fuller, Weber, Rhoads & Samkoff, 2012).

It can be seen from the foregoing discussion that asking students to express their level of conviction in arguments and/or proofs supplied by the researcher does not do enough to involve students in the manifold of activities and processes involved in proving. Furthermore, a study by Pfeiffer (2010) has revealed that although the processes of validating and composing a mathematical proof entail each other, it is more difficult to construct than to read a proof. So engaging students in proof constructions is more likely to generate more insights into the cognitive processes involved in proving. The idea of exploring students' thinking on the basis of their actual constructions has been reinforced by Davis, Maher, and Noddings (1990) cited in Greenes (2009) who suggest that to know mathematics individuals "make constructions using mathematical objects in a mathematical community" (p. 56).

To further emphasize the need to develop an understanding of student teachers' conceptualisations through their own *voices*, I draw ideas from Boero (1999) and Mejia-Ramos (2008). Mejia-Ramos articulated three activities involved in argumentation process that mathematicians engage in when proving: constructing a novel argument, presenting an already existing argument and reading an available argument. I reiterate that constructing a novel argument is more likely to generate more insights into students' thinking processes than any of the other two activities involved in mathematical proving and hence it became the major focal activity of this research. I emphasize the point made earlier that constructing a proof is different from reading an available argument.

Commenting on the typology of warrant types involved in proving, Boero (1999) noted that while empirical justifications and structural-intuitive arguments are useful in some stages of conjecturing and proving they do not appear in the products of these two processes, that is, conjectures and proofs of theorems. The point drawn from this piece of literature in connection with the research problem is that exposing students to the *front* of mathematics which is typical of undergraduate teaching and learning of proof will obstruct these conjecturing and proving activities which might be important in revealing students' thinking processes.

Consequently, the current study builds on existing research studies on students' understanding of mathematical proof by examining undergraduate mathematics education students' mental constructs around the notion of mathematical proof as well as developing an understanding of students' experiences within the universe of discourse (phenomenon of interest), which is mathematical proof. This exploratory study of undergraduate students' mental constructs around the notion of mathematical proof will be driven by the following main research question.

1.3 Main Research Question

The overarching goal of this main research question is to identify the critical elements of the knowledge of processes involved in mathematical proof and proving. This goal will be pursued by addressing the research question: *In what terms do Zimbabwean undergraduates think about proving in mathematics?* In view of the superficial understanding of mathematics highlighted, this study seeks to explore the modes (kinds) of undergraduate student teachers' mathematical proof schemes and how such proof schemes ultimately emerge among undergraduate student teachers. Ontology is the systematic account of existence of the being, that is, a study of what ultimately exists (Porta & Keating, 2008, p. 21). It is the study of the fundamental modes or kinds of being. Consequently, the set of student teachers' ontological commitments, that is, the terms in which they think about mathematical proving and how the students' proof schemes develop will be investigated through the following sub-questions.

(i) What kinds of proof schemes characterise undergraduate student teachers' conceptualisations of a mathematical proof?

(ii) How do the undergraduate student teachers develop their proof schemes?

To elaborate on each sub-question stated, I first refer to the fundamental ontological question. Ontology is a study that addresses the question; "What can we know?" (Porta & Keating, 2008, p.

21). Briefly, ontology is about what exists and can be potentially talked about the object of investigation (Corbetta, 2003, p. 13). We distinguish ontology from epistemology which deals with the question of how a phenomenon can be known, that is, epistemology is related to the possibility of knowing through different types of inquiry. The focus of the study is therefore on what can be known (ontology) about the object of investigation, which are the proof schemes, precisely their schemes of argumentation in validating and refuting conjectures. I now elaborate on each sub-question.

The first sub-question is: *what kinds of proof schemes characterise undergraduate student teachers' conceptualisations of mathematical proof?* The basic ontological question is "how the world fits together and how we make sense of it" (Porta & Keating, 2008, p. 21). Thus ontology is about the form and nature of the universe of discourse (UoD), that is, the phenomenon of interest, which in this study is the proof scheme. This study therefore seeks to establish the modes or kinds of being of the students' proof schemes as a way of establishing their formal praxis, that is, established habitual practice regarding proof and proving in undergraduate mathematics. Ontology also deals with categories and the sort of "things" in the categories of being of an object. Therefore, as a way of understanding the basic social process, the study will try to find answers to the questions: what are the fundamental categories within the UoD— undergraduate students' proof schemes? In what sense can items in those categories be said to be? Porta and Keating (2008, p. 21) differentiate nominalists from realists in connection with the existence of categories of being, where on one hand, there are nominalists who posit that categories of existence are arbitrarily created and, on the other, we have realists, according to whom categories of being are there to be discovered. This study will adopt a realistic view of the existence of categories, these already exist among the students and this study attempts to discover the categories by examining the kind of proof schemes held by student teachers.

Another ontological issue of concern to this study is about ultimate existence of proof schemes. This concern about the emergence of proof schemes will be dealt with by addressing the sub-research question: *how do undergraduate student teachers develop their proof schemes?* It is possible for an individual to hold more than one proof scheme (CadawallladerOlsker, 2011, p. 45; Harel & Sowder, 1998). The switch from one proof scheme to another when confronted with a proof task is termed an ontological oscillation and signals inconsistencies or confusions among students. Whilst Harel and Sowder acknowledge such mutual interrelatedness in student teachers' proof scheme states in their taxonomy, they are silent about the nature and causes of such ontological oscillations. In this regard, the study seeks to unravel what ultimately can be said to exist about proof schemes of student teachers bearing in mind that ontology is about the ultimate existence of the ‘being’. Hence, the present study seeks to develop a proposition about the possible shifts in student teachers’ proof schemes in undergraduate mathematics.

1.4 Justification for the study

A central reason why it is important that students should be able to construct and evaluate mathematical proofs is that they can keep instances of misconception when constructing proofs to a minimum level. A misconception is a measure of the discrepancy that exists between what a concept is and what one thinks it is (Mejia-Ramos, Fuller, Rhoads, & Samkoff, 2012, p. 3). Determining student teachers’ experience with mathematical proof is crucial because it influences how proving skills can be promoted amongst teachers’ future students (Isekenderoglu & Baki, 2011). Student teachers’ mathematical proof proficiency affects their classroom activities when they teach it in future. Hence, determining students’ thinking about proofs is significant because it helps mathematics educators find and implement strategies intended to raise students’ proof competences.

The route followed by teachers in doing proofs influences classroom activities. The route depends on the truth-value of a mathematical claim confronted by the student. Accordingly, proving is the process of establishing the truth-value of a mathematical assertion by deductive means for a true conjecture and for false assertions, proving is conceived as the refuting of a mathematical claim by means of a counter example. The development of abilities to construct proofs is an important goal in mathematics education in light of the fact that students do not have an accurate conception of a mathematical proof (Oflaz, Bulut & Akcakin, 2016). Promoting and fostering student teachers’ proving abilities allow them to develop a strong command of mathematical proof, which is essential for inducing flexibility in learners’ thinking because proof is a tool for deep learning (Oflaz et al., 2016). If student teachers think in a flexible manner then they will be able to promote amongst their own future learners the essential crucial interplay between conceptual and procedural knowledge of mathematical concepts that underpin a given proof task or proposition whose truth value an individual seeks to establish.

Exploring student teachers’ proving abilities is important because the notion of proof is core to the development of critical mathematical thinking among student teachers from the perspective of proof learning. Critical thinking is a process by which a conjecture is justified to be true or refuted through use of counter examples. Critical thinking also involves questioning the implications of procedure taken to prove a mathematical statement (Alcock, 2010; Fukawa-Conelly, 2012). Hence, critical thinking abilities enable secondary school mathematics teachers to facilitate the processes of formulating and proving conjectures amongst learners. Inducing habits of criticalness about mathematical proof among learners will allow proofs to play a prominent role in classrooms so that the classrooms become communities of inquiry. To accomplish the goal of turning classrooms into communities of inquiry, student teachers need to build a profound understanding of processes involved in proof constructions. Hence, creating communities of inquiry in classrooms can help mathematics educators overcome frequently insurmountable difficulties that students face when learning mathematical proof.

It was anticipated that by engaging student teachers with proving activities the student teachers would develop an appreciation of the crucial roles played by proof in mathematics such as providing insights and explanations as to why mathematical propositions (claims) can be refuted or accepted as accurate mathematical statements (Weber, Inglis & Mejia-Ramos, 2014). From this standpoint, the intent is to introduce undergraduate student teachers to the socio-mathematical norms of practicing mathematicians thereby narrowing the distance between students and expert conceptions of mathematical proof. For instance, by involving students in proof construction, it is anticipated that the study would generate insights about the kinds of student teachers' mental constructs around the notion of a proof. An understanding of the nature of such mental constructs held by the undergraduate mathematics education students in connection with the notion of mathematical proof may contribute towards efforts to raise students' proving competence levels to those of research mathematicians, that is, experts who compose proofs.

The justification for the current study of student teachers' kinds of proof schemes can be viewed from a meta-mathematical level. It can be noted that although proof is at the core of the development of mathematical knowledge, most students have a fragile grasp of what doing mathematics entails (Azrou, 2015). Reflecting on undergraduate student teachers' basis for rendering mathematics conjectures into facts or their schemes of argumentation in refuting false propositions can be useful to mathematics educators by generating insights that may be applicable to teaching of mathematical proof. Thinking through student teachers' proof constructions can generate insights that influence the role that mathematical proof comes to play in their classrooms. For example, reflecting on students' proof experiences can reveal reasons for lack of a profound understanding of mathematical proof and the tendency to feel insecure with concept of proof. Hence, reflecting on the kinds of students' schemes of argumentation can contribute towards an enhanced practice in the learning of proof from an emphasis on regurgitation of routine instructors' notes to construction of meaning as the student teachers engage in processes of conjecturing and proving.

There are two fundamental notions in social science research, namely, basic social problem and basic social process (Charmaz, 2006). A basic social problem refers to a problematic phenomenon from the point of view of people being studied. For something to qualify as a basic social problem, it must not be short-lived (Charmaz, 2006; Silverman, 2010). Mathematical philosophers, mathematicians and mathematics educators have been grappling with the notion of proof for years and perhaps will continue to do so for ages. Basic social process, means what the participants (people being studied) essentially do in dealing with their basic social problem (Charmaz, 2006). In the context of this study the basic social problem is superficial understanding of mathematical proof by undergraduate mathematics student teachers as reflected through the students' rote memorisation of routine proofs, lack of intellectual curiosity and appreciation for meaning in proof constructions. In other words there is no profound (deep and vast) understanding of mathematical proof among Zimbabwean undergraduate student teachers.

Troutman and Litchmberg (1995) cited in du Toit (2009) have identified four types of learning activities in mathematics education which are developmental activities, reinforcement activities, drill- and- practice activities and problem solving activities. Developmental activities and problem solving activities culminate in conceptual knowledge whereas drill- and-practice and reinforcement are more inclined to procedural knowledge (Davis, 2005). Davis suggests that the sequence of learning activities should be to engage students first in developmental and problem solving activities so that they get to develop conceptual knowledge before being exposed to procedural knowledge for effective learning to take place. In other words, sequencing learning activities in the manner suggested would promote sense-making among students during proof construction. Yet the teaching and learning of mathematical proof has generally been characterised by:

- Mathematical proof is presented as a finished product in which the learner is a passive knowledge recipient (Harel & Sowder, 1998).
- The teaching/learning of proof has not given emphasis on "why" and "when" mathematicians do proofs (Harel & Sowder, 1998, p. 247). Thus more emphasis has been given to facts and procedures rather than to reasoning behind the facts and procedures.
- Insistence by students during learning, on being told the proof for regurgitation later on, rather than taking part in the proof construction process (Harel & Sowder, 1998).

To shed more light on the importance of the study, I briefly recap main ideas of ontology. First, I recall that ontology is the theory of objects, which in this case are the mathematical proof schemes and their ties, that is, their dependencies and relations. Second. It can be recalled that ontology is a branch of metaphysics that examines the fundamental properties, modes and aspects of 'being" of the Universe Of Discourse (UoD). Third, ontology is the systematic study of the nature of existence of the UoD. Guided by these main ideas about ontology, the current study examines the basic social process, that is, what student teachers essentially do in dealing with meaning construction during proving by addressing these ontological issues (Wand, Storey & Weber, 1999). In this regard, this study seeks to establish the modes or kinds of being of student teachers' proof schemes. The study will investigate the nature of students' thoughts about mathematical proof and how such thoughts evolve by focusing on the following pertinent questions:

- What are student teachers’ schemes of argumentation?
- Why do the student teachers construct proofs in the manner they do?
- How does students’ thinking about mathematical proof evolve?

By seeking answers to these questions it is anticipated that the study could potentially develop an explanatory theory about kinds of proof schemes held by the student teachers and could also generate a possible proposition about how student teachers’ thoughts about mathematical proof emerge.

Finally, the justification for the current study relates to efforts to develop an understanding of student teachers’ struggles with mathematical proving. With previous studies much emphasis has been on checking the fact that a given proof is true, that is, emphasis has been on validating a given argument (e.g., Bleiler et al., 2014; Selden & Selden, 2003). Previous research has documented impasses reached by students when composing proofs (e.g., Varghese, 2009), but causal links within the impasses have not adequately researched. Hence, the current study seeks to develop an explanatory theory for the kinds of proof schemes held by the students based on their active proof production as opposed to validating purported proof texts supplied by researchers. In other words, the intent of the current is to uncover the salient features of students’ thinking about proof and the causal links that give rise to those features. It is hoped that by examining the sort of “things” in those categories (kinds of proof schemes), the study could explicate the causes of the student teachers’ struggles with mathematical proof.

1.5 General approach to the study

The study intends to establish the terms in which undergraduate student teachers think of mathematics by exploring the student teachers’ proof construction abilities and their proof experiences in the context of a teaching experiment in which concepts drawn from Real Analysis will be taught. The intent of the current study is to accomplish the following objectives;

(i) Establishing the basis for the student teachers’ schemes of argumentation for rendering mathematics conjectures into mathematical facts or alternatively the grounds for rejecting conjectures in undergraduate mathematics.

(ii) Identifying and explaining the different trajectories through which the students’ proof schemes emerge.

It is hoped that the study will accomplish its major aim of establishing the students’ formal praxis with respect to the notion mathematical proof. In other words, the study seeks to explore the student teachers’ set of ontological commitments, that is, their mental constructs around the notion of proof. A realist process approach treats proof events and processes as real observable entities that are causally relevant for student proof behaviour. Following Maxwell and Mittapalli (2010), a realist approach will be employed to explore undergraduate student teachers’ schemes of argumentation for rendering mathematical propositions into facts or alternatively refuting them through direct and indirect methods of mathematical proof.

It has been claimed that the proof scheme held by a person is indispensable from his/her perception of what it means to do mathematics (Harel & Sowder, 1998). For example, a student with a deeply rooted ritual proof scheme is persuaded to evaluate proof of a proposition by an appeal to its surface appearance (ritualistic aspects) (Martin & Harel, 1989). Such students may doubt the truth of arguments presented without symbol manipulations. The events described here, that is, exploring student teachers' thoughts about mathematical proof will lead to textual data. A realist process analytic framework developed using underpinnings of mathematical proof such as the notions of conceptual insights and technical handles, warrant types in proof and proving, and syntactic and semantic proof construction will be applied to the textual data using directed and summative content analysis techniques to observe proof events and processes in an endeavour to get sensitising insights about the nature of the student teachers' basic social problem of superficial understanding of mathematical proof (Charmaz, 2006; Punch, 1998, 2005). The sensitising insights may lead to identifying essential features of the proof schemes as opposed to accidental attributes of arguments found to be convincing to the student himself or herself (ascertaining) and by other students (persuading).

1.6 Delimitations

Undergraduate mathematics education students who would have studied Calculus courses participated in the study. Calculus courses are pre-requisite for many undergraduate mathematics courses. The notion of mathematical proof is a crucial and fundamental learning aspect in such topics at undergraduate level and hence the choice of such topics. Hence it was expected that involving students in the study would give rich data in the sense suggested by (Charmaz, 2006). The study took place in the context of teaching experiment in which a proof laden course Real Analysis Course ―an advanced form of Calculus was taught to undergraduate mathematics education at one state university in Zimbabwe.

1.7 Outline of thesis

In this chapter, I explain the motivation of the study and context, the research problem, the research questions, aims and objectives of the study as well as the significance or importance of the study. Also included in this chapter is an effort to bring clarity to study intentions by describing the context of the study and the general approach to the study of the kinds of mathematical object held by the student teachers and how the student teachers' proof schemes evolve.

In Chapter 2, I discuss the Theoretical and the Conceptual Frames and the guiding philosophy of the study. With regard to the Theoretical Framework, underpinnings about the notion of mathematical proof are explained and an effort to relate them to the study context. The mathematical underpinnings explained include the meaning and nature of mathematical statements, the idea of conjectures and an elaboration on the structure of mathematical proof. There is also focus on different understandings of mathematical proof in order to bring out the basis of argumentation that can be made in a given method of proof. I then turn to the question: which theories tackle the issue? Specifically, the question deals with major aspects of related studies which include: a theoretical construct called the behavioural schemas in proving (Selden & Selden, 2011), taxonomies of proof schemes and theories on mental processes underlying exploration of the problem situation during conjecture formulation and the subsequent proof construction. Other constructs examined include the concept of key ideas (Raman, 2003; Sandefur et al., 2013). The scientific realist philosophy that treats mental events and processes as real observable phenomena that are causally relevant to the explanation of individual and social behaviour (Maxwell, 2004) is applied in determining the kinds of proof schemes held by students and how these evolve among students. Finally, the Conceptual framework is presented essentially as a fusion of ideas from the mathematical underpinnings and scientific realist ideas.

In Chapter 3, two major sections are presented. First, specific studies on mathematical proof were critiqued for the purpose of explicating the research gap. The second section focuses on other ideas examined from related studies with the intent of illuminating the need to investigate students' thinking about mathematical proof.

In Chapter 4, methodological issues are examined. Ideas derived from Chapters 2 and 3 are used to discuss aspects related to methods such as the research design. Attempts are made to justify the use of the case study design using scientific realist positions. Research instruments are described together with data collection procedures and data analysis techniques. The chapter then concludes with a discussion of ethical matters that characterised the research process.

In Chapter 5, results of the study are presented in the form of data matrices where analytic tools drawn from existing studies (e.g., Doruk & Kaplan, 2015; Stavrou, 2014; Uǧurel et al., 2016) and primary sources (e.g., Berg, 2009; Corbin & Strauss, 2008; Hennink, Hutter & Bailey, 2013) were used to draw meaning about how student teachers think around the idea of mathematical proof and how students' thinking evolves based on tabulated transcription texts. Results are presented per research question as follows. First, directed content analysis technique is applied to written responses and transcriptions of chalkboard and reflective interviews on students' proof attempts to address research question one. Second, with respect to research question two summative content analysis technique is applied to make inferences from textual data from reflective interviews about how the student teachers' proof schemes emerge.

Finally, in Chapter 6, results presented in Chapter 5 are discussed within the main realist analytic framework using theoretical constructs such the idea of micro-reasoning, conceptual insights and technical handles and notion of intellectual challenge (Duval, 2006; Koichu, 2012; Sandefur et al., 2013) in order to address the two sub-research questions. Main findings are then discussed and main conclusion is formulated about terms in which the student teachers think about mathematical proof. Implications for theory, teaching and research methodology are suggested. Chapter 6 also points out limitations of the study and loose ends are identified for further studies. The chapter ends with my personal reflections of the whole research process.

Chapter Two

Theoretical Framework

In philosophical realism ontology refers to a set of terms and their associated definitions used to describe the Universe of Discourse (UoD)— the phenomenon of interest, which in my case is the notion of a mathematical proof. Dunn and Mearman (2006) recognize the need to determine the set of theories used to describe the UoD in order to promote deep engagement with it. Hence, in this chapter, I discuss theories embedding the notion of mathematical proof. The underlying mathematical ideas of proof and proving to be examined in this section include:

- Different understandings of mathematics
- The idea of a conjecture and its properties including novelty, certainty, originality.
- Meaning and different forms of mathematical statements.
- Definition of mathematical proof to include formal and practical meanings of proof
- What counts as proof?

2.1 Different perspectives on mathematics

This section examines different understandings of mathematics with the view to explicating the basis of argumentations that can be deemed mathematically sound in a given method of proof. There are different mathematical views or understandings such as the toolbox/instrumentalist, Platonist or objectivistic, system, and the process/problem solving views of mathematics. Other prominent understandings of mathematics are constructivism and formalism. I now comment on some of the mathematical philosophies.

First, we have the toolbox/instrumentalist school of mathematical thought. The toolbox/ instrumentalist understanding of mathematics is the view that mathematics is an accumulation of skills, facts, and rules to be learnt in pursuance of something external (Torner, 1998; Viholainen, 2011, p. 310). Learning of mathematics is construed as a passive reception of knowledge. Accordingly the purpose of a mathematical argument is to verify correctness of a mathematical statement.

Second, there is the Platonist or objectivistic worldview of mathematics which is defined by Ernest (1989) as an assertion that mathematics is a unified body of static certain knowledge. Mathematical objects and entities are an objective reality whose existence is independent of the human mind (Davis & Hersh, 1981). In other words, the Universe of Discourse (UoD), which in the current case is the proof scheme, consists of distinct properties independent of the enquirer. Put differently, we are saying that the meaning of the UoD is inherent in the phenomenon, thus pointing to the existence of absolute unconditional truth.

Third, there is constructivism— which in the context of mathematics is a school of thought founded by Luitzen Egbertus Jan Brouwer as suggested by Davis and Hersh (1981). Davis and Hersh write that constructivism recognizes natural numbers as the fundamental basic notions so any meaningful mathematics must hinge on natural numbers. A corollary of this premise is that one cannot establish the truth of a proposition by showing that its negation leads to a contradiction. Hence, the constructivist mathematician does not accept indirect methods of proving such as proof by contradiction and use of contrapositive. The reason for rejecting these methods of proving is the constructivist philosophical stance that all meaningful constructions should hinge on natural numbers. The direct proof method by mathematical induction is an example of a method of proof with an inclination towards the constructivist perspective.

The fourth mathematical school of thought is the system view alternatively referred to as formalism. The brainchild of David Hilbert, formalism is a philosophy in which definitions and axioms are the fundamental building blocks (Torner & Trigushct, 1994). Proofs depend ultimately on underlying definitions (Lay, 2009). The word definition of a concept refers to the form of words used to specify the concept (Vinner & Hershkowitz, 1980). Definitions need to be precise so that they lead to better proofs (Lay, 2009). For example the exponent x^n is defined by (Lay, 2009) as the number of times 1 is multiplied by x. Such a definition is valid because it allows learners to deal with cases when $n = 0$ and when $n = 1$. For example x^3 is interpreted as 1 is multiplied by x three times, that is, $x^3 = 1.x.x.x$.

Axioms are self-evident truths, that is, statements which do not require proof (Hersh & Davies, 1981, p. 412). Precisely an axiom is a statement that cannot be demonstrated in terms of other simpler concepts (Haggarty, 1992). Axioms are rock bottom, self-evident facts upon which mathematical structures rest held together by bolts of logic. Thus axioms are building blocks of mathematics theory. Axioms possess characteristics of consistency, independence, and adequacy. By consistency is meant that there is no sentence that should be demonstrated to be both true and false at the same time from our set of axioms. An example is that of a set X of vectors in R^n, one should not be able to show that X is both linearly independent and linearly dependent using vector space axioms of vector addition and multiplication of vector by a by scalar. When we refer to axioms as being independent, we mean no statement from our set of axioms should be derivable from other axioms in the set. Finally, by being adequate we mean axioms should be sufficiently many to constitute a theory. That means we should be able to derive as many results about our system from the set of axioms (Haggarty, 1992; Kirkwood, 1992).

The system/formalist view of mathematics states that mathematics is a system of logical rigorous constructions based on axioms. The central role of this view is the systematization role. That means various previously known facts or results are brought together using logical rules of inference. An exemplification of the systematization role is Euclid's Elements (CadawalladerOlsker, 2011; de Villers, 1999; Harel & Sowder, 1998) where many results previously proven by the Greeks were brought together and ordered to form a deductive system based on a collection of axioms, definitions and postulates.

The fifth form of mathematical understanding is the problem solving or process view. Here mathematics is seen as thinking and learning process that requires both conceptual and holistic understanding (Viholainen, 2011, p. 312). Mathematical arguments deemed to be sound according to the process or problem solving perspective of mathematics are those based on more concrete representations of the mathematical objects involved in the statement to be proved. According to this view, mathematics is a dynamic and continually expanding field of human creation and invention. The crucial role of argumentation is to help an individual understand reasons why a statement is true, that is to provide insight into why a statement follows from given data (Weber & Alcock, 2004). The process/problem solving perspective of mathematics thus seeks to engender understanding of mathematics by explaining (Viholainen, 2011, p. 309). Finally, there is the fallibalistic view of Mathematics which according to Lakatos in Ernest (1991, p. 19) is the philosophy that there are no authoritative sources of knowledge and no source is entirely reliable. The Fallibalists consider mathematics as an essentially human pursuit, invented by humans and therefore prey to human fallibility.

The significance or relevance of these different mathematical views to this study of mathematical proof schemes is that the form of arguments posited by undergraduate student teachers depends on their views of mathematics. Hence, such arguments should illuminate the terms in which the Zimbabwean undergraduate students think about the notion of proof which is therefore indicative of their kinds of proof schemes. Further, recall that ontology is the systematic study of what ultimately exists, in terms of categories of proof schemes and their elements. Thus the kinds of arguments put forward by undergraduate student teachers may provide insights into how their proof schemes eventually emerge.

2.2 The Idea of Conjectures

I begin this section by making some exemplifications and clarifications on the ideas of observation and conjecture as a build up to the definition of mathematical proof. An observation is a deliberate and conscious human act, which can be a mere recognition. For instance, the realisation that 1 is not a prime number is an example of an observation a learner could make after some learning experience/encounter on prime numbers (Harel & Sowder, 1998) or based on lifetime experiences. However, despite sensory or passive connotations apparently implied in the description of observations, they are a result of people's reconstructions (Piaget & Inhelder, 1967 in Harel & Sowder, 1998). Another crucial point to make about observations is novelty. An observation made by an individual is novel to him/her because the person was not aware of it until he/she has realised it. The significance of this notion to this study is that effective learning of proof ought to engage the students in situations where they discover relationships as opposed to mere regurgitation of facts or procedures. Characteristics of observations include originality, mode of thought, and certainty (Harel & Sowder, 1998).

Originality in an observation can be illustrated with the following example. Consider the statement: *A bounded monotone increasing sequence converges to the least upper bound of the sequence.* A learner can generate the proof on his/her own by capitalising on definitions of monotone increasing, convergence, least upper bound and limit of a sequence and produce the proof on his/her own. On the other hand, the learner can reproduce the proof from lecture notes or alternatively from peer discussions of the same proof. We accordingly classify observations as being innovative and imitative. An innovative observation originates with the observer whilst an imitative observation is communicated to the observer by others as exemplified above.

Mode of thought refers to how observations conceptually emerge, which could be by abstracting a phenomenon from several empirical observations or from thought processes with no mediation of empirical activities (Martin & Harel, 1989; Stylianides, 2007). An observation can be made during inductive explorations as the learners observe patterns and mathematical relationships through use of specific examples (Alcock & Inglis, 2008). Alternatively an observation can be made by thinking through mathematical objects. For example, by reflecting on definitions of a closed set and limit point of a set a learner might then observe that a closed set contains all its limit points.

Finally, certainty is concerned about how observations are evaluated. Accordingly we distinguish between a conjecture and a fact. A conjecture is an observation made by an individual with doubts about its truth value (Harel & Sowder, 1998, p. 241) whereas a fact is an observation made by an individual who is convinced by its truth or falsity. Hence, the learner's classification of an observation either as a conjecture or fact depends on the person's conviction in its truth. Consider the theorem: *There exists a real number, x, such that $x^2 = 2$.* In the proof construction process there is a stage where we write; Let $S = \{y \in R: 0 < y: y^2 \leq 2\}$ and the claim that clearly S is non-empty is then made. While the observation that S has elements may be apparent to the teacher, and therefore the claim can be labelled a fact from the teacher's point of view, there may be lack of clarity on the part of the learner. Such a claim can be considered to be a conjecture from the learner's perspective. The discussion so far has been focused on laying the basis for the definition of mathematical proof which is now presented.

2.3 Underpinnings of mathematical proof

2.3.1 Definition of mathematical proof

Definitions express the properties that characterise the objects of a given theory and express a network of relationships shared by objects in the theory (Wilkerson-Jerde & Wilensky, 2011). Hence, definitions can be thought of as complete descriptions of the behaviour, structure, or properties of the focal mathematical idea (i.e., mathematical proof) that accounts for all instances of that idea. For example, an odd number x is defined as $x = 2k + 1$, where $k \in \mathbb{Z}$. The definition just given captures all instances of the focal mathematical idea— an odd number. Mathematical proving is defined as, the process employed by an individual to eliminate one's doubts about the truth|falsity of a mathematical conjecture (Harel & Sowder, 1998, p. 241). Proving can thus be viewed as a process of converting a conjecture into a fact. A mathematical observation ceases to be a conjecture and rendered into fact (or refuted) in the view of the learner once he/she becomes certain of its truth value. Mathematical proof is an argument needed to validate a mathematical statement. Hence, mathematical proving is a truth seeking exercise and a mathematical proof is the product of the proving effort. By a mathematical argument we refer to a connected sequence of assertions for or against a mathematical claim (Stylianides, 2007). In the sequence of assertions deductive logic acts as a norm for warranting mathematical conjectures into mathematical facts or alternatively refuting the conjectures (Selden & Selden, 2003).

Proving involves two sub-processes of ascertaining and persuading. Ascertaining is the process employed by an individual to remove his/her doubts about the truth of an observation, while persuading is the process a person engages in convincing other people about the truth-value of a conjecture (Harel & Sowder, 1998, p. 243). The process of rendering a conjecture into a mathematical fact henceforth referred to as proving, can be done formally or informally (CadawalladerOlsker, 2011). We accordingly distinguish between the formal meaning and practical or informal meaning of a mathematical proof.

The formal meaning of a mathematical proof is consistent with most definitions suggested by (Selden & Selden, 2003; Stylianides & Stylianides, 2009). A formal proof is a carefully reasoned argument that validates a conjecture relative to a set of assumptions, definitions and axioms (CadawalladerOlsker, 2011; Haggarty, 1992). This formal definition of mathematical proof is shared by Duval (2002) who defines a mathematical proof as a specialised form of argumentation in which deductive logic acts as norm in warranting or validating conjectures. The term deductive logic connotes use of axioms, definitions and previously proven theorems in eliminating one's doubts about a conjecture. A formal proof is alternatively referred to as a public or syntactic proof (CadawalladerOlsker, 2011; Weber & Alcock, 2004). On the other hand, the practical definition of mathematical proof is informal and imprecise (CadawallaskerOlsker, 2011). It is essentially about what we do to convince other people, that is, make them believe our own proofs of theorems. Thus, the informal or practical meaning of mathematical proof is somewhat subjective. Further, while there is use of axioms in formal proofs, practical proofs exploit mental instantiations— graphs and other visual representations of the mathematical statement (Presmeg, 2006 as cited by Viholainen, 2011, p. 4).

A practical or informal proof is alternatively referred to as a semantic or private proof (Raman, 2003; Weber & Alcock, 2004). The central idea in both formal and practical definitions of mathematical proof is elimination of one's doubts about the truth of a proposition. It should be reiterated also that there is a distinction between mathematical proof and mathematical proving. Proving is a process or activity of removing doubts and may include trains of thoughts which may ultimately lead nowhere. Proof on the other hand is the product of the argumentation process which may lead to refuting or, alternatively, turning a conjecture into mathematical fact.

2.3.2 Forms of mathematical statements and sound mathematical reasoning

A mathematical statement or proposition is a sentence that is either true or false but not both (Bolstock, Chandler, & Rouke, 1992; Haggarty, 1992). For example the sentence: *5 is a prime factor of 40* is a proposition whilst the sentence; "*what is a subspace?"* is not a proposition because it cannot be evaluated to be either true or false. A proposition should therefore contain sufficient information as well as context in which its truth or falsity is to be determined. Mathematical statements can be classified as tautologies and predicates. The truth/falsity of a sentence involving a variable, x, depends on the context in which the variable is defined, for example, the statement $\int \frac{1}{x} dx = ln x + K$ is true if x is a positive real number but it is false if x is an element of all real numbers. Such statements are called predicates. On the other hand we have tautologies which are either true or false statements regardless of the context in which the statement appears (Haggarty, 1992). An example of a tautology is that of the fundamental trigonometric identity: $sin^2\alpha + cos^2\alpha \equiv 1$ for all values of the arbitrary angle α. The identity $sin^2\alpha + cos^2\alpha \equiv 1$ is a tautology because it holds for any arbitrary value of the angle picked unlike a predicate whose truth is determined by input values from the scope of the statement.

Several statements can be combined to produce a larger composite statement whose truth/falsity is determined by the constituent statements in addition to the manner in which the component statements are combined (Haggarty, 1992). This means careful attention must be given to propositional connectives. By propositional connectives, we refer to those words that link simpler statements to produce composite statements (Haggarty, 1992; Kirkwood, 1992). Examples of propositional connectives are: if….then, some or all, if and only if. Most mathematical statements can be classified as conditional or biconditional or non-conditional statements. This classification is described in the following section.

A conditional statement, alternatively referred to as an (if…then) statement, consists of two components, namely, the antecedent, alternatively known as a hypothesis or the "if part" and the conclusion, also referred to as the consequent component, that is, the "then part." The hypothesis is a statement that is assumed to be true (Bolstock , Chandler, & Rouke, 1992; Kirkwood, 1992). The consequent is a statement that can be deduced if the conditions of the hypothesis hold. The conditional statement: p implies q is symbolically represented $(p \Rightarrow q)$. An example of a conditional proposition can be illustrated by considering two fundamental concepts of continuity and differentiability of a real-valued function. Consider two statements p and q defined as $p\colon f$ is a differentiable function on an interval (a, b) and $q\colon f$ is a continuous function over $[a, b]$. The conditional statement formed by joining p and q is: If f is a differentiable function on (a, b) then f is continuous on $[a, b]$. This can be written as $p \Rightarrow q$.

To each conditional statement is a statement formed by interchanging the statements p and q, called the converse of the implication statement. It is written q implies p or p is implied by q. Symbolically the proposition p is implied by q is written $q \Rightarrow p$. We need to observe that a conditional statement and its converse are not necessarily of the same truth value. The converse of the above conditional statement is: If f is a continuous function on $[a, b]$ then f is a differentiable function over (a, b)— which is not a generally true statement because a continuous function is not always differentiable (Kirkwood, 1992). Another example is p: $x^2 = 4$; $q\colon x = 2$. The implication statement is: If p: $x^2 = 4$ then $x = 2$, which is necessarily not true because x can also take the value -2.

One can distinguish between the contrapositive and the inverse of a conditional statement. We define the inverse and the contrapositive of p implies q as follows. If both p and q are negated then the implication statement $\sim p \ => \sim q$ is called the inverse of $p \Rightarrow q$. Also if both p and q are negated then the implication statement written $\sim q \Rightarrow \sim p$ is referred to as a contrapositive statement. I conclude this section by observing that a statement and its contrapositve are logically equivalent. That means a statement and its contrapositive are either, both true or false (Bolstock, Chandler, & Rouke, 1992; Haggarty, 1992; Kirkwood, 1992). An example of the contrapositive of the composite statement: *If f is a continuous function on a connected set A then $f(A)$ is connected* is; *if $f(A)$ is not connected then is f not continuous on the connected set A*. The two statements are logically equivalent. That means in order to prove that; if f is a continuous function on a connected set A, then $f(A)$ is connected, one needs to prove the contrapositive of the statement. Next, I describe an unconditional statement. Consider statements such as p: the set of prime numbers is infinite and q: there is no smallest positive real number (Archimedean principle). These are examples of single component statements whose truth value can be determined using the same rules of inference applied to conditional statements (Bolstock, Chandler, & Rouke, 1992; Kirkwood, 1992).

A biconditional statement is now considered. Here statements such as $p: P(x, y)$ is a point on the circle, radius 2 and centre O and $q: x^2 + y^2 = 4$ were considered. The truth value of p and q when considered separately is unknown but we can say: if $P(x, y)$ is a point on the circle, centre O, and radius 2 then $x^2 + y^2 = 4$. The converse of this composite statement is also true provided the implication statement $p \Rightarrow q$ is true. Such a statement is called a biconditional statement. It is important to note that in a biconditional statement the conditional statement $p \Rightarrow q$ can be truthfully reversed provided that $p \Rightarrow q$ and $q \Rightarrow p$ are either both true or false (Haggarty, 1992). The linguistic equivalence of the statement should be: p implies and is implied by q or alternatively; p is a necessary and sufficient condition for q.

I now focus on what is meant by sound mathematical reasoning and typology of warrant types in mathematical arguments. To explain the meaning of sound mathematical reasoning, I begin by describing the structure of a mathematical argument. A mathematical argument is a set of propositions, one of which is the conclusion, and the rest of which constitute what is known as the premises (Curd, 1992). The premises are meant to support the consequent statement (conclusion), that is, the premises provide valid reasons for inferring that the conclusion is true. An argument is said to be valid if it is deductive and provides conclusive evidence about the truth of a conjecture (Stylianides & Stylianides, 2009, p. 239). A mathematical argument is said to be valid if it contains no errors (Weber & Mejia-Ramos, 2015). A valid argument is one in which the premises logically entail or imply the conclusion. An argument is deemed to be sound when it is valid and all its premises are true. The current study of student teachers' thoughts about mathematical proving sought to investigate the validity of students' arguments from students' proof attempts.

The formal meaning and practical meaning of mathematical proof both involve use of arguments in getting rid of the prover's doubts. As noted earlier an argument is a sequence of assertions in support of or against a mathematical statement. An argument can be conceived both as an element and a product of mathematical reasoning. It is thus crucial to examine the notion of argumentation in some detail. To see why and how an argumentation can be considered both as a product and an element of reasoning I refer to Toulmin's (2003) model of argumentation which stipulates that the goal of an argumentation is to construct an explanation, alternatively referred to as a warrant, for why the information pertaining the initial state (data) necessitates the statement which is being argued (conclusion). Toulmin further points out that sometimes the support of an authority (a backing) is needed in the argumentation process. On the basis of Toulmin's ideas presented above it can be noted that the goal of argumentation is to develop an explanation that justifies why the premises logically lead to the conclusion, which implies reasoning. Arguments can be categorised as formal or informal arguments.

An argument is said to be formal if its warrants are based on axioms, definitions and previously proven theorems, lemmas and corollaries (Weber & Mejia-Ramos, 2015). A formal argument usually entails rigor and detail and removes all doubts about the truth of a conjecture. So for true mathematical assertions, we can bestow formal deductive warrant types with a more prominent role in ensuring that one is convinced that an assertion is true because they provides conclusive evidence for the accuracy of an assertion (Weber & Mejia-Ramos, 2011).

On the other hand, an informal argument is one in which its warrants are based on concrete representations (imagery— visual or other representations) of the mathematical object (Presmeg, 2006). Raman (2003) classified arguments as private or public arguments. A public argument has similar features with a formal argument in the sense that it is based on rigorous constructions and rules of inference which must be shown step by step during its construction (Raman, 2003). Similarly, a private argument has the same properties as an informal argument in that it is based on empirical representation of mathematical objects like visual illustrations. The main purpose of an informal argument is to engender understanding by allowing holistic conceptualisations of mathematical concepts (Raman, 2003). Next, I examine warrant types a prover can produce to justify the truth or falsity of a mathematical claim.

Weber and Mejia-Ramos (2011) distinguish four types of arguments that an individual can use when proving. First, one can produce empirical evidence. An empirical justification involves use of mathematical objects drawn from a proper subset of the scope of the statement being proved. Second, an arguer can increase his/her confidence that a claim is true because an authoritative source such as the teacher has endorsed it. Third, one may attempt to determine the accuracy of an assertion by checking if the assertion is a consequence or property of the mental models one associates with concepts that constitutes the assertions. We call this warrant type the structural-intuitive where the prover refers to informal representations of the mathematical conjecture such as diagrams or images that one associates with those concepts. Finally, one may produce or observe a deductive argument that derives the claim from axioms, definitions and lemmas using socially acceptable mathematical techniques. Warrant types are now evaluated in terms of their capacity to validate given conjectures.

Empirical evidence, structural-intuitive warrants, and authoritative endorsements have significant limitations in mathematical argumentation. Empirical arguments may lead to false conclusions because the assertion might be true for the examples that one happened to consider but false for just one example that was not picked from the scope of the statement. This limitation about empirical evidence points to the critical role played by a counter-example in refuting mathematical claims. Similarly, a structural-intuitive justification may be misleading because one's mental models of the mathematical domain being studied might be inaccurate as suggested by Weber and Mejia- Ramos (2011). Weber and Mejia-Ramos assert that authoritative endorsements might be equally deceiving because the authoritative source might be mistaken. So for true deductive assertions, empirical, structural-intuitive and authoritative warrants should not be regarded as conclusive. However, Boero (1999) comments that empirical and structural-intuitive arguments are often useful in some stages of proving and conjecturing though they do not appear in the products of these processes (i.e., conjectures and proofs). In other words both structural-intuitive and empirical arguments do not result in absolute conviction regarding the truth-value of a deductive task.

Harel and Sowder (1998, 2007) reported that a mathematical claim remains a conjecture until one has acquired absolute conviction about its truth, otherwise one is said to hold relative conviction about the truth of a mathematical claim (Weber & Mejia-Ramos, 2015). A prover is said to have relative conviction in a mathematical claim if the subjective probability he/she will attribute to the statement being true exceeds a certain threshold to provide a warrant for further attempts to prove it (Weber & Mejia-Ramos, 2015). These warrant types are regarded here as real observable mechanisms that influence proof attempts by the student teachers.

I conclude this section by recalling that ontology is sometimes defined as a set of terms and their associated definitions needed to describe the phenomenon of interest. Thus the relevance of classification of arguments to the study of proof schemes of Zimbabwean undergraduate student teachers is that the kinds of arguments generated by students will be examined to address the research questions raised. This study will consider questions such as: what kinds of arguments do students produce when constructing proofs of mathematical statements? I now focus on the question: what methods are available for proving mathematical statements?

2.4 Methods of proving

We distinguish between direct and indirect methods of argumentations in proving (Bolstock, Chandler & Rouke, 1992, p. 155; Kirkwood, 1992). Under direct proof we have proof by direct deduction and proof by mathematical induction. Methods that employ indirect arguments include method of proof by contradiction and proof by use of contrapositive of the given conditional statement. I briefly describe each method of proving conjectures. Many theorems in mathematics take the form $p \Rightarrow q$, and to show that $p \Rightarrow q$ is true one usually adopts one of the following schemas.

2.4.1 Proof by direct deduction

First, we have the method of proof by direct deduction of the conditional statement $p \Rightarrow q$. With this schema one assumes that p is true and then, endeavors with the aid of some processes, to show that q is also true. We note that with this schema, since $p \Rightarrow q$ is false whenever one of either p or q is false, there is no need to consider the case where p is false. Second, we have the method of mathematical induction.

2.4.2 Method of mathematical induction

This schema is based on the well ordering property of natural numbers (Haggarty, 1992; Kirkwood, 1992). The well ordering property of natural numbers states that every nonempty subset of natural numbers has a least element and this is fundamental property upon which the schema is hinged. The principle of mathematical induction (PMI) (as the schema is often referred) comprises the following steps:

- Empirical explorations of the mathematical statement in specific instances, usually by finding the numerical value for $n = 1$. The statement can, however, be quantitatively evaluated for other natural numbers not equal to 1 depending on context of the problem. For example in the proof of the statement: Prove that $7^k - 6k - 1$ is divisible by 36 for all integers greater than 1. In this example, empirical verifications hold when the least value of natural numbers is 2, according to the well ordering property of subsets of the set of natural numbers.

- Upon completion of empirical verifications, one then makes an induction hypothesis. Here one assumes that the statement holds for $n = k,$ once again a natural number.
- Finally, one proves that the statement holds for $n = k + 1$. The stage just described is called the induction thesis stage.

After successfully going through the following steps: the base step that involves empirical explorations, inductive hypothesis stage and finally the induction thesis stage one then concludes that: because the proposition is valid for a possible initial value n_o and $n = k + 1$ after assuming that it holds for $n = k$, it can be concluded that the statement holds for all natural numbers greater than or equal to the least natural number from which empirical verifications can be performed— of course basing on well ordering property of the subsets of natural numbers. I now turn to indirect methods of proof.

2.4.3 Proving by use of contrapositive of the statement

This schema is an indirect method of rendering conjectures into mathematical facts (theorems, lemmas, and corollaries) by capitalising on the fact that a conditional statement and its contrapositive are mathematically equivalent. In other words, we are saying the conditional statement $p \Rightarrow q$ and its contrapositive, $\sim q \Rightarrow \sim p$ are both either true or false. Therefore to prove that $p \Rightarrow q$, one assumes that q is false and then establish through direct deduction described above that the negation of p is also false.

2.4.4 Method of proof by contradiction

The schema is also referred to as reducto ad absurdum (Bolstock, Chandler, & Rourke, 1992, p. 169; Haggarty, 1992). Let us begin this section by defining a contradiction. A contradiction is a statement that is always false regardless of the truth value of its constituent elements. Before describing the method, we need to recall that the negation of implication statement $p \Rightarrow q$ given by $\sim p \Rightarrow \sim q$ is logically equivalent to p and not q. To prove the conditional statement $p \Rightarrow q$, we first negate the implication statement $p \Rightarrow q$ and then replace the negation ($\sim (p \Rightarrow q)$) by its less clumsy representation p and $\sim q$ (Haggarty, 1992). Thus, for this argument, we assume that p is true and q is false and show by direct deduction a false statement referred to as a contradiction. This illustrates that the original hypothesis (p and ($\sim q$)) must be false. This then establishes that the statement (~p and ($\sim q$)) is false. But this is logically equivalent to $p \Rightarrow q$, which completes the argumentation process. A concrete illustration of this schema is proof of the Archimedean principle in Real Analysis. The Archimedean theorem states that the set of real numbers is unbounded. To prove the theorem, we let $S = \{ka: k \varepsilon N\}$ where a is a real number is. A prover then makes an assumption that S is bounded. We then use the axiom of completeness to show that this assumption will lead to a contradiction and hence S is indeed unbounded. Studies have demonstrated that students find this schema difficult to apply when constructing proofs.

2.4.5 Proof by use of counter example

Consider a statement p which is assumed to be true. To prove that p is false all we need to do is to produce one case that shows that p is false. This single case is called a counter example (Bolstock, Chandler, & Rourke, 1992, p. 171; Haggarty, 1992; Harel, & Sowder, 1998; Kirkwood, 1992). This schema also helps to reinforce the assertion that empirical verifications need not be elevated to the status of a mathematical proof as the following example will illustrate. For the conditional statement $p\colon x^2 = y^2$ implies $x = y$, the single case $x = 3$ and $y = -3$ can be used to show that p as stated is a false statement.

I conclude on methods of proving by observing that the method of proving by direct deduction is a common technique in the method of proof by contradiction, proof through use of contrapositive and method of proof by principle of mathematical induction. Perhaps this serves to explain why these methods often collectively referred to as deductive methods (Haggarty, 1992). Finally, I justify inclusion of the methods of proof construction though such an effort has been made already in the introduction. Let us recall that ontology is the study of objects and their ties. In mathematical proof construction (involving processes and relations referred to as ties in ontological terms), there is careful attention to detail as regards how basic mathematical objects lead to intricate generalisations, that is, ultimate existence, which is another ontological concern. Therefore arguments generated by Zimbabwean students will be examined for their rendition of detail in the proof construction processes.

Another point to make on how proofs are constructed is that while proof by use of counter example provides complete and conclusive evidence about the truth value of a mathematical conjecture, there is, need however, to exercise caution and not confuse empirical arguments (inductive explorations) with generic proofs. In generic proofs, proving of mathematical generalisations is demonstrated in particular instances. In a generic proof the validity of a mathematical statement is established through transformations on a mathematical object considered as the typical representation of the mathematical object involved in the conjecture (Morselli, 2006, p. 7). Unlike in inductive explorations, particular instances exploited in generic proofs offer complete and conclusive evidence about the truth value of a proposition. An example of a generic proof is that involving proof of the proposition: The set of real numbers is uncountable. To demonstrate that the set of real numbers is uncountable, the interval $I = [0,1]$ is used. That is a particular case is used and yet it offers complete and conclusive evidence that the set of real numbers is uncountable.

2.5 Processes involved in proving mathematical statements

In this section I focus on the question: how are proofs constructed? Before I address this question I begin by putting into proper context what is meant by proving a mathematical statement. Proving can be seen as a process of constructing an argument that justifies the truth-value of a given statement. The idea of an argument is used in the sense of justifying a conclusion based on data (Inglis & Mejia-Ramos, 2009; Toulmin, 2003). In this sense, mathematical argumentation can be viewed as the social activity of reasoning aimed at increasing or decreasing the acceptability of a controversial mathematical stand point. The controversial stand point is taken by the learner with respect to a given conjecture. Acceptability or alternatively refuting of the conjecture is decided on the basis of a constellation of propositions (data) that are adduced before a rational judge (Ubuz, Dincer, & B$\ddot{u}$b$\ddot{u}l$, 2013). The rational judge implied by Ubuz et al. (2013) is a member of the community of research mathematicians and educators whom I represented in the current study. Briefly, proving can be defined as the process of building an explanation about the truth value of a mathematical proposition. But how do proof aspects and modes of reasoning relate when constructing these explanations?

The process of proof construction depends on mathematical reasoning involved. We accordingly distinguish between syntactic and semantic proof productions (CadawalladerOlsker, 2011, p. 38). CadawalladerOlsker writes that syntactic proofs are those guided by formal rules of logic in an axiomatic system. In other words, in syntactic proof constructions the prover employs formal arguments, using symbolic manipulation of definitions, axioms and previously proven theorems in a logically permissible way. In semantic or referential proof productions there is use of different kinds of internally meaningful representations or mental images (instantiations) to guide reasoning in proof constructions (Alcock & Weber, 2005, p. 33). Therefore in semantic proof production, there is use of informal or private arguments. An important observation made from my interrogation of literature with respect to syntactic and semantic methods of proof construction is that there is an intricate interplay between semantic and semantic modes of proof production. The two methods of proof construction complement each other during proving. Although the two forms of reasoning usually support each other in the process of proving, research mathematicians often use semantic reasoning methods to identify and make sense of the mathematical properties and relationships they describe. I elucidate on provers' thinking processes by discussing the prevailing view among research mathematicians and mathematics educators about proof construction by considering Goethe and Friend (2010)'s description of analytic and axiomatic proofs.

The distinction between semantic and syntactic approaches to proof construction is strikingly similar to the distinction between the axiomatic and analytic methods of proof writing (Goethe &

Friend, 2010). Goethe and Friend define an axiomatic proof construction method as one that proceeds via axioms, definitions and previously established theorems to lead to a conclusion using sanctioned rules of logical inference. On the other hand, Goethe and Friend claim that an analytic method of proving resolves a mathematical problem by using a mixture of deductive moves and "induction" means to construct an explanation, (i.e., proving) about the accuracy (or lack thereof) of a mathematical conjecture. The word "induction" captured in Goethe and Friend is interpreted to refer to tables, diagrams and other visual displays which in this sense are some informal representations of the mathematical claim whose accuracy a prover seeks to establish.

Furthermore, Goethe and Friend (2010) describe analytic proofs as derivations of plausible hypotheses to mathematical problems. In the authors' sense a hypothesis is any means (whether deductive or inductive) of solving a problem where a problem is viewed as an open mathematical question. Goethe and Friend's description of a problem informed selection proof tasks as I ensured that tasks used for data collection were open to invite plurality in students' responses (Mamona-Downs & Downs, 2013). For a hypothesis to be referred to as being plausible it is necessary and sufficient that the hypothesis is compatible with existing data. Data denote all pertinent mathematical ideas that exist at the time of trying to establish the proof. In other words, data are the mathematical facts that are a foundation to our claim (Ubuz, Dincer, & B*übü*l, 2013, p. 134). I now comment on literature ideas on analytic and axiomatic methods of proving.

First, because the analytic method employs both deductive and "induction" moves the axiomatic method can be seen as unjustified truncation of the analytic method of proof construction since the axiomatic method of composing proofs is only a part of the analytic method where hypotheses stated at a certain stage in the form of axioms and definitions are unduly considered as an absolute starting point for the proving process (Goethe & Friend, 2010). Hence, along with Stylianides' (2007) view that a proof is a sequence of assertions for or against a mathematical proposition, I can deduce that the axiomatic method of proof is a subsequence of the sequence of assertions derived in an analytic manner.

Second, another comment I make here relates to similarities and differences observed between the analytic and axiomatic proof methods and the semantic and syntactic methods of proof discussed earlier. I observe first that the axiomatic method is similar in many respects to syntactic method in the sense that both methods emphasize the prominent role played by axioms and definitions and the structural mode of reasoning, which is the vehicle to proof construction with axiomatic and syntactic methods of proving.

Next, one can also discern that the semantic and analytic means of producing proofs share many similarities as implied by use of "induction" means which are construed to refer to tables, diagrams, empirical verifications which are particular instantiations captured in the definition of semantic methods of proving. Another strikingly similar feature of the semantic and syntactic methods of proving is that both are used by expert mathematicians. The use of semantic and analytic means to validate mathematical statements can be inferred from the following authors' remarks. Goethe and Friend (2010) say in workshops of research mathematicians who write proofs we generally find analytic proofs not axiomatic proofs. The point I would like to make here is that although many proofs are usually presented in syntactic form, expert efforts that go into their production involve the interplay between intuitive mathematical thoughts (semantic proving) and rigorous logical reasoning that corresponds to part of analytic thinking processes. Hence, both analytic and semantic proof methods uphold the importance of informal representations of mathematical ideas pertinent to the proof task faced by the arguer (Kidron & Dreyfus, 2014). Finally a key distinguishing feature between analytic and semantic methods of proof construction is that the analytic method explicitly calls for the use of axioms whilst the semantic method of proof insists on use of the referent mathematical objects in the scope of the mathematical statement.

I conclude on methods of proof construction by indicating that the literature on proof methods was important to this study because it provided a window used to determine students' thinking as they engaged with proof tasks assigned. For instance, the mode of reasoning involved in students' proof attempts could be determined by mapping students' written responses with Goethe and Friend's (2010) concepts of axiomatic and analytic methods of proof discussed here. Hence, this literature on methods and thinking processes involved in proving was important in determining the kinds of proof schemes held by student teachers which precisely was the focus of research question one.

To compose proofs, there are aspects provers need to handle technically using certain modes of reasoning. My theoretical perspective on the technical aspects of a proof and the modes of reasoning involved in proof construction draws from the works of Selden and Selden (2009) and Alcock (2010) respectively. Selden and Selden (2009) define five aspects of a proof an arguer needs to attend to when writing proofs: hierarchical structure, construction path, formal rhetoric part, the proof framework, and the problem oriented part. These aspects are now described.

Selden and Selden (2009) describe the hierarchical structure of a proof as developing awareness of what the proving efforts seek to accomplish. Such awareness includes being able to coordinate and construct sub-proofs and lemmas that are relevant to the proof task. For example, the cut property in $\mathbb{R}$ states that: *If an ordered pair* (A, B) *of nonempty subsets of* $\mathbb{R}$ *forms a cut in* $\mathbb{R}$ *then there* *exists a unique element* $\varepsilon \in \mathbb{R}$ *such that* $a \leq \varepsilon$ *for all* $a \in A$ *and* $\varepsilon \leq b$ *for all* $b \in B$. The hierarchical structure of the cut property includes being aware that ε is unique and also developing awareness of the need to satisfy the conditions $a \leq \varepsilon$ and $\varepsilon \leq b$. The other aspects that need to be coordinated in the proof construction exercise include the completeness axiom for bounded subsets of $\mathbb{R}$ and a call for the definition of a cut in $\mathbb{R}$. Further, the hierarchical structure also includes a call for the prover to coordinate definition of a cut, the order axiom and the rational density theorem that says: *Let* $x, y \in \mathbb{R}$ *with* $x < y$*, then there is a rational number* r *such that* $x < r < y$.

(Selden & Seden, 2009, p. 196) assert that the proof framework is the collection of the conventions of proving theorems in mathematics but it does not require one to define any terms that are associated with the logical structure of different methods of proving. For instance, for the proof method by contrapositive the prover is expected to be aware that one should start by negating the implication statement and then use the method of proof by direct deduction to lead to the conclusion, that is, the negation of the implication statement is used to derive the if part (i.e., antecedent statement), but one is not compelled to define what is meant by a contrapositive.

The construction path refers to the means by which the proof is actually produced. According to Selden and Selden (2009, p. 340) the construction path can be adequately described by efforts of an idealised prover who has never erred or followed false leads when composing proofs. Selden and Selden define the formal rhetoric component of a proof as one with a focus on predominantly behavioural aspects of a proving activity. It includes the ability to do algebraic and technical symbolic manipulations that are performed within the formal structure (scope) of the mathematical conjecture (Selden & Selden, 2009; Weber & Mejia-Ramos, 2015). So the emphasis on technical symbolic manipulations performed within the formal structure of the mathematical claim shows that the formal-rhetorical component is one the learner can produce by appealing to logic, definitions, other theorems without recourse to conceptual knowledge (Selden & Selden, 2009).

Finally, a prover is expected to handle the problem centred aspect of a proof. This component covers proving matters to do with conceptual understanding and problem solving activities pertinent to the proof task. Precisely, the problem oriented part deals with matters that are non-routine and for which there are no standard solution strategies. The problem oriented aspect calls for the prover's mathematical intuition and the ability to deploy the right resources at the right time (Selden & Selden, 2009; Fukawa-Conelly, 2012). Hence, different skills are required to construct the different aspects. However the two aspects are inter-related in the sense that by writing the formal–rhetorical part the other part, that is, the problem to be solved is exposed.

If we reconcile Sandefur, Mason, Stylianides and Watson's (2013) manipulating (M), getting a sense-of- (G) articulating (A), that is, (MGA) construct with Selden and Selden's (2009) aspects of a mathematical proof we can conceive manipulating (M) and articulating (A) as proof mechanisms or processes connected to the formal-rhetoric aspect of a proof, whilst getting-a- sense-of, (G), is a thinking process involved in proof construction tied to the problem-oriented part. Having described the components of a proof, I now consider how proofs are constructed, that is, how the components come into existence.

The aspects of proof in the foregoing discussion are supported by certain modes of thought which will be the focus of this section. There are four modes of reasoning put forward by Alcock (2010, p. 78), namely: instantiating, critical thinking, and creative thinking and structural mode of reasoning. A description of each mode of reasoning is now presented. Instantiating is when a prover meaningfully constructs and understands proof of a mathematical claim by manipulating mathematical objects drawn from the scope of the focal mathematical statement (Fukawa-Conelly. 2012; Weber & Mejia-Ramos, 2015). The mathematical objects consist of graphs, diagrams, numeric values. So instantiating as a mode of argumentation employed in proving is similar to the structural-intuitive justification type where a prover examines examples of the mathematical conjecture to see if it is a consequence or a property of the mental models one associates with the concepts embedded in the conjecture (Weber & Mejia-Ramos, 2011). Hence, instantiating as a mode of thought involved in proving is alternatively called the referential mode of thought by virtue of making reference to specific configurations to which the statement applies.

Creative thinking is defined as a process involving examining particular instantiations with the goal of determining the mathematical property that can form the crux of the subsequent proving process (Fukawa-Conelly, 2012). Finally, critical thinking is used as a mode of reasoning in those situations where a conjecture is converted into a fact, that is, it is proved through such means as questioning the implications of the mathematical claim or envisioning an example that would undermine the conjecture. In other words, the conjecture is refuted by searching for a counter example (Fukawa-Conelly, 2012).

The three modes of thought described so far belong to the semantic or referential mode of proof construction (Weber & Alcock, 2004, 2009). In a referential mode of proof construction the prover attempts to produce a proof through some transformations using particular or generic examples drawn from the scope of the mathematical statement (CadawalladerOlsker, 2011; Weber & Alcock, 2009). In other words, a prover attempts to produce a proof by linking aspects of a mathematical claim to configurations in another representation system of the statement such as diagrams to build an informal explanation (Kidron & Dreyfus, 2014). The informal explanation will then be expressed in a formal manner at a later stage.

Another approach to proof construction qualitatively distinct from the semantic approach is the syntactic approach. In a syntactic proof construction, a prover starts from axioms, definitions and previously proven theorems and builds an argument within the representation system, that is, the scope of the statement until a conclusion can be deduced in a logically permissible formal deductive manner without instantiating (Kidron & Dreyfus, 2014; Fukawa-Conelly, 2012; Weber & Alcock, 2009). A formal deductive justification (i.e., public argument) supporting a mathematical claim is a sequence of assertions that concludes with a mathematical statement. According to Kidron and Dreyfus (2014), each assertion from the sequence is a claim that is known or is assumed to be true or is purported to be a logically necessary consequence of the preceding assertion (Weber & Mejia-Ramos, 2015). The validity of a deductive argument depends on whether each assertion in the sequence contains an error. So if each assertion does not contain an error then we call it a valid deductive argument.

Further, one of the assumptions that undergird research in mathematics educational practice is that a valid deductive argument is the final word on the matter concerning the proof because it provides conclusive evidence about the accuracy of a mathematical claim. In other words, it is a socio-mathematical norm of the mathematics education community of research mathematicians who write proofs that once a valid deductive argument has been generated it then becomes superfluous to seek further confirmatory evidence to establish the truth-value of the mathematical statement (Goethe & Friend, 2010; Weber & Mejia-Ramos, 2015). Hence, once a valid deductive conclusion has been established one is expected to hold absolute conviction about the truth-value of a claim. However, in stark contrast even when learners become capable of formal deductive reasoning the effects of lower level proof schemes continue to linger in the minds of the learners (Fischbein, 1994 cited in Kidron & Dreyfus, 2014). Having described the syntactic approach to proof construction, I now examine the mode of thought associated with this approach.

We call the mode of reasoning associated with syntactic proof construction the structural thinking mode according to Alcock (2010, p. 78). Alcock describes structural thinking as that mode of reasoning in which one draws on definitions, axioms, lemmas and other pertinent ideas to the conjecture to build justifications in a purely public or formal manner (Fukawa-Conelly, 2012; Raman, 2003). In other words, in structural thinking proof construction occurs within the formal structure of the scope of the mathematical proposition without recourse to informal modes of thinking such as instantiations of relevant mathematical ideas of the conjecture.

2.6 The notion of Cognitive Unity of Theorems

In this section I first examine an important theoretical construct known as the Cognitive Unity (CU) of theorems. Next I discuss crucial processes involved in conjecture formulation and the related proving process. Such processes include the notion of dynamic exploration of the problem situation and transformational reasoning. The CU of a theorem is an attempt to describe and interpret processes involved in conjecture formulation and also an attempt to explain proving process (Garuti, Boero & Lemut, 1998, p. 345). Key emphasis of the construct is on continuity between the formulation of a mathematical conjecture and the possible construction of its proof. Basic ideas of the construct are summarised by (Garuti et al., 1998) as

> during the production of a conjecture, the student progressively works his/her statement through an intensive argumentative activity functionally intermingled with the justifications of the plausibility of his/her choices. During the subsequent statement proving stage, the student links up with this process in a coherent way, organizing some of the previously produced arguments (p. 345).

The intensive argumentative activities that are a typical feature of the Cognitive Unity of theorem construct involve dynamic explorations of the problem situation by the student. Dynamic explorations can be conceived as imagined or concretely performed transformations on space configurations (Garuti et al., 1998). Dynamic explorations may be either goal-oriented or non-goal directed during conjecturing and proving. The distinction between the two kinds of exploratory activities is that in non-goal directed argumentation the learner would be 'testing the ground' without knowing exactly what to find. This usually occurs at the initial phases of proof construction. Typical proving activities at this phase may be in the form of numerical tests and reflecting on those examples (Morselli, 2006, p. 6). The process of reflecting on examples when exploring the problem situation is essential as it helps in identifying the underlying mathematical property that forms the crux of the proof.

On the other hand, goal-oriented exploratory activities in proof construction are anticipatory in nature and may involve pictorial representation of mental actions in the sense suggested by (Thompson, 1994). Here the learner strives to derive relevant information that deepens one's understanding of the problem situation thereby potentially leading to proof construction or refutation. Exploration of the problem situation at this level can lead to the generation of new statements easier to prove (Boero, 1999).

Transformational reasoning (CadawalladerOlsker, 2011; Harel & Sowder, 1998) is closely connected to the exploration of problem situation. Transformational reasoning involves envisioning the change of a mathematical situation and anticipating results of such a change. A particular instance involves the transformational use of symbols in proof construction. In such cases proofs are accomplished by translating from verbal language to algebraic language after which standard algebraic manipulations are performed. One's interpretation of the final formula then validates the statement (Garuti et al., 1998; Stylianides, 2009). In other cases the learner may not know algebraic language and in such circumstances transformational reasoning is done using natural language. In such situations the learner explores the statement using natural language before transforming it (Boero, 1999).

2.7 What counts as a mathematic proof?

Having constructed the proof by either syntactic or semantic means or even by both forms of reasoning the next concern is on determining the status of a proof. The question here was: which forms of arguments qualify as proof? The concern here was on evaluating the status of a mathematical proof. A mathematical proof employs forms of reasoning (modes of thoughts or argumentation) that are valid and known to or within the conceptual reach of the mathematical community, that is, the proof should be acceptable to mathematicians of impeccable reputation (Stylianides, 2011, p. 2; Weber & Mejia-Ramos, 2015). In other words, the argumentation should be consistent with the conventional understanding of mathematics as proposed by Stylianides (2011, p. 2). It uses statements that are accepted by the classroom community- a social process (Stylianides, 2007). Such statements are considered as true and available without further justification.

We distinguish arguments just described from empirical verifications/arguments. Empirical arguments are inductive explorations which are based on the use of specific examples that offer confirming and yet incomplete evidence about the truth or falsity of a mathematical statement (Stylianides, 2009). Consider again the conjecture: Prove that 7^k – 6k -1 is divisible by 36 for all integers greater than 1. Empirical arguments consist of verifying that the statement holds for $k = 2$, and, $k = 3$. These arguments do not qualify as proofs because they do not provide complete evidence that the statement is a multiple of 36. Hence, such arguments should not be elevated to the status of a proof despite their importance in identifying patterns, generating conjectures and giving insights on what needs to be proved.

2.8 The notion of a proof scheme

2.8.1 Taxonomies of proof schemes

Ideas discussed in this section will assist in evaluating students' arguments in this study on the kinds of proof schemes and how such proof schemes develop. Several studies have led to different classifications of students' responses to tasks on proofs of mathematical statements. These include van Dormolen's (1977) taxonomy cited and Balacheff's (1988) taxonomy of mathematical proof, Harel and Sowder's (1998, 2007) taxonomy of proof schemes. I briefly describe the proof scheme classifications.

2.8.2 van Dormolen's (1977) taxonomy of mathematical proof

Varghese (2011, p.182) posits that van Dormolen (1977) differentiated three categories of mathematical proof which are outlined below.

- Use of a particular example.
- Use of example as a generic embodiment of a mathematical proof.
- Use of general and deductive arguments in proving.

2.8.3 Balacheff's (1988) taxonomy of mathematical proof

According to Varghese (2011, p. 182), Balacheff's taxonomy of mathematical proof is in fact an extension of van Dormolen's categorisation scheme. Balacheff created two broad categories by distinguishing between pragmatic and conceptual mathematical justifications. In pragmatic justifications a student' doubts about a mathematical conjecture are eliminated by focusing on use of examples, actions or "showings." That is, there is use of empirical explorations in establishing the truth or possibly refuting a conjecture. Three pragmatic justifications in hierarchical order are naïve empiricism, crucial experiment, and generic example.

Within the Naïve empirical proof scheme, a mathematical conjecture is validated by checking in a proper subset of all possible cases in its domain. The inductive explorations in this category are selected primarily on the basis of a few examples chosen out of convenience (Stylianides, 2011, p. 1). The crucial experiment class of proof scheme is also characterised by empirical verifications in a proper subset of all possible but there is one major distinction. In the crucial experiment proof scheme, the choice of cases is now based on some rationale such as the need for a counter example, as opposed to mere convenience as in the former. In a generic proof as alluded to earlier general arguments are illustrated in a particular case seen as a prototype (Harel & Sowder, 1998, p. 243; Stylianides, 2011, p.1; Varghese, 2011, p.184). A generic proof does not have empirical status as it provides adequate conclusive evidence that a mathematical claim is true. The highest level in Balacheff's taxonomy in terms of mathematical sophistication is the thought experiment which is composed of conceptual or intellectual justifications. With thought experiments, actions are internalised and dissociated from specific examples considered (Stylianides, 2011, p.1).

Both van Dormolen and Balacheff's taxonomies show increasing levels of mathematical sophistication. The two taxonomies demonstrate a natural tendency in student thinking to move from inductive towards the deductive modes of thought and towards greater generality (Varghese, 2011, p.184).

2.8.4 Harel and Sowder's taxonomy

Harel and Sowder (1998, 2007) give a detailed proof scheme taxonomy comprising three main categories which in turn consists of numerous subcategories. The categories are mutually interrelated and represent cognitive levels ―intellectual abilities. The main categories in hierarchical order are: the external conviction proof scheme, the empirical proof scheme and the analytical proof scheme.

To recall, the external conviction proof scheme is the lowest cognitive level where conjectures are validated through

- an appeal to the form or appearance of an argument without due regard to correctness of the argument, Harel and Sowder describe this subcategory as the ritual proof scheme,
- approval of an authority such as a textbook or a word uttered by a teacher or an individual the learner perceives to be highly knowledgeable in the discipline and accordingly was named the authoritative proof scheme,
- Symbolic reasoning: Here there is manipulation of mathematical symbols without paying regard to meaning of the symbols, that is, without establishing a coherent image of the problem context in the sense of (Harel & Sowder, 1998, 2007) ―symbolic proof scheme.
- The next cognitive level is the empirical proof scheme. The empirical proof scheme (CadwalladerOlsker, 2007; Harel & Sowder, 1998) is one by which conjectures are validated, impugned and/or subverted by an appeal to sensory experiences or physical facts. This cognitive level of the proving process has two subcategories, namely, inductive and perceptual proof scheme. The inductive proof scheme is an empirical proof scheme wherein conjectures are validated by quantitatively evaluating them in one or more specific instances. Such arguments by their nature should not be elevated to the status of a proof as discussed earlier because they do not provide complete and conclusive evidence about the truth of a conjecture (Stylianides, 2009). Harel and Sowder have identified some interrelated factors that foster the inductive proof scheme. Among them is the psychologically natural tendency to evaluate conjectures probabilistically, the adverse effects of the authoritarian proof scheme, the absence or lack of advanced proof scheme, and the natural dislike by learners of method of proof by contradiction.

In the perceptual proof scheme, there is use by the learner of rudimentary mental images which consist of perception and a coordination of such perceptions. However the images lack the ability to transform or anticipate the results of a transformation (Weber & Alcock, 2007). By mental images we mean the ones (Thurston, 1994) refers to as supporting “thought experiments,” that is, they facilitate mathematical reasoning. The development of a perceptual proof scheme can be considered to be the initiation of higher order theoretical proof schemes which I now turn to.

The third main category of proof schemes is the analytical proof scheme with which the learner renders conjectures into mathematical facts. The analytical proof scheme has two broad subcategories namely, the transformational proof scheme and the axiomatic proof scheme (CadwalladerOlsker, 2011; Harel & Sowder, 1998). The idea of transformational proof scheme as examined under transformational reasoning involves use of mental images. Transformational proof schemes are characterised by operations on objects and an anticipation of the results of the operations (Harel & Sowder, 1998). Harel and Sowder point out that the transformational proof scheme is the foundation of all theoretical proof schemes. Among the various subcategories of the transformational poof scheme we have are, the interiorised proof scheme, the internalised proof scheme and the contextual proof scheme. The axiomatic proof scheme, a higher cognitive level scheme than the transformational proof scheme, is one by which conjectures are rendered into facts or possibly rejected by basing logical deductions on a collection of axioms (basic principles comprising undefined terms and defined terms in the sense of Harel and Sowder.

As mentioned earlier the taxonomies of proof schemes represent cognitive levels which show increasing levels of mathematical sophistication. The main goal of this study is to gain insights into the kinds of proof schemes held by undergraduate student teachers as well as developing hypothesis as to how those proof schemes emerge. In order to realise this goal I intend to examine the kinds of proof schemes by mapping students’ arguments to Harel and Sowder’s (1998, 2007) taxonomy. The mapping of students’ responses will be done in an effort to address various ontological concerns such as: what ultimately will exist (development of proof schemes) and in what terms do the students think about the notion of proof that is, addressing the question of what constitute the students’ mental constructs around the notion of mathematical proof. The classification by Harel and Sowder is preferred to other proof scheme categorisations because it has been noted to be detailed and highly comprehensive (Balacheff, 2008, p. 504; CadawalladerOlsker, 2011, p. 44).

As concluding remarks on this section I note that there are other categorisations of proof scheme such as the one by Stylianides and Stylianides (2009). Such a taxonomy also show similar characteristic to those described under van Dormolen , Balacheff and Harel and Sowder taxonomies wherein empirical verifications form the lowest level of mathematical reasoning whilst the highest level is dominated by conceptual justifications.

2.8.5 How student teachers' proof schemes could be inferred?

This study intends to explore students' thinking around the notion of proof. It seeks to develop an explanation, grounded in data, for the kinds of proof schemes held by Zimbabwean undergraduate student teachers as well as developing a proposition about how students' proof schemes emerge. The crucial question here was: how could we infer the kinds of undergraduate students' proof schemes from their argumentations about mathematical statements given the latent nature of the universe of discourse (UoD), which is the proof scheme? Put differently, the concern of the study was: how could undergraduate student teachers' thoughts of mathematical proof be evaluated? In other words, what could be the philosophical guide used in drawing inferences about the terms in which undergraduate student teachers think of mathematical proof?

Dunn and Mearman (2006) define ontology as the general nature of reality and the types of entities existing in it. Ontology is also about the ultimate existence of reality. The word reality refers to whatever it is in the universe; forces, structures, concepts which causes the phenomena we perceive with our senses. Schwandt (1997) describes scientific realism as a school of thought that denotes the precise position to the question of how a scientific theory should be conceived. In the context of this study the word reality refers to the object of investigation, that is, the concept of mathematical proof and its underpinnings. Drawing on Dunn and Mearman's definition of ontology and Schwandt's perspective of scientific realism, I can say that the concern of the study was to determine the nature of reality (i.e., ontology of proof scheme) and how it evolves as captured by the phrase "ultimate existence of reality" in Dunn and Mearman's ideas about ontology. Hence, there is need to determine the nature of students' thinking around the concept of mathematical proof in order to determine how a deep understanding of mathematical proof can be enhanced amongst the learners.

Ontology is the structure of the nature of reality according to Lawson (2009). Further, phenomenon at one level may warrant explanation in terms of phenomena lying at a deeper level. Lawson here cautions against the assumption and temptation to assume that all causes lie at the surface. Hence, the point is that a deep understanding of students' argumentation schemes can be developed if qualitative data are analysed within the framework of the realist methodology. The term "scientific activity" in the context of this study refers to proof construction activities and processes of understanding by the students. Further, "scientific activity" is also used in this study to describe how their proof experiences shape their conceptualisations of mathematical proof.

In realism a distinction can be made between the object of knowledge (social entity) and the social process that produces that knowledge as pointed out by Dunn and Mearman (2006). In scientific realism this distinction is called the intransitive and transitive dimensions of knowledge. This is a critical point in realism because it allows researchers to differentiate social entities from the knowledge we have about these entities and from the conventions of science that produced that knowledge. The notion of intransitive and transitive dimensions of knowledge is applicable to this study in the following manner. The object of knowledge or "social entities" captured in this definition refer to knowledge of mathematical proof and the conventions of science that produce this knowledge designated here as social processes refer to the mathematics community which includes research mathematicians and students (Stylianides, 2007). In this study the undergraduate students who participated in this study and I formed the mathematics community. According to philosophical realism the mathematical entities, referring here to concept of mathematical proof and its underpinnings have an objective existence as can be inferred from Maxwell and Mittapalli's (2007, 2010) comment that the world is the way it is but is not objectively knowable. This school of thought is called realist ontology which is a commitment to the existence of an objective reality that is however not objectively knowable (Maxwell & Mittapalli, 2007, 2010). Thus, scientific realism asserts that there can be more than one correct way into which reality can divided into categories (Maxwell, 2004). This realist position is in stark contrast to scientific objectivism which asserts there is only one correct way of knowing the real world.

The realist position presented implies that reality has an independent existence. Hence, in van Fraassen's (1980) view, a realist with respect to a given theory or discourse holds that, the sentences making that theory are either true or false. van Fraassen writes that what makes these sentences either true or false is something external, that is, the truth-value of a mathematical theory from a realist perspective is not determined by our sense data whether real or imagined or our language. An implication of this realist stance for the study is that the term "something external" signifies the objective existence of mathematical knowledge that is represented by the proof conceptions of the community of mathematicians who compose proofs. I represented the community of mathematicians. I emphasize the point that there can be more than one scientifically correct way of acquiring this knowledge which exists independently of our minds (Maxwell, 2004; Maxwell & Mittapallli, 2010). The terms "our sense data" and "language" refer to students' proof attempts and utterances as they engaged with the object of investigation. The primary goal of the study was to measure the discrepancies between research mathematicians' conceptions and the students' conceptions.

I followed van Fraassen's (1980) conceptualisation of scientific realism. Van Fraassen asserts that scientific realism provides an answer to the question of what it is to accept or hold as scientific theory. The importance of van Fraassen's ideas to the study is that the realist process approach that will be employed in this study will allow me to determine the kinds of proof schemes held by the students as well as enabling me to trace the emergence of students' proof schemes. However, it is still not very clear how scientific realist ideas will make it possible to pursue these research goals so I now address the question; how will the realist process theory approach make it possible to address the research questions?

The realist process approach employed in this study asserts that mental events and processes involved in proving are real observable mechanisms and processes (Maxwell, 2004; Maxwell & Mittapali, 2007, 2010; Patton, 2001). A huge question that remains to be tackled is: how can the observation of causal relationships be made in light of the caution by Dunn (1978) in Maxwell (2004) who warns that "there are still no cheap ways to deep knowledge about other persons and the causes of their actions" (p. 171). Similarly, Becker (1966, p. 69) cited in Maxwell (2004) commenting on direct observation of events and processes in scientific inquiries argues that "Observable, yes, but not easily observable, at least for scientific purposes." Hence, in an attempt to address these concerns about direct observation of causative mechanisms and processes from students' proof events, I will capitalise on a key psychological construct called a schema and its implications for mathematical proving.

A schema is a key psychological framework concerned with the issue of organized content knowledge in the human mind (Tall, 2008). In simple terms a schema is a coherent collection of objects and processes (Tall, 2008). A schema is a cluster of organised knowledge, in its various forms such as strategic and syntactic, that helps learners to understand and represent a given problem and provide cues for the activation of relevant strategies during the solution process (Tall, 2008). Further, we observe that a person's knowledge of a particular mathematical concept refers to the individual's tendency to invoke a particular schema in order to deal with, understand, or make sense out of a perceived problem (Tall, 2008). It can be discerned here that the learner's schema cannot be observed directly because of its internal nature— in the human mind. The nature of one's proof scheme can only be inferred from an individual's actions as he or she may or may not bring his schema to bear on problems. It follows therefore that the kinds of proof scheme held by undergraduate student teachers will be inferred from their actions expressed in written, verbal or behavioural forms as the learners interact with proof tasks. Scientific realism asserts that mental events can be inferred from one's behaviour (Maxwell, 2004).

2.9 Mathematical understanding

This study aims to explore students' thinking about the concept of mathematical proving and how students' experiences with the notion of mathematical proof shape their conceptualisations of mathematical proof. This goal of the study was pursued by evaluating students' understanding as they constructed mathematical proofs; hence it is crucial to address the question: what is mathematical understanding? Mathematical understanding can be seen when a student brings together different types of mathematical resources to engage in the complex interaction between rigorous and intuitive thought that is characteristic of the mathematicians' practice. Further, mathematical understanding is not only a matter of possessing and connecting between varied knowledge resources (i.e., "items"), but also requires an awareness of exactly what purposes are served by these resources (Michner, 1978; Wilkerson-Jerde & Wilensky, 2011). Hence, in the next section I discuss and illustrate key notions involved in use of mathematical resources as a way of showing how mathematical understanding can be discerned.

To clarify the meaning of what understanding mathematical proof entails, I draw on Hanna's (2000) idea of mathematical understanding and appreciation to describe the key elements of proof using the notions of key ideas in the form of conceptual insights and technical handles, depth and width of a proof, rote and generational memory (Hanna & Mason, 2014; Sandefur, Mason, Stylianides & Watson, 2013, p. 329; Raman, 2003). Hanna suggests that a mathematical proof is most valuable to a prover when it leads to real mathematical understanding. Real understanding of mathematical proof is differentiated from superficial understanding of mathematical proof which is characterised by rote memorisation and reproduction of uncoordinated and unrelated facts about proof. In this section I discuss with exemplifications the key elements of proof which are the potential indicators of a profound mathematical understanding.

Tim Gowers in Hanna and Mason (2014) uses the metaphor of width to describe the pseudo-measure of distinct pieces of information one has to keep in mind at any one time in order to be able to (re) construct and follow reasoning in proofs. Depth of a proof is captured by fresh ideas and conceptual insights that come to mind in sequence in order to complete a proof (Hanna & Mason, 2014). Fresh ideas or conceptual insights associated with the notion of depth of a proof are in Raman's (2003) terms known as key ideas. Key ideas are subdivided into conceptual insights and technical handles (Hanna & Mason, 2014, p. 146; Sandefur et al., 2013 p. 328). A conceptual insight alternatively known as a heuristic idea or 'surprise calculation' in Gowers' terms is an idea that aid thinking by illuminating a sense of the structural relationships that indicate why a mathematical conjecture is likely to be true or false (Sandefur et al., 2013, p.328). The term technical handle, also known as a procedural idea, is used to denote the specific techniques for manipulating symbols or diagrams and specific forms of calculations used to map a heuristic idea into a mathematical proof (Raman, 2003; Sandefur et al., 2013, p. 328). Examples of technical handles in mathematics include: the techniques for solving a system of linear equations by Gauss-Jordan elimination, technique for finding an orthogonal basis for a subset S of a finite-dimensional vector space V over a field K using the Gram-Schmidt algorithm, and the method for finding a basis and the dimension of the solution space of a homogeneous system of linear equations.

The notions of conceptual insights and technical handles (key ideas) provide some measure of degree of precision for (re) constructability of a proof. Key ideas are indicative of depth of a proof. The notions of conceptual insights and technical handles are the major focus of this study that aims to determine the kinds of proof schemes held by Zimbabwean undergraduate pre-service teachers and the manner in which those proof schemes emerge. The utilization or non-utilization of key ideas by undergraduate students will provide a window through which the nature of theorem and proof images can be determined. To increase clarity on what proof understanding entails, the ideas of depth and width and key ideas of a proof illustrated in the following examples. I re-cap that heuristic ideas and technical handles are collectively known as key ideas.

An example on the concepts of depth and width of a proof adapted from Hanna and Mason (2013, p. 6) is now shown.

$$\begin{array}{r} 47 \\ \times 63 \\ \hline \end{array} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

$$\begin{array}{r} 2820 \\ +\ 141 \\ \hline 2961 \end{array} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

$$47 \times 63 = (50-3)(50+3) + 10 \times 47 \ldots\ldots\ldots\ldots\ldots\ldots (iii)$$

$$= 2500 - 9 + 470 \ldots\ldots\ldots\ldots\ldots\ldots (iv)$$

$$= 2961 \ldots\ldots\ldots\ldots\ldots\ldots (v)$$

The need for multiple digits in steps (i) and (ii) of the computation has been removed in steps (iii) and (iv) through the evocation of the insightful idea of concept of difference of two squares. In Gowers' terms the conceptual idea, that is, difference of two squares has reduced the large-width operation. In other words width has been replaced by depth. We now state and exemplify how conceptual insights and technical handles are employed to construct proofs by discussing the following theorem and its proof.

Theorem: *There is a real number, x, such that $x^2 = 2$*

Proof

Let $S = \{y\varepsilon R : 0 \leq y, y^2 \leq 2\}$. First observe that S $\neq \emptyset$, e.g., 1 ∈S Next, observe that S is bounded above e.g., it is bounded above by 2. If 2 was not an upper bound then $\exists s \in S$ such that $2 < s$ which implies that 4< $s^2 \leq 2$. Here 4 is strictly less than 2, a contradiction and hence 2 is an upper bound. So by the axiom of completeness, S has a least upper bound. Let this be x. Clearly, $x > 0$, 1 is in S and x is an upper bound. Claim: $x^2 = 2$. If claim was false then either $x^2 < 2$ or $x^2 > 2$. We want to prove that neither of the two assertions is true.

If $x^2 < 2$ then 2 - $x^2 > 0$ leading to $\frac{2-x^2}{2x+1}$> 0. ……………………………………………(i)

By corollary 3 to the Archimedean principle, there exists a natural number, n, such that $\frac{1}{n} < \frac{2-x^2}{2x+1}$ which simplifies to $x^2 + \frac{2x}{n} + \frac{1}{n} < 2$ leading to

$x^2 + \frac{2x}{n} + \frac{1}{n^2} < x^2 + \frac{2x}{n} + \frac{1}{n} < 2$…………………………………………………….(ii)

Factorizing we obtain $\left(x + \frac{1}{n}\right)^2 < 2$. Hence, $\left(x + \frac{1}{n}\right) \in S$. This is a contradiction since x is a supremum of S, so the assertion that $x^2 < 2$ is false.

If $x^2 > 2$ then $x^2 - 2 > 0$ leading to $\frac{x^2-2}{2x}$>0. ……………… (iii)

Once again by corollary 3 to the Archimedean principle ∃ a natural number m such that $\frac{1}{m} < \frac{x^2-2}{2x}$, which simplifies to 2< x^2-$\frac{2x}{m}$.

Now 2< x^2-$\frac{2x}{m} + \frac{1}{m^2}$ ………………………………………………………………...(iv)

and factorizing gives 2 < $\left(x - \frac{1}{m}\right)^2$. Since $x$is a supremum of S, $\exists s \in S$ such that $2 < \left(x - \frac{1}{m}\right)^2 < s^2$, a contradiction since 2 is an upper bound of S and hence the assertion that $x^2 > 2$ is false. Therefore the claim that $x^2 = 2$ is true, and hence there is a real number x such that x = $\sqrt{2}$.

In Gowers' terms there are steps in the proof of the existence of $\sqrt{2}$ that seem to "spring' from nowhere" such as the choice to divide 2 – x^2 by the quantity (the notion of a 'surprise calculation') $2x + 1$ in step (i), to get $\frac{2-x^2}{2x+1} > 0$. This is a crucial conceptual insight which allows application of technical handles that is, the subsequent factorization process. Dividing by $2x + 1$ is both convincing and legitimate (Hanna, 2000) because we see here the order properties of R being utilized in proof of square root of 2. Ideas being employed in the "surprise calculations" $\frac{2-x^2}{2x+1} >$

0 and $\frac{x^2-2}{2x} > 0$ emanate from a consideration of R as an ordered field. The specific order property giving the conceptual insight is: $If\, a > b$ and $c > 0$ then $ac > bc$ where a, and b are arbitrary elements picked from the real field R. Other conceptual insights include: $\frac{x^2-2}{2x} > 0$, $\frac{1}{n^2} < \frac{1}{n}$. Such conceptual insights allow the prover to apply results of the axiom of completeness in the proof of the existence of $\sqrt{2}$.

The next question is: how do the key ideas arise during proving? Key ideas come to mind either through rote memory or generational memory. When key ideas strike a prover's mind through generational memory he/she just remembers a few important ideas about a proof and then develops a technical skill to convert them quickly into a formal deductive proof. With generational memory ideas do arise naturally as if without effort (Hanna & Mason, 2014). The proof construction process is generational when there is no expenditure of intentional and explicit effort as connoted by the term "as if without effort" in the definition. Generational memory is differentiated from rote memorisation by noting that in rote memorisation there is recourse to one's ability to remember unrelated and uncoordinated steps (Hanna & Mason, 2014). For instance from personal experience of undergraduate mathematical analysis, I always remember with excitement my last minute exigency in memorising the Bolzano-Weistrass theorem in order to replicate it in the examination without appreciating the depth of reasoning involved. Thus we should not equate the ability to replicate a theorem and its proof with the ability to re-construct and hence understand the proof (Hanna & Mason, 2014). From this discussion of key elements of mathematical proof construction, I present the definition of mathematical proof understanding that will guide this study.

Hanna and Mason (2014) define the term understanding of a mathematical proof as having access to a theorem so as to be able to make use of both the form or method of proof and the result itself (the conclusion) in other situations as well as to be able to read, follow, reason with, appreciate, comprehend and re-construct other theorems that depend on the theorem (Balacheff, 2008; Hanna & Mason, 2014). The term accessing a theorem means that the conceptual insights forming the depth of the proof come to the mind of the prover in sequence as if without effort and are accompanied by relevant and accessible technical handles (Hanna & Mason, 2014; Sandefur et al., 2013). From this definition it can be seen that understanding of proof is closely aligned to generational memory. However, this definition is used while being well aware that the terms mathematical understanding and explanation are somewhat elusive terms. According to Hanna (2000) mathematicians and mathematics educators acknowledge that there is such a thing, most of them actually share the view that proof becomes both convincing and legitimate to a prover when it leads to understanding, that is, helping the individual to think clearly when solving mathematical problems (Hanna, 2000, p. 7).

Hanna points out that thinking more clearly and effectively about mathematics is in its own sense elusive just as the word understanding and hence difficult to ascertain. It is perhaps useful to turn to Tim Gowers' conceptualisation of mathematical understanding.

Hanna (2000, p. 7) cited in Gowers (2007) holds the view that mathematicians must see proof not as a syntactic derivation, that is, as a sequence of sentences in which each sentence is an axiom or the immediate consequence of preceding sentences by application of rules of inference. Hanna proposes that mathematicians should see proofs as primarily conceptual, with the specific approach being secondary. Hanna seems here to give prominence to conceptual insights and devalue technical handles, although lack of technical facility can be just as much an obstacle as not having conceptual insights proving (Hanna & Mason, 2014). Hence, I emphasise here the importance of the intricate interaction between the intuitive semantic mode of reasoning and rigorous reasoning. Intuitive reasoning can be seen by the utilisation of conceptual insights whilst the later form of reasoning can be seen by manipulation of axioms, definitions and previously proven mathematical results.

The discussion of mathematical understanding has revealed that mathematical knowledge can be seen as a network of resources in the form of conceptual insights and technical handles between which mathematicians traverse when composing proofs. We can therefore assert that learning new mathematics can be thought of as the creation of a network of mathematical resources as suggested by Sierpinska (1994). Duffin and Simpson (2000) in Wilkerson-Jerde and Wilensky (2011) further categorise mathematical understanding by differentiating building, having, and enacting as different components of mathematical understanding. Duffin and Simpson describe building as an aspect that refer to the process of developing the connections, having denotes the state of the connections and enacting is used to describe the process of applying the connections available to solve a problem. We observe that the link between Sierpinska's description of the network of mathematical resources and Duffin and Simpson's categories of mathematical understanding is that having; building and enacting can be viewed as mechanisms by which relationships in the network can be established and used in resolving proof tasks. I leverage on this literature on mathematical understanding by using it as a window to generate insights on students' conceptualisations of mathematical proof. I now elaborate on Sierpinska's conceptualisation of mathematics proof learning as the creation of a network using some concepts from the Real Analysis course. A network of concepts from Real Analysis at undergraduate is presented in Figure 1.

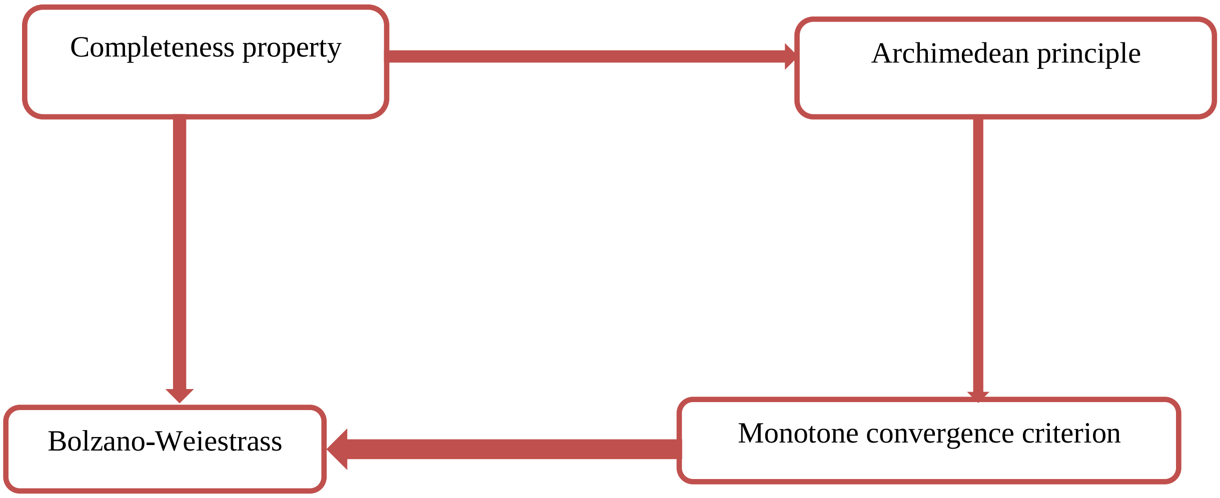

Figure 1: Mathematical proof as a network of concepts

Figure 1 illustrates how central the axiom of completeness is in proof and proving. The axiom of completeness states that *every bounded subset of real numbers has a least upper bound (supremum) or a greatest lower bound (infimum) accordingly as it is bounded above or below*. This notion is needed in the proof of the Archimedean principle by contradiction. The Archimedean principle says the set of real numbers is unbounded. The Archimedean principle is used together in re-constructing the proof of the theorem: *a bounded monotone sequence of real numbers converges either to its least upper bound or infimum,* depending on whether it is a monotone increasing or decreasing sequence. This gives us our first criterion for convergence of a sequence. The theorem on bounded monotone sequence can then be used to prove Bolzano Weistrass theorem: *a bounded infinite set of real numbers has at least one limit point.* Thus knowledge of bounded monotone sequences is used in conjunction with ideas of a limit point of a set and a sub-sequential limit point to construct the Bolzano-Weistrass Theorem which will be used to read, follow reasoning, and re-construct other theorems that depend on it.

Informed by the discussion of the term mathematical understanding and metaphors of depth, width, and the notion of mathematical knowledge as a network of connected resources, the meaning of mathematical proof understanding that was considered in the study refers to having access to key ideas (in the form of technical handles and conceptual insights) forming the depth of a proof in order to construct, validate, and complete proof tasks. This definition was used in the study to determine how student teachers' proof construction attempts illuminated the kinds of proof schemes held by the pre-service teachers— corresponding to research question one. Students' narrations of their proof experiences as they went through various scholastic levels were scrutinized in order to determine how students' proof schemes evolved— corresponding to research question two. I used the perspective of mathematical understanding described here, bearing in mind the following considerations. Proving as a mathematical practice should strive to achieve a balance between two often competing and conflicting considerations: mathematics as a discipline and students as learners (Ball et al. 2008; Stylianides & Stylianides, 2009, p. 239).

Regarding the consideration of mathematics as a discipline, a mathematical proof should provide complete and conclusive evidence about the truth of a conjecture through deductive means (Stylianides & Stylianides, 2009) for true mathematical assertions while the search for counterexamples should prevail when a prover is dealing with false assertions. The notion of mathematics as a discipline thus denotes use of axioms and definitions and formal deductions as valid modes of argument presentation and appropriate modes of argument presentation. Here mathematical proof needs to be conceived in the context of what is typically agreed in the domain of mathematical theories— the so called conventional understanding of mathematics suggested by Stylianides (2011, p. 2). Regarding the consideration of students as learners, it is desirable that proof and proving activities be within the students' conceptual reach. The concerns of this study with respect to the two considerations were: identifying proof qualities resembling those associated with formal deductive arguments among undergraduate student teachers' constructions and validations that may be closely aligned to semantic or private arguments (Raman, 2003). Qualities identified were used to determine the terms in which undergraduate student teachers think around the notion of mathematical proof. According to realism such universals exist independent of the act of perception and hence they had to be unravelled through examining students' argumentations on tasks involving proof and proving.

2.10 Conceptual Framework

A conceptual framework is a representation in narrative or graphical form of the main concepts and/or variables involved and their presumed relationships (Punch, 1998, 2005). A conceptual framework is defined as a collection of interrelated concepts which guide the study, determining which variables or concepts a researcher will investigate and how she/he will interpret data. In the current study, the guiding philosophy is scientific realism whose focus is on investigating causal relationships through direct observation of mechanisms and processes connecting students' proof events. The conceptual framework refers to a collection of interrelated concepts that are the focus of the study. I reiterate that the concern of the investigation was not measuring the influence of one variable on another. In other words, I did not employ the variance theory approach (Mohr, 1982, 1996). Instead, a realist process understanding of causality was employed to observe causal links that could account for the kinds of students' proof schemes as well as how their proof experiences could be used to determine how proof schemes evolve. These goals of the study were compatible with the realist process approach. The thesis title is: *Undergraduate student teachers' conceptualisations of mathematical proof.* From this title the main constructs investigated are:

(i) mathematical proof and its underpinnings including the concept of a proof scheme,

(ii) students' pre-university proof experiences,

(iii) students' undergraduate proof experiences

(iv) students' proof construction abilities

(v) students' conceptualisations of mathematical proof.

The main conceptual strands are mathematical underpinnings, student proof construction abilities, students' proof experiences, taxonomies of proof schemes. What is the ontology of the student's proof scheme, that is, the structure of nature of existence of the student's proof scheme? How can we characterise the mathematical object in terms of underpinning concepts? What can be known about these constructs amongst Zimbabwean undergraduate student teachers'? The underlying assumption was that by addressing these questions I would generate insights into students' conceptualisations of mathematical proof. Figure 2 shows the conceptual framework in graphical form which I constructed from the theoretical underpinnings of the concept of mathematical proving within the realist process perspective.

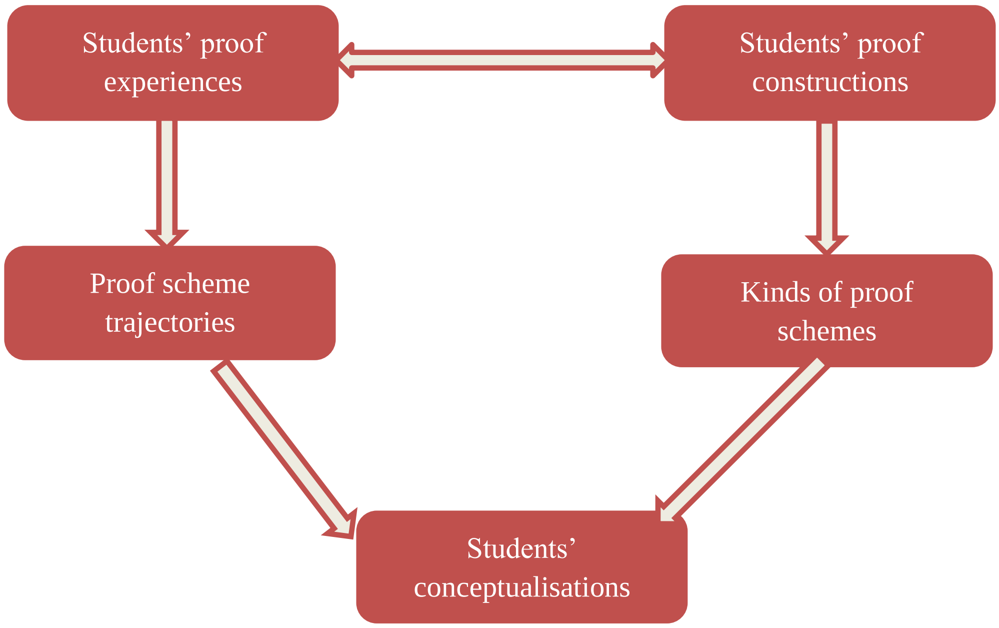

Figure 2: Conceptual framework of students' conceptualisations of mathematical proof

The line with arrows at both ends connecting the constructs: proof experiences and proof construction abilities is illustrating the interrelatedness between the two constructs investigated. I argue that students' pre-university and university experiences influence their proof construction abilities and vice versa. Some of the theoretical constructs that guided the study are Ausbel's theory of learning and Tall's (2008) notion of a met-before. I drew on these constructs to illustrate the connections between a learner's proof experiences and his/her proof construction ability. Ausbel's theory of learning points out that learning takes place by assimilating new information into existing conceptual structures of the learner (Varghese, 2009, p. 6). This process of information filtering depends on students' met-befores, which is a collection of the student's prior knowledge, beliefs, prejudices, preconceptions and misconceptions (Tall, 2008). An exemplification of a robust met-before in proving is the tendency to remove doubts about conjectures through empirical verifications and this has been reported to resist strongly any attempts to switch to conceptual justifications. A student with a robust empirical proof scheme is likely to exhibit a propensity to evaluate mathematical statements through use specific examples from the scope of the mathematical proposition.

Students' expressions of terms in which they think about the idea of mathematical proof could be inferred from their proof experiences as well as from proof productions examined through the lenses of taxonomies of proof schemes. For example, Harel and Sowder (1998, p. 243) point out that it is possible to switch from one level of proof scheme to another. Further, it is possible for a student to hold more than one proof scheme during the same encounter. However, the taxonomy is silent about the nature of such consistencies and inconsistencies in student's proof schemes henceforth called ontological oscillations (CadawalladerOlsker, 2011; Wand, Storey & Weber, 1999). Hence, this study aimed at accounting for students' ontological oscillations through the realist process approach that holds that these students' behavioural tendencies are real observable phenomena that can be explained in terms of causative mechanisms and processes (Maxwell, 2004, Maxwell & Mittapalli, 2007; Yin, 2009).

Given the impossibility to directly observe the development of students' concept images on mathematical proof, we shall use the idea of a learning event. A learning event is said to have occurred when a student communicates and applies his/her new understanding of the concept of proof. Closely linked to the notion of a learning event is also the idea of concept usage drawn from Moore's (1994) model of concept-understanding schemata. Concept usage refers to the ways a student uses a concept in generating examples or in doing proofs (Housman & Porter, 1997, p. 141). The constructs described guide this study of the kinds of proof schemes held by undergraduate student teachers. The students' proof events were scrutinised through the lenses of the underpinnings of mathematical proof such as technical handles, conceptual insights, micro reasoning and intellectual need in order to determine their attention to detail in an effort to determine the nature or kinds of proof schemes that characterise undergraduate student teachers' conceptualisations of mathematical proof.

I conclude this chapter by summarising the main goals of this study of students' proof schemes which are;
(i) What is the nature of Zimbabwean undergraduate student teachers' proof schemes and what can be known about them in terms of modes of being, characteristics? The objective here is to get insight into the nature of the students' formal praxis, that is, established habitual practice with respect to the notion of proof construction and develop an explanation for the formal praxis so established.

(ii) Further, the study also sought to determine how the mathematical object evolves amongst undergraduate students. In the process of achieving this goal I attempted to address pertinent questions such: what features of proof schemes are essential as opposed to mere accidental attributes among undergraduate student teachers' emerging categories of proof schemes? The categories are the ways in which the proof scheme can be addressed such as its ‘’ whatness’’ or essence. The idea, here, is to get insights into the structure and form of the mathematical object as it evolves.

Finally, by merging an explanation of the formal praxis (i.e., the kinds of proof schemes), and an account of how the proof schemes emerge I intend to develop an explanation about the terms in which students think around the concept of mathematical proof. Precisely the aim was to establish a set a causal links in the ways student teachers think of mathematical proof. In other words, the main goal was to develop an explanatory theory about students' conceptualisations of mathematical proof.

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Chapter Three

Related Studies

This chapter presents results of literature survey in connection with students' learning of mathematical of proof. In the process of examining related literature an attempt was made to explicate the gap the study sought to fill. There are two main sections covered by this chapter. The first section focuses on specific previous studies with a discussion of each piece of literature reviewed structured under the following headings:

- The research problem,
- What the study sought to accomplish including theoretical considerations that informed efforts to realise the articulated goals,
- Data collection procedures,
- Data analysis,
- Findings/conclusions from the study.

When each piece of literature was examined, I paid particular attention to flaws and strengths of the study in relation to the current study thereby illuminating the research gap as well as informing research design and analytical framework used in the present study. The second section covers other pieces of literature reviewed about mathematical proof and proving for the same purpose of explicating the research focus of this study in the following areas: research instruments, data collection procedures, data analysis and reliability and validity matters.

3.1 Studies on mathematical proof and proving

First, I reviewed an article by Weber and Mejia-Ramos (2011) on, *why and how mathematicians read proofs: an exploratory study*. With the goal of establishing research mathematicians' motivations and strategies for reading published proofs, Weber and Mejia- Ramos used semi-structured interview guides to capture reasons why research mathematicians read proofs and how they behave when trying to comprehend published proofs. To accomplish this goal the semi-structured interview guide included the following question, *what do you think it means to understand a proof?* The purpose of posing the question was to tease out the sort of behaviour demonstrated by the research mathematicians when reading proofs.

I drew from Weber and Mejia-Ramos' semi-structured interview guide. I modified the question item just described in order to explore students' proof behaviour during reflective interviewing. Hence, a similar item in the reflective interview guide was crafted as: *In your opinion what is a mathematical proof?* It was anticipated that students descriptions of their conceptions of mathematical proof would be parallel with proof behaviour demonstrated during their proof attempt to given tasks. The connection between students' conceptions of proof and proof behaviour is in line with the realist stance that mental events and processes are real entities that are causally relevant to the explanation of individual and social behaviour (Maxwell & Mittapalli, 2007).

For data analysis Weber and Mejia-Ramos used the following technique. The question posed was first stated. Then an excerpt of the exchange that took place between the research mathematician (interviewee) and the interviewer was presented. For instance the following excerpt involving the interviewer (1) and one of the research participants with the code (M4) went on something like this:

> 1: What do you hope to gain when you read this proof?
>
> M4: Okay. Two things. One is I would like to find out whether their asserted result is true, or whether I should believe that it's true. And that might help me, if it's something I'd like to use, then knowing it's true frees me to use it. If I don't follow their proof then I would be psychologically disabled from using it. Even if somebody I respect immensely believes that it's true. More importantly, I want to understand the proof technique in case I can use bits and pieces of that proof technique to prove something that they haven't yet, that the original author hasn't yet proved (p. 334).

This exchange was then followed by researchers' comments on the participant's motivation and strategies for reading published proofs (for details see Weber and Mejia-Ramos (2011, p. 334)). The data analysis strategy just described was found strategic for my study when drawing meaning from verbatim transcription texts of reflective interviews on students' proof attempts to assigned proof tasks and recorded student utterances of students' proof experiences. That is, I treated written responses and verbatim transcriptions in a similar manner and then followed by researcher comments just as was the case with Weber and Mejia-Ramos (2015).

A perusal of Weber and Mejia-Ramos' result section revealed that their study uncovered many reasons for research mathematicians' motivations and strategies for reading published proofs such as searching for new ideas, checking for utility and originality of ideas generated in theorems read (pp. 335-336). While I acknowledge that these are important insights with respect to efforts to understand the learning of mathematical proof I would like to take cognisance of the fact that these findings were based on participants' evaluations or validations of proofs supplied to them and not their own productions. Azrou (2016) has emphasised that it is important "to analyse students written tasks produced individually" (p. 81). Therefore the current study responded to this dearth in research by exploring student teachers thinking about mathematical proof based on students' own "voices" by engaging students with proof tasks with the intention of eliciting students' thoughts about proof and proving.

Second, a report of a research study by Stavrou (2014) on: *Common errors and misconceptions in undergraduate mathematical proving by education undergraduates*, was reviewed. The study that took place in the context of two proof laden courses, namely, Number Theory and Abstract Algebra had two main goals. One of the main goals was to identify common errors and misconceptions made by the students when proving. The other major goal was to determine how students' proof behaviour would change when student teachers were made aware of those errors. Accordingly data collection proceeded in two phases that I now describe.

Phase one of the data collection procedure involved examining homework given to 97 students for the purpose of compiling errors made by the students in the homework. I observed some flaws in this procedure. The students worked on assigned tasks at home, so independent reasoning was most likely to have been compromised as students could present 'workings' from other sources. Novelty of proof tasks assigned was also likely to have been compromised because Stavrou (2014) stated that participants proved routine statements covering basic Number Theory and Abstract Algebra (p. 2). In the current study, such flaws were avoided by asking students to work on assigned proof tasks individually in the mathematics lecture room in order to get independent students' reasoning.

Phase two of data collection involved 91 new students who had been informed of the common errors compiled from phase one responses. The purpose of informing them of errors was to detect any changes in proof behaviour as a result of that awareness. The participants were then asked to work on the assigned task after being made aware of the errors made by the previous group.

For data analysis, Stavrou presented the students' written responses and used those written solutions to describe the common errors and misconceptions as well as identifying emerging patterns in students' proof behaviour after they had been made aware of common errors and misconceptions. Similar to the analytic technique employed by Weber and Mejia-Ramos (2011) presented solutions were then followed by researcher comments that consisted of Stavrou's descriptions of common misconceptions identified from presented solutions and changes in students proof behaviour. Stavrou's analytic techniques were considered strategic in my study that sought to determine the kinds of students' proof schemes in the students. Along with Stavrou's data analytic techniques, written responses were presented in a similar manner and then used to construct students' proving profiles. However, as discussed earlier some flaws were noted in the methodology that includes assigning tasks as homework that could have compromised students' actual voices. Informed by such flaws, I then sought to explore students' mental constructs around the notion of a proof on the basis of students independent reasoning (Duval, 2006). Major findings from Stavrou's study are now briefly described.

With respect to research question one to which Stavrou sought to identify common errors made by students, the most common error was violation of proof framework (Selden & Selden, 2009). Students assumed that the conclusion was true before producing evidence to support that the conclusion was true yet the premises should logically imply the conclusion (Styliandles & Stylianides, 2009). Another common error was that students only proved the implication statement of a biconditional statement and did not consider the converse of the statement. To find out how students' proof behaviour change as a result of being aware of common errors made by the previous group, seven prospective teachers identified using categories of errors that emerged from the study of written responses were then interviewed using a semi-structured interview. With regards to research question 2 to which Stavrou sought to determine changes in undergraduate students' proof behaviour, Stavrou's study revealed that students proceeded to use specific examples after producing valid deductive arguments further revealing the tenacity of the empirical proof scheme among students. In other words, the education undergraduate students had relative conviction about proofs of statements produced (Weber & Mejia-Ramos, 2015). Findings from the current study would be compared with these findings for similarities and differences for the purpose of building an explanation about the kinds of proof schemes held by student teachers and how the proof schemes emerged (Corbin & Strauss, 2008).

Third, Doruk and Kaplan's (2015) research report on*: Prospective mathematics teachers' difficulties in doing proofs and causes of their struggles with proofs,* was reviewed. Before discussing Doruk and Kaplan's study under the headings outlined earlier as research goal (problem), participants, data collection, data analysis and results, I begin by outlining students' difficulties with proof documented in Doruk and Kaplan's research report. Some of the difficulties students have with proofs include;

- Not knowing how to make a proof structure using definitions
- Being unable to use concept images
- Not knowing how to begin the proof construction process

Doruk and Kaplan's main goal was to uncover prospective teachers' difficulties with mathematical proof as well as revealing reasons for such difficulties from the perspective of the actors, i.e., the prospective teachers. A case study was considered strategic for the purpose of studying prospective teachers' difficulties and their evaluations of the causes of such difficulties.

With respect to data collection the prospective teachers were asked to prove a theorem drawn from topology of real numbers. Precisely the students were asked to prove that, *every neighbourhood is an open set.* Students' responses were in written form. The data collection technique by Doruk and Kaplan was considered to be also strategic for the current study because documenting students' proof attempts have been suggested to be effective in illuminating students thinking (Manilla and Wallin, 2009). Hence, the same data collection technique was adopted in the current study. However I somewhat questioned the novelty of the statement given that it appears to be routine since it is one of the basic ideas in most mathematics literature texts on topology of the real line. Written responses from 121 prospective teachers were assessed for correctness with the assistance of other specialists in Real Analysis.

Doruk and Kaplan's semi-structured interview guide included the question: *what is the reason for this difficulty in a person?* This question was posed with the intent to develop an account of the difficulties noted in written responses to the task that required students to prove that, *every neighbourhood is an open set.* I found this data collection technique to be strategic for the current study that sought to account for the emergence of proof schemes among students. Doruk and Kapalan's interview guide informed the formulation of the reflective interview guide questions used to address research question two in the current study, *how do undergraduate students develop their proof schemes?* For instance, a review of Doruk and Kaplan's data collection section helped me to craft questions such as: *what differences have you noticed at various scholastic levels in the way proofs are done?*

For content analysis of written responses, Doruk and Kaplan first presented the written responses and then described the difficulties they inferred from the student's proof attempts. Doruk and Kaplan's data analytic technique was also considered to be strategic for the current study. However I hold the opinion that numbers involved in Doruk and Kaplan's study were too large for one to apply content analysis efficiently. The study involved 121 students during the initial phase. Emphasis in case studies should be on in depth study of few cases. The researchers involved 7 students in the interviews for the purpose of developing an account of the difficulties identified when research question one was addressed. This was a strategic case study data collection technique which the current study derived from since it involved 10 students.

Major findings from Doruk and Kaplan's study are now described. Regarding difficulties encountered with proof the study revealed that prospective teachers were unsuccessful in proving. One of the difficulties encountered in proving related to use of definitions in proving. Prospective teachers failed to state definitions correctly and failed to organize the definitions into a valid proof.

Second, prospective teachers failed to use mathematical language correctly. For example, students confused notations and language because they did not grasp ideas involved in the propositions. In such situations pre-service teachers failed to pay attention to the scope of the statement to be proved. The consequences of such superficial understanding of proof concepts resulted in pre-service teachers resorting to proving other propositions instead of the statement in question.

Interview transcription texts were used to determine students' thoughts of the causes of their difficulties with mathematical proof. Interview data revealed that prospective teachers were aware of difficulties they encountered with proofs. The teachers expressed that they experienced negative feelings when they struggle with proofs. Negativity about proving stemmed from the teachers' belief that proving was a useless activity that had no practical value in their lives. So one's belief has an influence on one's ability to prove statements. This finding parallels Furinghetti and Morselli's (2009) finding. Interview data also revealed that the pre-service teachers preferred to learn proofs by rote memorisation of facts and they invested little study time to mathematical proof and proving. Pre-service teachers attributed the difficulties they experienced with proofs to the way proofs are taught. These findings would be useful in interpreting students' proof experiences at various scholastic levels from transcription texts from reflective interview data.

Fourth, Uğurel, Morali, Koyunkaya and Karahan's (2016) study on, *Pre-service secondary teachers' behaviours in the proving process*, was examined. The major goal of the study was to generate insights about the kinds of proof behaviours revealed by pre-service teachers when attempting to prove a given proposition. Uğurel et al. studied 15 volunteer pre-service teachers of whom 5 were male and 10 were female. To solicit data, each pre-service teacher was asked to prove a given proposition on the chalkboard by thinking aloud as the student teacher presented the proof. Researchers used the first 5-7 minutes to remind the participant of the purpose and content of the study. Researchers present in the room in which the pre-service teacher was presenting encouraged the student to explain his/her thoughts during the think aloud interview protocol.

The data collection technique employed was considered strategic in conducting the current study. It would be used in the current study for students' chalkboard demonstrations of their proof attempts during both mid-instruction and end of instruction assessment data collection phases. However, for the purposes of triangulating data, that is, locating an unknown point from more than two known points, the chalkboard demonstrations were followed by reflective interview audits of the chalkboard demonstrations, something missing from the study by Uğurel et al. (2016). During data collection Uğurel et al. did not impose time restriction on students as they engaged in think aloud interview protocols. Along with Uğurel et al. the current study would not also impose time restrictions on students during chalkboard demonstrations. However, in the study by Uǧurel et al. data collection was a one day event which made it impossible for the study to derive benefits associated with prolonged engagement in the research setting that would have allowed Uǧurel et al. to uncover less visible aspects about students' proof behaviour difficult to unravel in a single day (Lewis, 2009; Maxwell, 2004; Maxwell & Mittapalli, 2007). Further, collecting data on a single day was likely to result in conversational fatigue on the part of participants and even the researchers themselves. To counter the effects described here, data collection for the current study would take place over four days.

For data analysis, the researchers started by transcribing data from the video recordings of the think aloud protocols. Uǧurel et al. then applied content analysis to the transcription texts. The researchers applied content analysis separately and then met to discuss emerging themes and codes. They agreed on three main themes from their separate content analyses of the data namely; behaviours shown before proving, behaviours exhibited when proving, and use of particular instantiations throughout the proving process. Each emerging theme had two or more sub-themes (for details see Uǧurel et al. 2016, p. 210). For the current study, the three major themes from Uǧurel et al. informed the crafting of the observation guide for the chalkboard demonstrations in that I divided the observation guide into sections that captured student proof behaviour before, and during chalkboard demonstrations. Along with Uǧurel et al., I also checked for instances in which students used particular instantiations during written responses and chalkboard demonstrations.

For data presentation and analysis Uǧurel et al. used a table that had 2 columns in which the first column identified the participant and second column entries consisted student's utterances from think aloud protocols. The tables used by these researchers to display their results formed the basis of the data matrices I employed to draw meaning from students' written responses and chalkboard demonstrations about the kinds of proof schemes held by the students and how those proof schemes evolved among the students. Similar to Weber and Mejia-Ramos' (2011) and Stravrou's (2014) analytic techniques, the tables by Uǧurel et al. were accompanied by some researcher comments. The researcher comments consisted of descriptions of student proof behaviour inferred from column 2 transcription texts captured from the think aloud protocols. I would employ the same analytic techniques in the current study for the purpose of describing proof scheme elements identified which would then be subsequently mapped to existing literature during efforts to develop an explanatory theory about students' schemes of argumentation when proving mathematical statements.

From the results section, the major proof behaviours revealed by Uğurel et al. are now described. Regarding the first theme, main findings include reading the given proposition aloud and making some comments aimed at comprehending the given proposition. Proof behaviour demonstrated before proving also included efforts to express the proposition in students' own words. Efforts to express the proposition in students' own words included attempts to interpret language and meaning of mathematical symbols. Regarding the second theme, that is proof behaviours shown during proving, pre-service teachers experienced difficulties in expressing their own opinions about proof construction particularly, when those opinions involved abstract ideas concerning the proposition. Further, ambiguous statements were a common feature of this major theme (see Uğurel. et al., 2016, p. 216).

With respect to use of particular instantiations when proving, Uğurel et al. found that some students could produce examples by themselves while others could do so through some guidance from the researchers. It was also concluded that the pre-service teachers resorted to use of examples after failing to use mathematical representations to construct the proof of the proposition. The purpose of employing examples was to concretise the proposition in order to interpret it. These findings were significant to the current study since they provided some lens to evaluate students' proof attempts to the assigned proof tasks.

In conclusion it can be noted that while I appreciate efforts by Uğurel et al. to apply the emic approach, that is, to determine students' proof behaviour by engaging students in proof construction process during the think aloud interview protocols, the fact that data collection took place in a single day and that just a single proposition was used in data collection point to the need for further studies that will allow the 'voice' of the student to be heard in order to illuminate the kinds of student's schemes of argumentation during proving. Hence, more studies in which the student's voice is prominent are vital to avoid what Hennink, Hutter and Bailey (2013) call a mere passing mention of an event with respect to proof and proving in mathematics education. So in the current study, I responded to this paucity in studies based on the student's actual proof productions through prolonged involvement in the research setting.

Finally, another piece of literature reviewed is a study that focused on: *Providing written feedback on students' mathematical arguments: proof validations of prospective secondary mathematics teachers* by Bleiler, Thompson and Krajčvski (2014). The study had two aims, one of which was to evaluate the effectiveness of an instructional sequence in improving pre-service teachers' abilities to validate arguments produced by high school students. Another aim of the study was to determine the sort of errors pre-service mathematics teachers, (PSMTs), attend to when validating mathematical arguments purported to be proofs by high school students. To pursue these goals the researchers raised the following research questions. (i). *In what ways do PSMTs' validations of students' written arguments differ before and after the implementation of a set of structured activities? (ii). To which errors in authentic samples of high school students' written mathematical arguments do PSMTs attend when engaging in proof validation?*

Bleiler et al. then designed and implemented a sequence of intervention activities with the intent of increasing the pre-service teachers' awareness and skills in validating mathematical arguments. The design of the sequence of activities was informed by two critical ideas drawn from the three researchers' survey of literature on students' behaviour during proof validations. First, students have shown a tendency to use inductive arguments to prove mathematical propositions. Second, the instructional activities were influenced by the finding that teachers tend to focus on local components of an argument rather than focusing on reasoning and the logic sustaining the entire argument (Knuth, 2002). The consequences of focusing on the specifics as opposed to considering the proof as a holistic entity include the following proof behavioural tendencies. In some cases a biconditional mathematical statement was considered to be valid when a proof of the implication statement $p \Rightarrow q$ has been provided without proof of its converse, $q \Rightarrow p$. The purpose of the study by Bleiler et al. was thus to determine the effectiveness of an instructional sequence that gave particular attention to these limitations in students' proof behaviour. Precisely, the study sought to determine whether the instructional intervention could ameliorate the proclivity by students to use particular instantiations and to evaluate the tendency to focus on local aspects instead of considering the proof as a holistic entity. These critical ideas from literature provided a window through which students' proof behaviour could be determined from the data sources for the current study about students' thinking of mathematical proof and how such thinking evolved among students.

Data collection took place over a period of 3 semesters. During the 15 week long semester, researchers met with the students for 3 hours per week. Unlike in studies of Doruk and Kaplan (2015) and Uǧurel et al. (2016), Bleiler et al. had a prolonged engagement with the pre-service teachers. The current study drew on the idea of prolonged interaction with study participants. In the study by Bleiler et al. the instructional sequence was in implemented in 5 phases which I now describe.

During phase one (pre-instruction phase) of the instructional sequence, pre-service teachers reflected on high school students' written arguments. In the subsequent three phases pre-service teachers engaged in activities meant to change their proof validation behaviour in light of critical ideas derived from literature; the tendency to focus on local components instead of global features of an argument and the tenacity of the empirical inductive proof scheme among pre-service teachers. For example, during phases 2,3 and 4 the instructors held discussions with the pre-service teachers in which the intent was to motivate contexts on which the PSMTs could employ proof methods by deduction and refutation. For instance, in phase 2, pre-service teachers discussed and validated arguments used in Martin and Harel's (1989) study.

Finally, there was phase 5 (post-instruction) where PSMTs evaluated mathematical arguments for the purpose of determining the impact the instructional sequence had on proof validation behaviour. Both the pre-instruction and post-instruction proof validation activities had explicit instructions that pre-service teachers had to follow. These include the fact that a specific grading criterion for the arguments had to be followed by the PSMTs. First, the pre-service teacher was required to assign a numeric score to the argument that ranged from 1 up to 4. Scores 1and 2 were used to rate unsuccessful proof attempts while scores 3 and 4 were used for successful proof attempts. A score of 4 was used for an error-free proof construction effort (see Bleiler et al. 2014, p. 113 for details on scoring criterion). After scoring, the PSMTs were each asked to write a short paragraph justifying the scoring he/she had done. Further, for wrong answers, PSMTs were asked to explain why the argument was wrong and to suggest how students could improve their solutions.

I employed a similar data collection technique with undergraduate student teachers involved in the current study when conducting reflective interviews on students' proof attempts. During the reflective interviews, students were asked to explain why they had decided that the given statements were either true or false and provide some justifications (further details are in the data collection procedure section of Methodology in Chapter Four).

The data presentation style by Bleiler, Thompson and Krajčvski (2014) was similar to the way Uǧurel et al. (2016) presented their data, that is, tables were used to facilitate the data reduction process (Miles, Huberman, & Saldana, 2014). Bleiler et al. presented data in tables that contained the argument validated by the pre-service teachers. The researchers' judgements of proof validation effort were then shown in parentheses. The data reduction technique by these researchers informed the construction of data matrices in the current study. However, the tables were modified to include a separate column for researcher comments and another column for researcher inferences of proof scheme elements instead of using parentheses. For further details about how tables from studies by Bleiler et al. and Uǧurel et al. (2016) were modified to facilitate data reduction in the current study refer to Chapter Four on data analysis procedures section. I now describe the data analysis procedure.

Data analysis was conducted per research question. Research question one sought to determine the discrepancies between pre-service teachers' pre-instruction and post-instruction proof validations. To address research question 1 pre-instruction PSMTs' proof validations were compared with post-instruction PSMTs' argument reflections for similarities and differences. It can be inferred that Bleiler et al applied Corbin and Strauss' (2008) constant comparison analytic tool. The current study would employ the same analytic tool used by Bleiler et al. to compare Mid-instruction assessment verbatim transcriptions with End-of-instruction assessment transcription texts of written responses to proof tasks. A similar process would be used in the current study for the purpose of detecting changes in students' proving behaviour as a result of the teaching experiment conducted in the current study. Data analytical tools for research question two, *To which errors in authentic samples of high school students' written mathematical arguments do PSMTs attend when engaging in proof validation?,* are now described.

With respect to research question 2 to which Bleiler et al aimed to determine the sort of errors to which the pre-service teachers attended to, the researchers were assisted by two mathematics educators and one mathematician in answering it. The researchers and the assistants created a list of errors separately and then met to discuss their compilations. When a similar error was identified they used 'expert consensus' to formulate a composite statement that could be used to characterise that error. The activity of comparing individually compiled lists of errors for the purpose of composing was some form of data reduction technique. The current study would be influenced by this notion of composite ideas by Bleiler et al when main observations from student proof profiles would be used to construct composite tables of students' proof behaviours (for further details see data analysis section of the methodology chapter).

I emphasise the point that the study by Bleiler et al. used data based on pre-service teachers' proof validations of arguments supplied to them by the researchers. This justified the call for more studies based on students' own proof constructions instead of reflecting on arguments supplied by the researchers. Hence, the current study sought to fill this gap in studies based on students' own constructions. I now describe efforts by Beiler et al. to validate their findings.

To validate their inferences from data, Bleiler, et al. used their expert consensus, that is, the researchers met and agreed on emerging categories from the data about the effectiveness of the instruction sequence and the kinds of errors PSMTs attended when validating arguments given to them. In place of expert consensus I would consider using member checks and conference presentation critiques by peers to validate conclusions drawn from data about the kinds of proof schemes held by students as well as how the proof schemes emerged. Main findings from the reviewed research piece are now described.

Regarding research question one that focused on student proof behaviour as a result of implementing the instruction sequence, the following findings were obtained. Bleiler et al reported that the instructional sequence increased students' awareness of the fundamental limitation of the empirical proof scheme as shown by the evidence produced by PSMTs when validating empirical arguments produced by high school students. In stark contrast, Bleiler et al. reported that PSMTs' superficial understanding of indirect methods of proving was persistent. For instance, with respect to method of proof by contrapositive PSMTs did not realise the need to start by negating the consequent part of the implication statement, $p \Rightarrow q$, that could then lead to $\sim q \Rightarrow \sim p$.

Research question two focused on the sort of errors PSMTs attended to when they validated arguments assigned to them. The study revealed that students did not draw meaning from mathematical objects constructed. For example, some students did not provide a justification why the fraction $\frac{ky+x}{y}$ is irrational when given that x is irrational and y is a rational. Other errors also include endorsing arguments with imprecise or incorrect definitions and violation of the proof framework when proving (for exemplifications see Bleiler et al, 2014, p. 117). These findings provided a window through which students' written responses and chalkboard demonstrations were examined in order to determine the kinds of proof schemes held by students.

While the studies discussed thus far pointed to the need for more studies that examine students' thinking as they engage with mathematical proof, I considered it necessary to survey other studies of mathematical proof in order to develop a comprehensive view of students' proof behaviour. Hence, the need to further explicate the research gap motivated me to examine other research studies on mathematical proof. So, the next section focuses on other ideas gathered from literature search of students' proof behaviour. I reiterate that the realist position that emphasises that mental events and processes are causally relevant to the explanation of student behaviour informed literature search.

3.2 Studies on student teachers' proof behaviour

Studies on students' understandings of mathematical proof run the gamut from university level where studies have involved pre-service mathematics education student teachers and mathematics majors (e.g., Housman & Porter, 2003; Stylianou, Chae & Blanton, 2006; Varghese, 2009) to the secondary level where the thrust has been proof validations, wherein students reflected on and evaluated arguments supplied by the researcher (Balacheff, 1988; Knuth, 2002; Stylianides &

Stylianides, 2009, p. 328) and to the perspective of pre-service elementary teachers (e.g., Martin & Harel, 1989). Some studies have uncovered a variety of phenomena regarding the ways students comprehend and appreciate the notion of mathematical proof that include holding an empirical conception of mathematical proof (e.g., Stylianides & Stylianides, 2009; Stylianides, 2011). Other studies have also clarified the nature of patterns of that exist between students' proof schemes and problem solving strategies (Housman & Porter, 2003), as well as revealing student teachers' superficial understanding of the notion of a counter example. In the following section, I review those studies and relate them to thrust of the present study.

Recio and Godino (2001, p. 83) with the aim of determining the nature of arguments student teachers find convincing in different institutional contexts studied proof schemes of lower level (year one) mathematics students at University of C'ordoba in Spain. During data analysis students' proof schemes were then related to different contexts of mathematics, such as daily life, experimental sciences, logic and mathematical foundations. Their main conclusion pointed to students' difficulty with axiomatic proofs. The study was done at transitory phase, that is, a few days after commencement of university mathematics courses. Many studies, and indeed from my personal experience with the concept of mathematical proof, have shown that both elementary and secondary school mathematics curricula are characterised by low intensity of mathematical proof activities (Stylianides, 2007, p. 289, Stylianides & Stylianides, 2009, p. 237). The current study seeks to uncover the kinds of proof schemes held by undergraduate student teachers after those students would have learnt many introductory courses involving the notion of mathematical proof.

Housman and Porter (2003) investigated patterns among proof schemes of 11 above average mathematics students and strategies the students used to learn a new mathematical concept. Housman and Porter used task-based interviews to elicit students expressions of mathematical arguments they find convincing and these expressions were classified according to Harel and Sowder's (1998) taxonomy of proof schemes. Task-based interviews, were also used to elicit expressions that contained strategies used by upper level undergraduate mathematics students to learn a new mathematical concept and those expressions were categorised according to a taxonomy of learning strategies suggested by (Dahlberg & Housman, 1997). Housman and Porter went on to examine patterns among learning strategies and proof schemes and the following were observations were made. The study revealed an increased awareness of the fundamental limitation of inductive explorations as only one student exhibited an empirical conception of proof. The student with empirical conception, as if conforming to the dictates of the proof scheme category, generated examples far more than other students. All except one student exhibited two or more proof schemes and one student actually exhibited four different proof schemes thereby revealing a high intensity of ontological oscillations. The students who demonstrated an external conviction proof scheme reached an impasse in proving as they could not generate examples, reformulate concepts and let alone concept usage. The same pattern regarding use of learning strategies observed with students in the external conviction scheme was also noted by (Stylianou, Chae, & Blanton, 2006, p. 57). Students who exhibited the axiomatic proof scheme were successful in reformulating concepts, and could engage in example usage. Further, students within the axiomatic proof scheme demonstrated high mathematical sophistication in their concept understanding schemata (Moore, 1994) in the sense that the students did not generate examples and non-examples spontaneously, but rather examples were generated only when it was necessary to disprove conjectures (Housman & Porter, 2003, p. 155).

Such research studies as (Housman & Porter, 2003; Recio & Godino, 2001) while providing useful information for mathematics educators and mathematicians on the nature of proof schemes held by students, they do not elaborate on use of key ideas in characterising student teachers' proof schemes. So the current study sought to elucidate on how students access and use technical handles and evoke conceptual insights when involved with proof construction tasks. It was therefore anticipated that studying students' use of key ideas could then shed light on kinds of proof schemes held by students. Further, an examination of student actions, utterances and written responses during their engagement with the proof tasks could provide important insights into whether the tasks were completed through either or a mixture of rote memorisation or generational memory in the sense suggested by Gowers (2007, p. 40).

Informed by the notion that the process of constructing a mathematical proof is similar to the process of solving a mathematical problem, Stylianou, Chae, and Blanton (2006) studied 34 undergraduate mathematics students enrolled in a first year course with a strong emphasis on mathematical proof. Transcribed audiotaped interviews were examined to identify, for each student, his/her proof scheme using Harel and Sowder's (1998) taxonomy. Stylianou et al. went on to look for possible patterns of the proof schemes observed. Results revealed the dominance of the empirical proof scheme (18 out of 34 students) and a significant proportion (12 out of 34 students) were in the external conviction proof scheme category. Further, students who had exhibited external conviction proof scheme reproduced definitions but there was no further explanation or discussion to fit those definitions into the relevant problem situations and the students reached an impasse (Stylianou, Chae, & Blanton, 2006, p. 58). This result parallels Doruk and Kaplan's (2015) finding that pre-service student teachers had difficulties in using definitions correctly to prove theorems and propositions. Students with empirical proof schemes differed widely from the aforementioned students in their proof behaviour. Students within the empirical proof scheme category tried many numeric examples but lacked symbolic representations of the problem. Few students (2 out of 34) who exhibited an axiomatic proof scheme introduced relevant definitions and produced longer segments of analysis and could link new information to the initial problem (Stylianou, Chae, & Blanton, 2006, p. 58).

While, the research generated useful information on problem solving patterns within a proof scheme category, it did not shade light on light on the level of accessibility of conceptual insights and technical handles during students' attempts to resolve the proof tasks. For instance, for those students with an empirical conception of proof, what is the nature of the students' key ideas that potentially could have hindered progression to higher proof scheme levels such as the transformational proof scheme? The current study aims to contribute to our understanding of students' schemes of argumentation by developing an explanation about why undergraduate student teachers would produce proofs in the manner they would do as they engaged with proof tasks assigned.

The theoretical framework guiding the current study is scientific realism that holds that the universals (properties) of the entity, that is, the proof scheme have an objective reality. However, according to realism there can be more than one way of knowing this reality (Maxwell, 2004). Hence, the purpose this study was to uncover the nature of existence of this reality among the Zimbabwean undergraduate students using different sources of data. The word reality here is used to refer to the nature of students' thinking about mathematical proof. It was anticipated this would be illuminated through student proof behaviour revealed as students engaged in mathematical proving (Maxwell, 2004; Pawson & Tilley, 2004).

Varghese (2009) took a case study approach involving 17 prospective mathematics student teachers to examine both students' conceptions and their ability to construct proofs of given mathematical statements (Varghese, 2009, p. 3). All the students were mathematics majors who had completed undergraduate courses and just commenced studies on pre-service teacher education. The study findings regarding students' conceptions of mathematical proof indicated that the dominant meaning of proof was one in which proof is viewed as serving the verification purpose (9 out of 17) and the least number of responses was in the category where proof was considered by students as a tool for explaining and discovering of mathematical knowledge. Regarding proof construction process, 13 out of 17 suggested a teacher guided step- by- step procedure as the way to complete proof tasks. It can thus be inferred that Varghese's study revealed the dominance of the external conviction proof scheme, specifically the authoritative sub-proof scheme where the teacher and the textbooks were authorities for the right answer as the step-by-step procedures suggested by the students were supposed to be led by the teacher. Proof behaviours revealed from Varghese's study were important lens for explaining proof behaviours demonstrated by the students involved in this study. Further, along with Varghese, the current study also employed the case study to allow prolonged engagement with students.

Motivated by the desire to confront Balacheff's (1988) categorisation of proof schemes, Varghese (2011) administered the tasks used by Balacheff (1988) to a group of university students who had rich proof experiences. Varghese's decision to evaluate Balacheff's (1988) taxonomy of proof schemes had stemmed from the realisation that Balacheff had earlier worked with thirteen and fourteen year old children who had limited experience with mathematical proof. The study showed that most students' written responses and verbal utterances indicated traces of the thought experiment, the highest level of Balacheff's taxonomy of proof schemes in terms of mathematical sophistication corresponding to the axiomatic proof scheme in Harel and Sowder's taxonomy of proof schemes. The results were not surprising given the high intensity of mathematical proof in the university curriculum. The current study sought to establish the kinds of students' thinking at university level in light of the proof behaviour revealed by participants involved in Varghese's study.

In a study involving 39 prospective elementary mathematics education students (Stylianides & Stylianides, 2009) examined the students' ability to construct and evaluate proofs. The combined 'construction-evaluation' activity employed by Stylianides and Stylianides in their study helped to illuminate prospective teachers' understandings of proof that tend to defy scrutiny when using the technique of asking students to indicate arguments they find convincing from a set of arguments supplied by researchers. In other studies students were required to construct proofs of given statements without being required to evaluate the arguments. In Stylianides and Stylianides study some student teachers provided empirical arguments as evidence of mathematical proofs. However an analysis of their evaluations revealed that the students were conscious of the limitation that inductive explorations, even though readily accessible to students as a method of verifying mathematical propositions, do not count as proofs since they do not offer complete and conclusive evidence about the truth of a conjecture. Thus, an empirical argument is not a valid general argument in the sense suggested by (Stylianides & Stylianides, 2009, p. 239). Being valid means the mode of argumentation involves use of deductive logical inferences from a set of axioms, definitions and previously proven theorems. In other words, a general mathematical argument makes use of all cases from the scope of the statement. This property of an argument is usually satisfied by picking arbitrary elements from the domain of the statement (Stylianides, 2007, p. 292; Stylianides & Stylianides, 2009, p .239). Stylianides and Stylianides noted that it is highly probable that if students' proof constructions had been considered without the accompanying evaluations a false conclusion that students have an empirical conception could have been reached.

From the above discussion it can be observed that the combined 'construction-evaluation' technique used by Stylianides and Stylianides (2009) helped to illuminate a correct proof conception by teachers of the distinction between proofs and empirical verifications. This suggests that an element of student self-evaluation of the validity of one's own proof construction is valuable. Hence, the present study would employ a similar strategy whereby student teacher informants would be asked to evaluate their written responses, transcriptions from chalkboard demonstrations and reflective interviews for the purpose of determining the kinds of Zimbabwean undergraduate student teachers' conceptualisations of mathematical proof.

In another study by Iaanone and Inglis (2011, p. 1), prospective elementary school teachers showed awareness of the limitation of empirical arguments. In their analysis of 222 proof attempts produced by 74 first year mathematics students the researchers found that from the onset of year one most students in their sample could associate the request for a mathematical proof with the production of a formal deductive argument. Findings by Iaanone and Inglis (2011) and Stylianides and Stylianides (2009) indicate lack of development in student teachers' proof schemes beyond the empirical proof scheme. In spite of their understanding of the limitations of empirical arguments, students justified mathematical conjectures through empirical arguments by default, implying that these are the only forms of mathematical reasoning accessible to them, that is, within their conceptual reach. There is therefore, an intellectual need for advancement in students' proof schemes beyond use of particular instances. Such an objective can eventually be realised perhaps by first increasing our understanding of the nature of students' thinking about mathematical proof and how such thinking evolves.

Furthermore, manifestations of a deeply rooted empirical proof scheme have been unearthed by a several studies. For example, Harel and Sowder (1998, p. 254) reported that there is a natural tendency to evaluate conjectures probabilistically, that is, through use specific instances. Lovell (1971) as cited in Harel and Sowder (1998, p. 254) noted a high percentage of 14-15 year-old students in his study employed a sequence of particular examples to derive the truth of a mathematical statement. In yet another study by Martin and Harel's (1989), out of the 101 pre-service student teachers who wrote a written test in which the students had to evaluate given arguments, Martin and Harel found that as many as 80% of the elementary student teachers relied on empirical explorations to prove mathematical statements. Interestingly, to reach conviction about the truth of a mathematical statement the students did not employ a multitude of cases but rather a single example was used. Students believed that since the single example was randomly chosen and it satisfied the general statement, then it qualifies as 'proof' of the statement. I reiterate that empirical arguments do not count as proofs. Stylianides (2011, p. 2) has reported that there is lack of grasp of this fundamental limitation of inductive argumentation among pre-service teachers.

However, by presenting the above severe limitation of empirical proofs, I had no intention to devalue the importance of inductive explorations in conjecturing and proving in mathematics. Some benefits derived from empirical explorations include, identifying patterns or finding the property leading to formulating and communicating the conjecture. Put another way, use of specific examples promotes understanding and appreciation of the problem (Stylianides, 2007, p. 290; Morselli, 2006, p. 185). Other benefits of inductive thinking are: offering insights into what needs to be proved through exploring the conjecture and discovering theoretical arguments that can potentially be mobilised in the subsequent proof construction process (Morselli, 2006, p.185). Further, example usage has been shown to have explanatory power in mathematics in general and in particular in proof and proving activities.

Kedem (1982) cited in Harel and Sowder (1998, p. 235) found that students did not have an appreciation and understanding of proof as a valid general deductive argument that does not require further empirical verifications. Thus even after producing a valid general deductive argument of a mathematical statement, the students still wanted to verify the proven result using one or more specific examples. Some studies also found that students have a shaky grasp of the notion of a counter example. The students have been reported to hold the conviction that even if a counter example to mathematical assertion has been found; the statement still stands, because the counter example is just an exception (Harel & Sowder, 1998, p. 235). The aforementioned proof conceptions are serious threats to students' opportunities to learn proof. I now discuss another major finding reported in literature that pertains to how student teachers evaluate mathematical arguments.

Knuth's (2002) study involving 16 secondary mathematics teachers revealed that a large proportion of the teachers (10 out of 16) failed to identify an invalid proof. The teachers in Knuth's study validated a biconditional statement in which only the proof of the converse of the statement had been given. A proof to a biconditional statement consists of proofs to both the implication and the converse of the statement. Knuth attributed this proof behaviour to a tendency by teachers to focus on local details of a proof instead of adopting a holistic approach where the overall reasoning and assumptions should show an alignment with the global components of the argument. Knuth's finding parallels Bleiler, Thompson and Krajčevski's (2014) finding on pre-service mathematics teachers' proof validation efforts. According to realism mathematical entities have an objective existence independent of the human mind. A biconditional mathematical statement being one of such mathematical entities, have an objective truth independent of our conceptual schemes. Ontology is about the nature of relations and categories of being of the entities. What is the level of awareness of these entities (biconditional statements) among undergraduate student teachers in Zimbabwe? Techniques such as the one described can be useful in determining students' thinking about mathematical proof in situations involving biconditional statements. Hence, this research finding influenced my selection of proof tasks included in Mid-assessment and End-of-instruction assessment data collection tools. I anticipated that arguments produced by students as proofs of biconditional statements would help to reveal the kinds of proof schemes held by the students.

Proof scheme taxonomies discussed in Chapter Two, offer a comprehensive view of the notion of a mathematical proof in several ways that include the fact that the taxonomies all indicate increasing levels in mathematical sophistication. The proof classification schemes concur that mathematical justifications are likely to proceed from inductive towards the deductive end (Stylianides, 2011, p. 2). However the focus of the taxonomies has been mainly on the types of arguments of arguments students find convincing. Stylianides and Stylianides (2009) as stated before have employed the combined 'construction-evaluation' strategy. Other studies have focused on strategies used by students to resolve 'prove that …' tasks (Iaanone & Inglis, 2011). Further, Iaanone and Inglis have suggested that current research focus should be on process used by students to produce deductive arguments. But in many studies little if any attention has been made to characterise the kinds of proof schemes held by students in terms of students' own proof attempts.

The present study is focused on Zimbabwean undergraduate students' conceptualisations of mathematical proof. Based on a disciplined study of a phenomenon that goes beyond mere thick descriptions of what is being studied, in conceptualising one develops one or more concepts to explain the causal relationships in the phenomenon being studied (Punch, 2005; Yin, 2009). Accordingly the study seeks to explore the kinds of students' mental constructs around the notion of proof and explain in terms of theoretical constructs such as technical handles and conceptual insights, how students develop their proof schemes.

The terms proving and proof assume different meanings depending on the specific contexts in which the concepts are considered. Varghese (2009, p. 4) posits that the definition of a proof may be based on the purpose of teaching proof, the forms of reasoning involved in the proving process, and the needs the process of proving is seen to address such as verification, explanation, systematization, and intellectual challenge (Pfeiffer in Bleiler, Thompson & Krajčevski, 2014, p. 3).

It can be argued that just as there are concept images associated with student understandings and appreciation of a technical term, there are also theorem images and proof images (in the form of diagrams, associations) which parallel and support undergraduate student teachers' understandings of mathematical proof (Hanna & Mason, 2014). In this regard the current study is interested in the forms of reasoning (modes of argumentation) leading to student teachers poof and theorem images (Stylianides, 2007, p. 291). Thus the study aims to explore students' thoughts about proof and proving as illuminated in their proof and theorem images, designated in this study as proof schemes. Specifically, the study aims to develop a characterisation of students' proof schemes in terms the notions key ideas, proof framework, hierarchical order (Raman, 2003; Selden & Selden, 2009). Thus, the study aims at developing an explanation grounded in the data about the kinds of proof schemes held by student teachers.

The key ideas provide a skeleton of the reasoning involved in the construction of proof. How can the undergraduate Zimbabwean student teachers' proof schemes be characterised in terms of key ideas referring here to conceptual insights and technical handles) of a proof (Sandefur, Mason, Stylianides & Watson, 2013). Much of the focus of the several proof scheme taxonomies has been on the type of arguments students find convincing from those supplied by researchers (Balacheff, 1998; Harel & Sowder, 1998; Housman & Porter, 2003). Little attention has been given to processes students use to produce arguments meant to validate conjectures. This review of literature has shown an increased awareness among students that valid deductive argument constitute proof and at the same time that empirical arguments should not be elevated to the status of proof. Why then do student teachers continue to resort to use of empirical verifications as proofs of conjectures? It is therefore the goal of this study to explore students' thinking processes as they engage with proof tasks.

It can be seen from the foregoing discussion that few studies have focused on students' justification efforts on the basis of students' actual proof constructions (e.g., Doruk & Kaplan, 2015; Stylianides, 2009). Further, even in such cases when attempts were made to determine students' proof behaviour on the basis of their own constructions (e.g., Doruk & Kaplan, 2015; Uǧurel at al. (2016)), this Chapter has uncovered flaws that were characteristic of such studies. Flaws include lack of long term involvement of the researchers, and limitation associated with data collection tools such assigning tasks that seemed to be routine exercises in Topology. Hence, this study is part of efforts to respond to the paucity of studies that explore students' thinking based on their own productions and guided by awareness of flaws in methodologies discussed in this chapter.

Chapter Four

Research Methodology

4.1 General approach

Blumer (1969) cited in Corbin and Strauss (2008, p. 65), commenting on the purpose of an exploratory research study writes:

> The purpose of an exploratory investigation is to move towards a clearer understanding of how one's problem is to be posed, to learn what are the appropriate data, to develop ideas about significant lines of relation and to evolve one's conceptual tools in line of what one is learning about in the area of life.

One of the goals of the study was to establish the kinds of proof schemes that characterise student teachers' conceptualisations of mathematical proof. The other goal of the study was to generate insights about how the mathematical object (proof scheme) emerges. Scientific realism was considered to be strategic for this investigation into students' thinking around mathematical proof. van Fraassen (1980) asserts that scientific realism is used to denote the precise position on the question of how a 'scientific theory' should be understood and what scientific activity really entails. In the current study the "scientific theory" is the student teacher's conceptualisation of mathematical proof. The term "scientific activity" in the context of this study refers to proof construction activities and processes of proof understanding by the student teachers. Further, "scientific activity" is also used in this study to describe how students' proof experiences shape their conceptualisations of mathematical proof. The question of how the student's conceptualisation of mathematical proof should be understood can be answered by ascertaining the distance between expert conceptualisations and learners' conceptualisations of mathematical proof. Huge discrepancies between students' conceptualisations and those held by research mathematicians indicate limited command of proof knowledge. Ideally, students should develop proof understandings held by research mathematicians (Weber & Mejia-Ramos, 2015).

Scientific realism enables a researcher to study what Maxwell (2004) calls local causality. Local causality refers to specific events and processes that illuminate the nature of the basic social problem (Charmaz, 2006; 2014). The basic social problem in this regard was students' superficial understanding of mathematical proof. So by applying scientific realist ideas I could determine the nature of the basic social process (Charmaz, 2006, 2014). The basic social process refers to what students actually do in dealing with the basic social problem. Hence, the ontology, that is, the structure of the nature of the basic social problem was determined by applying realist ideas.

Scientific realism holds that the context of the phenomenon being investigated is intrinsically connected to the causal explanation that can be developed (Maxwell, 2004; Maxwell & Mittapalli, 2010; Pawson & Tilley, 2004). This is a critical realist stance that allowed me to consider the context of the teaching experiment when trying to build causal explanations for the kinds of proof schemes held by undergraduate students. The influence of context on students' proof experiences was also considered. The influence of context as a key feature of scientific realism was crucial in analysing students' emotions, utterances during data analysis. Precisely, the scientific realist positions articulated here influenced my analytic framework.

Furthermore, in the realist theoretical perspective direct observation of causal mechanisms and processes that connect events in proof constructions is even possible in single cases and situations without requiring a comparison group in which the presumed causal effects are absent or present (Maxwell, 2004; Maxwell & Mittapali, 2007). This realist position affirms the value of case studies in developing causal explanations in qualitative studies which was really the method of this study. Hence, the realist process approach which can be used to unravel causative mechanisms and processes for the occurrence of individual and social phenomena in case studies was considered to be strategic for this study.

The argument presented here has shown that case studies producing textual forms of data can generate causal explanations through direct observation of causative mechanisms and processes connecting proof events. The main aim of this study was to generate an explanatory theory grounded in the data that accounts for the kinds of proof schemes held by undergraduate student teachers as well as developing a hypothesis about how students develop those proof schemes. The research methodology had to account for the nature of existence of a mathematical object (proof scheme in this case) by addressing the following research questions:

(i) *What kinds of proof schemes characterise undergraduate mathematics student teachers' conceptualisations of mathematical proof?*

(ii) *How do the undergraduate student teachers proof schemes come into being?*

The concern here was on the terms in which student teachers think of the mathematical object, that is, the concept of mathematical proof. Hence, the study focused on conceptualising the kinds of proof schemes and developing a hypothesis about how proof schemes emerge. Precisely, I specify the two objectives of this study. First, the intent of the study was to develop an explanation about the kinds of proof schemes held by students. Second, the goal was to build a hypothesis about how proof schemes emerge. The explanatory theory and the proposition are outputs from the research as opposed to being inputs to the research process as is the case with most quantitative research designs.

In the current study the hypotheses evolved from the study. This is compatible with qualitative research methodology that was used in this research with a focus on generating hypotheses as opposed to quantitative research methodology where the major goal is verification of pre-specified hypotheses. Second, the generalisations I would make from the study are called naturalistic generalisations (Starke, 1988 as cited by Punch, 2005) or alternatively referred to as theory-connected or analytic generalisations. Generalisations are described as such because they involve understandings that are furthered or abstracted from data. Hence, a qualitative research design was used in this study and a discussion of the research design now follows.

4.2 Research Design

A research design is an overall plan of how a piece of research will be executed (Punch, 2005, p. 241). The research design is concerned with a shift from what data will be collected to a consideration of how data will be collected and from who the data will be collected under a specific theoretical framework (Punch, 2005, p. 243). A case study research design was used in this study because of the following reasons. The aim of the study was to develop an in-depth understanding of the kinds of proof schemes that characterise students' conceptualisations of mathematical proof and how students' proof schemes emerge. In this study a design that employed a scientific realist approach was used. A case study allows an in-depth understanding of a case or perhaps a small number of cases using methods considered strategic by the researcher (Punch, 2005, p.143).

Miles and Huberman (1994) define a case as a phenomenon that exists within some bounded context. A case can assume several forms such as an individual, a role, a process, an incident or an event, a small community or a policy (Punch, 2005, p. 144). In the current study, the individual student teacher is a case implying that there were 10 cases involved in the study. The proof scheme, which consists of what is both ascertaining and persuading to an individual when validating mathematical conjectures, is the unit of analysis (i.e., object of study) for proving process for which the study intended to develop an in-depth understanding among student teachers at university level.

In as much as there are different themes that make up cases, e.g., process, policy, individual, incident, there are different types of case studies classified according to the purpose the study seeks to accomplish. Stark (1994) in Punch (2005) distinguishes three types of case studies. First, there is an intrinsic case study whose goal is to have an in-depth understanding of a case. The second type of a case study is called an instrumental case study where a researcher aims to generate insights on or refine theory using a particular case studied. Third, there is a collective instrumental case study which is an extension of the instrumental case study design that covers several cases in order to generate insights about a phenomenon (Punch, 2005, p. 144). Similarly, Yin (2009) distinguishes two case studies: an explanatory case study and an exploratory case study. Baxter and Jack (2008) write that the difference between the two types of case studies is that the former addresses the *why* question whereas the later addresses the *how* question. Hence, in the context of this study a collective instrumental case study with an explanatory bent was used to establish a set of causal links within the student teachers' proof schemes. Further, a collective instrumental case study with an exploratory orientation was used to address how proof schemes evolve among student teacher informants.

A scientific realist process approach treats mental events and processes as real observable phenomena which were causally relevant to the explanations of students' behaviour, emotions and actions as they engaged with proof tasks (Maxwell, 2004; Maxwell & Mittapali, 2007). Further, the realist research methodology allows the direct observation of mental processes and causative mechanisms involved in proving in a few cases or even single cases (Maxwell, 2004, Yin, 2009). Hence, we can infer that there is compatibility between the realist positions and the case study with regards to the two research questions the study addressed. Therefore, it was on the basis of these advantages of the case study that I considered it to be strategic in pursuing the goals of this study.

The purpose of the research design was therefore, to fit the research questions to data. 'Fitting' data to research questions means that relevant data were gathered to generate insights about undergraduate student teachers' conceptualisations of mathematical proof. In other words, it was anticipated that data would give some sense of students' proof images (Hanna & Mason, 2014). For instance, the assertion that students espouse an external conviction proof scheme could be evaluated on the basis of data elicited. Data were in the form of written responses to proof tasks and students' utterances during chalkboard demonstrations and interviews.

This study sought to explain the kinds of proof schemes held by undergraduate students. In other words, the focus was on establishing the causal links that characterise the kinds of proof schemes held by undergraduate student teachers. Precisely, this is the research objective of research question one: *What kinds of proof schemes characterise undergraduate students' conceptualisations of mathematical proof?* In this regard, the study aimed to develop an explanation about how the various categories of proof schemes can be differentiated from each other in terms of the manner in which students utilise underlying mathematical ideas involved in proof construction process. The underlying ideas include: key ideas, notion of intellectual need and epistemological justification, components of a proof and justification types (Koichu, 2012; Sandefur, Mason, Stylianides & Watson, 2013; Selden & Selden, 2009; Weber & Alcock, 2011). I re-cap here that proof scheme categories show increasing levels of mathematical sophistication from the external conviction proof scheme right through the empirical up to the analytic proof scheme. Hence, students' level of accessibility to key ideas and other underpinnings of the concept of proof in this explanatory case study provided a window through which I could differentiate categories of proof schemes. The key ideas refer to heuristic ideas (conceptual insights) and procedural ideas employed during proof construction. Hence, a collective instrumental case study that is explanatory in form was used to realise the goal stated, that is, to develop an explanation about the causal links within different proof scheme categories (Yin, 2009, p. 7).

A major goal of this study was to develop an explanation about the kinds of proof schemes held by undergraduate student teachers. It is therefore, important to clarify on what we mean by building an explanation in a case study. To explain a phenomenon is to stipulate a set of presumed casual links about it (Yin, 2009, p.141). By stipulating causal links one is required to provide an account of why and how something happened in the manner it did. For example, why is there so much tenacity in the empirical proof scheme and other cognitively lower order proof schemes? We describe tenacity as holding on to ideas and beliefs because they have been accepted as facts for a long time. Habits lead us to believe that something is true despite injections of contradictions to the fact. For instance, as observed in Chapter One empirical verifications continue to dominate student teachers' proof construction attempts despite efforts aimed at increasing grasp of the fundamental limitation of the inductive proof scheme. A detailed account of reasons for the persistent use of inductive explorations during proving constitutes an explanation. An explanation should therefore reflect critical insights that contribute to theory building (Yin, 2009, p. 141).

This study also intended to develop a conceptualisation of how proof schemes emerge. In other words, the intent was to develop an in-depth understanding of how students' proof schemes evolve, that is, how students develop their proof schemes. Punch (2005) clarifies the meaning of conceptualisation as follows:

> To conceptualize means on the basis of the disciplined study of this case and using methods for analysis which focus on conceptualizing rather than on describing [...], the researcher develops one or more new concepts to explain some of what has been studied (p. 146).

Hence, the objective of the present study in this regard was to develop a proposition about how proof schemes emerge in terms of mathematical underpinnings of a proof that include: technical handles, and conceptual insights, modes of reasoning in proof construction and theory of actions in proving (Alcock, 2010; Sandefur et al., 2013; Selden & Selden, 2009). In other words, I strived to conceptualise, that is, to develop one or more concepts that explain how students' proof schemes emerge. Hence, a collective instrumental case study case with an exploratory focus (Yin, 2009, p.

7), was used to address research question two: *how do undergraduate mathematics education students develop their proof schemes?*

In summary, it can be said that some features of scientific realism have been used in this section to justify the use of a collective instrumental case study design with a focus on generating an explanation, that is, to develop an explanatory theory about the kinds of proof schemes that characterise students' conceptualisations of mathematical proof. Concerning the emergence of proof schemes among students, a collective instrumental case study with an exploratory orientation has been enlisted to develop a proposition that accounts for the manner in which the proof schemes evolved.

4.3 Population and sampling

4.3.1 Participants: Pilot study

The study involved 20 undergraduate mathematics education student teachers at one university of science education in Zimbabwe who had done Calculus. Calculus courses cover many pre-requisite concepts needed to grasp mathematical underpinnings of proof in great depth and hence the decision to focus on mathematics education students who had studied Calculus. Upon graduating, these in-service teachers are expected to teach mathematics in high schools and their abilities at proving and conjecturing activities will determine how they will teach the same aspects to learners (Jones, 1997) and hence, the focus on undergraduate students' understanding of mathematical proof.

4.3.2. Sampling procedure

Participants involved in the pilot study were sampled from undergraduate mathematics education class I had taught the Real Analysis course for a semester at that university. Selection of study participants was in 2 phases. First, convenience sampling was used (Berg, 2009). Berg describes this sampling method as one which is based on participants being readily available and accessible. Following (Housman & Poter, 2003; Varghese, 2009), undergraduate student teachers who had been taught Real Analysis content were individually involved in task-based interviews as settings for exploring the student teachers' proof schemes. The initial phase involved 20 student teachers which is the usual class size for students studying proof related courses at undergraduate level in most university contexts in Zimbabwe. The next phase of the sampling procedure involved purposive selection of six undergraduate mathematics student teachers. Berg (2009) suggests that in purposive sampling, the researcher selects some group to represent population of interest basing on some expertise about the group. Usually, a purposive sample is selected after some investigation to ensure that individuals displaying desired attributes are involved in the study. This justifies use of task-based interviews with the undergraduate mathematics class in order to identify students from each of Harel and Sowder' (1998) broad categories of proof schemes namely external conviction, empirical, and deductive proof schemes. That is, two cases from each proof scheme category were selected for the subsequent interpretative inquiry of proof schemes. The aim was to generate rich data in the sense suggested by Charmaz (2006, 2014) so that explanations about the kinds of proof schemes held by Zimbabwean undergraduate students and proposition(s) about how those proof schemes emerge could be inductively inferred from those data. The 6 students identified were then involved in-depth reflective interviewing.

4.3.3. Participants: Main Study

Students who had enrolled for the Bachelor of Education Degree (BEd) in mathematics at a second comprehensive university participated in the study. The main study involved 10 BEd student teachers. There were 6 female and 4 male students. The study took place during first semester of their final year of studies. The BEd in-service programme of study has duration of two years of full-time study during the course of which student teachers study content and professional courses. Mathematics content taught at this level is pitched to the level of third year courses for a four year bachelor's degree for mathematics majors. The BEd programme of study includes Calculus and Real Analysis courses. Calculus which is a pre-requisite course for the proof laden Real Analysis course had been studied during the first year. BEd student teachers study professional courses in which pedagogical content knowledge (PCK) is emphasised. In addition to professional and subject content courses the in-service student teachers, are required to complete a research project in mathematics education. Research projects in the BEd programme of study focus on teaching and learning issues in mathematics.

Participants were holders of the Diploma in Education: eight students were holders of the diploma in education for secondary school level of teaching while 2 student teachers had diplomas in primary education. Both diplomas for secondary and primary education were awarded by a Department of Teacher Education of the university where the study was undertaken. Students who were holders of a diploma in secondary education could teach mathematics up to ordinary level for the local Zimbabwe School Examinations Council (ZIMSEC). ZIMSEC 'O' level of secondary school learning is similar to Cambridge 'O' level. The two students who were holders of the Diploma in Primary Education had majored in mathematics during initial teacher training. Hence, they had been exposed to the same subject and professional content as their companions from secondary teacher training colleges and so the two student teachers were eligible for the study. The major goal of BEd programme of study was to upgrade the students' subject content knowledge

(SCK) and their pedagogical content knowledge (PCK) in order to capacitate them to teach advanced level mathematics.

4.4 The study context

4.4.1Teaching experiment

The teaching experiment was seen here as a “crucible” for developing and testing theory about how proof schemes emerge and about the kinds of proof schemes held by undergraduate student teachers. The teaching experiment involved the teaching of the Real Analysis course to mathematics education undergraduates over the whole semester. Following Selden and Selden (2003), in this proof laden course undergraduate student teachers were provided with self-contained notes consisting of statements of theorems, definitions of concepts on proof, and requests for students to produce proofs of given tasks. The mechanisms and processes involved in the development of proof schemes were treated as real observable phenomena in line with the scientific realist process approach that was used in this study (Maxwell, 2004; Maxwell & Mittapalli, 2010).

I used the teaching experiment as setting for addressing the multi-faceted realist question, “What works for whom, in what circumstances and in what respects and how?” (Pawson & Tilley, 2004, p. 2). The purpose of posing such a question was to deal with the intricacy regarding the kinds of proof schemes held and how these objects emerge using a realist analytic framework. A realist approach is defined as that which seeks to explain social phenomenon by reference to mechanisms and causal processes below the surface contingent upon specific historical, local and even institutional contexts (Maxwell, 2004; Pawson & Tilley, 2004). The term historical context is relevant in this present study whose other main goal was to uncover how proof schemes emerge. Students’ proof experiences were traced from pre-university to undergraduate learning experiences and hence the use of the term historical context captured in the definition of the realist approach employed in this study was relevant to the study.

Realism asserts that causal mechanisms and processes can be directly observed rather than being inferred from measured co-variation of presumed causes and effects (Maxwell, 2004). Direct observation of causal mechanisms and processes involved in proof and proving activities was even possible in single cases without requiring comparison situations or some control group (Maxwell, 2004; Maxwell & Mittapalli, 2010). Pawson and Tilley (2004, p. 2) in their programme evaluation study define a mechanism as “what it is about programmes and interventions that brings about effects.” Maxwell (2004) refers to a mechanism as a detailed account of the behaviour or an account for the makeup, and interrelationships of those processes involved in proving. In this study, mechanisms refer to students’ behaviours and actions as they engaged with proofs. Mechanisms are often hidden in a similar manner to the workings of a clock which drive the patterned movement of the hands of the clock.

Another key feature of the realist approach that was employed in this teaching experiment is the realist argument that mental events and processes are real phenomena that can be the causes of behaviour. Mental concepts refer to real entities that are causally relevant to explanations of individual and social phenomena. For an example, the tendency to search for appropriate axioms and formal definitions (as seen through verbal responses, behaviour and actions) would be indicative of axiomatic proof scheme held, which is a mental construct. Further, realism regards beliefs and emotions as real phenomena. This assertion supports the essentially interpretative nature of meaning and intention within the realist approach (Blumer, 1956 in Maxwell, 2004).

Another important realist position that informed the study is the role of context in developing causal explanations. From a realist perspective, the mechanisms and processes that can be observed depend on context in which the proof schemes are observed (Maxwell, 2004; Pawson & Tilley, 2004). In this regard I considered both the physical and cultural contexts in which the proof schemes were studied (Lewis, 2009). I described the student teachers' actions and emotions and in that process I made efforts to ensure all the causes of 'what happened?' are captured (Lewis, 2009). As concluding remarks to this section on features of the realist approach the current study employed we note that, although causal processes and mechanisms are directly observable, they were not easily observable (Becker, 1966 in Maxwell, 2004). Dunn (1978, p. 171) reinforces the idea that causal processes and mechanisms are not easily identified when he argues that "there are still no cheap ways to deep knowledge about other persons and the causes of their actions." Dunn and Becker's comments were crucial to this current study as they sensitised me about the fact that employing the realist approach to observe causal mechanisms and processes responsible for the emergence of proof schemes, for instance was not just a simple exercise.

4.4.2 Curriculum content for the Real Analysis course

The curriculum for the course which ran parallel with my research has the following broad content areas: $\mathbb{R}$ as a field, Sequences of real numbers, Limits and continuity of functions, Differentiation and integration of real-valued functions. I now briefly describe major aspects covered in each learning area stated.

First, with respect to $\mathbb{R}$ as a field, algebraic, order and completeness properties of $\mathbb{R}$ are treated. Algebraic properties include addition and multiplicative axioms of elements in $\mathbb{R}$. Central ideas to the development of $\mathbb{R}$ as an algebraic structure include the concept of a binary operation. The addition and multiplicative axioms of $\mathbb{R}$ and the concept of a binary operation lead to the definition of a field. Under order properties, the closure properties under addition and multiplication of arbitrary elements of the subset P of a field F are considered as well as the Trichotomy rule of an ordered field F. An understanding of these basic ideas of an ordered field should lead to proofs of many theorems such as: If $a, b \in \mathbb{R}$ then $ab > 0 \Rightarrow a > 0$ and $b > 0$ or $a < 0$ and $b < 0$. The completeness properties of $\mathbb{R}$ as a field are dealt with where the notions of a least upper bound (supremum) and the infimum of a bounded subset of $\mathbb{R}$ are covered. The uniqueness of the least upper bound of a bounded subset of $\mathbb{R}$ is treated. The axiom of completeness is central here as it leads to several theorems such as the Archimedean principle and its four corollaries which form the bedrock of many other theorems such as the rational density theorem in $\mathbb{R}$, the nested cells property, the theorem on the characterisation of cuts in $\mathbb{R}$ and hence revealing the coherence of mathematics.

Second, another learning area covered involves the treatment of the Real Sequences in $\mathbb{R}$ where a sequence is conceived as a mapping with domain in $\aleph$ (natural numbers) and range in $\mathbb{R}$. Central ideas are the convergence of a sequence and uniqueness of the limit of a sequence. These fundamental concepts should lead to the formulation and proofs of many theorems on sequences such as the squeeze theorem, a theorem on the characterisation of convergence of sequence which stipulates that the interval $(L - \varepsilon,\ L + \varepsilon)$ contains infinitely many terms of a sequence (a_n). The notion of a monotone sequence is treated, in particular the convergence criterion for a bounded monotone sequence. Subsequences are also covered under this learning area. An understanding of the notion of subsequential limit points should then lead to the treatment of many theorems such as the Bolzano-Weiestrass Theorem.

Third, the notions of limit and continuity of a real-valued function are treated. The $\varepsilon - \delta$ conception of the key notions of limit and continuous functions should be developed. Further, the distinction between limit and continuity in terms of the notions of a deleted neighbourhood as opposed to a mere neighbourhood of an arbitrary point x_o in $\mathbb{R}$ should be fostered. The $\varepsilon - \delta$ conception of a limit should lead to proofs of many theorems such as those on products, sums, composite, and quotients of functions. The idea of continuity of a function is treated in a similar manner but however, it has some additional aspects. For instance, the concept of uniform continuity of a function on a set is treated.

Fourth, the Real Analysis course covers content on differentiation of a real-valued function. The conception of the derivative a limit of the quotient $\frac{\delta y}{\delta x}$ as $\delta x \to 0$ is emphasised. The $\varepsilon - \delta$ understanding of the derivative of a function should lead to proofs of many theorems such as: *If a function f is differentiable at x_o then f is continuous at x_0*. The converse of this implication statement is not necessarily true. Other theorems also covered under differentiation include: the mean value theorem, the extreme value theorem, and Rolle's theorem.

Finally, the course introduces undergraduate mathematics education students to the Riemann integral where key notions such as a partition of an interval, a refinement Q of a partition P of a closed interval, and the notion of mesh are treated. An understanding of these basic ideas should then lead to a construction of the Riemann integral in terms of the upper and lower Riemann sums of a given function. The Riemann condition for integrability of a function and closely related theorems are also treated here. The Real Analysis course is then concluded by introducing student teachers to the concepts of metric spaces and measure theory.

4.4.3 Theoretical considerations for the teaching experiment

The teaching experiment for the Real Analysis course was informed by the following theoretical positions;

(i). Theory of actions in proof constructions (Selden & Selden, 2011),

(ii). Cognitive analysis of argumentation in proving (Boero, 1999; Duval, 2002),

(iii). Manipulating-getting a sense of-articulating (MGA) notion in proof construction (Sandefur, Mason, Stylianides & Watson, 2013), and,

(iv). The notion of rationality in conjunction with the ideas of conceptual insight alternatively heuristic idea (CI) and technical handles also referred to as procedural idea (TH). (Sandefur, Mason, Stylianides & Watson, 2013; Raman, 2003).

According to the realist approach the theoretical positions articulated were used to account for processes and mechanisms involved in individual student teachers' proof behaviours in the following ways. The theory of actions posits that the proof construction process is a sequence of mental and physical actions such as drawing or visualizing a graph and reflecting on earlier proof attempts (Selden & Selden, 2011). When the student teacher gains experience the proof construction process becomes small situation-action pairs called behavioural schemas (Selden, Mckee & Selden, 2010). Behavioural schemas are persistent mental structures consisting of recognising a situation and then taking an appropriate physical or mental action. In order to develop beneficial behavioural schemas one should carry out an action correctly many times. In the teaching experiment I provided participants with a variety of proving opportunities to allow the growth of beneficial behavioural schemas. Students then presented their individual proof attempts in class. Class presentations were video-taped and then analysed to guide subsequent teaching as well as to observe mechanisms and processes by which proof schemes emerge.

Another theoretical consideration that guided the teaching of Real Analysis course is Duval's (2002) cognitive analysis of argumentation in proving. Ideas drawn from Duval's theory include the notion of micro reasoning. Micro reasoning refers to the student's ability to identify crucial elements in their reasoning. Micro reasoning also involves being mindful to check the conditions in which the theorem applies (Hoyles & Kuchemann, 2002). An example would be when a student is asked to construct proof of the theorem; *A bounded monotone sequence converges*. Of interest to the prover would be whether the sequence is monotone increasing or decreasing. Further, the student should then determine the least upper bound for a monotone increasing sequence and the greatest lower bound for a monotone decreasing sequence. Thus conditions of the Axiom of Completeness must be checked during the proof construction.

Duval (2002) also distinguishes between two levels of competency in geometric proving. In the first level students should demonstrate the ability to organize statements according to premises, and conclusion(s) into deductive steps. The second level involves turning the deductive steps into a proof. It is crucial to note that from the first step conclusion to the target conclusion valid deductive reasoning progresses either through substitution with intermediary conclusion or a coordination of conclusions. Duval points out that as students traverse between the two levels of geometric proof competency they may experience difficulties in their reasoning which may result in unsuccessful proof attempts. When addressing research question one it also became a strategy of the study to identify and describe the levels of geometric reasoning shown by the students during proving.

Yet another theoretical construct relevant to the way students learn how to construct proofs is a notion suggested by Sandefur, Mason, Stylianides and Watson (2013) called manipulating (M)-, getting a sense of (G)-articulating (A), (MGA). Basic ideas embedded within the construct are that when a student is confronted with a puzzling mathematical proof task, it is natural to search for something familiar to manipulate. Manipulating means resorting to the use of familiar mathematical objects as worked examples. This may involve acting on symbols and other representations for the purpose of getting a sense of (G) of the underlying mathematical structure, patterns or relationship. As the structure gets more coherent as the student gains experience, and through repetition of the manipulations, the student may be able to articulate (A) his understanding in verbal or written or visual form. Such articulations serve as manifestations of mental events and processes which are treated as real observable phenomena according to scientific realism (Maxwell, 2004; Maxwell & Mittapalli, 2007).

However, it is important to observe some flaw in the MGA idea when applied to a teaching experiment. Sandefur, Mason, Stylianides and Watson (2013) say not all manipulations no matter how intentional they might appear, illuminate a true sense of the mathematical relationship. Learners may disguise by following rules in doing proofs without making any contact with the underlying mathematical structure of the mathematical concepts involved (Ndemo & Mtetwa, 2015; Sandefur, Mason, Stylinides, & Watson, 2013).

Finally, another idea that guided the teaching of Real Analysis course is the concept of rationality suggested by Balacheff (2008). Balacheff defines rationality as the system of rules or criteria mobilised when one is advancing an argument. Balacheff further suggests that rationality depends on content and context and also that students do not use the same rules in proving different mathematical statements. An exemplification is noted in that rules and criteria mobilised in constructing proof by induction are different from those mobilised in doing proving by contradiction.

Closely related to the idea of rationality is the distinction between semantic and syntactic approaches to proofs (Alcock & Inglis, 2008; Sandefur, Mason, Stylianides & Watson, 2013; Weber & Alcock, 2004). Details of syntactic and semantic approaches to proof construction were presented in Chapter 2 so a brief re-cap of these concepts is given here. Briefly, syntactic approach is used to denote the manipulation of formal definitions and axioms within the given representation system (reference theory) of the mathematical proposition. On the other hand, a semantic or referential approach to proof entails use of referential objects such as graphs and other instantiations of the mathematical proposition to guide a prover's logical inferences (Weber & Alcock, 2004). The two approaches should be seen as complementary rather than dichotomous. The link with the idea of rationality is seen in the kinds of arguments mobilised when using either the referential or syntactic approaches to proof construction.

Also related to the concept of rationality are the notions of conceptual insight (CI) and technical handles (TH) suggested by (Hanna & Mason, 2014; Sandefur, Mason, Stylianides & Watson, 2013; Raman, 2003). Once again these ideas were discussed in Chapter 2 so here an effort to show their application in the teaching experiment is made. Conceptual insights are fresh ideas that come to mind through generational memory, that is, without recourse to rote memory by a prover. A conceptual insight is a sense of a structural relationship pertinent to a phenomenon of interest that indicates why a mathematical proposition is likely to be true (Birky et al., 2009). A technical handle or procedural idea is a way or technique of manipulating structural relationships that support the mapping or conversion of a CI into a proof. The focus of the study in this regard is on assessing students' level of accessibility to the key ideas from student demonstrations during the teaching experiment and the written tasks. The level of student accessibility to key ideas would then be used to characterise the kinds of proof schemes held by students. In other words, these theoretical constructs are windows that could be used to observe causative mechanisms and processes in proof events from written solutions, and chalkboard demonstrations.

Consequently the above discussion of theoretical considerations for the teaching experiment and key features of the realist approach influenced the construction of the following data collection tools: observation guide, the reflective interview guide, task-based interview guide and written task sheets within the context of the teaching experiment. Presented next is a discussion of these data collection instruments.

4.5 Research instruments

The instruments were intended to elicit data that could address the following research sub-questions:

- *What kinds of proof schemes characterise undergraduate student teachers' conceptualisations of mathematical proof?*
- *How do the undergraduate student teachers develop their proof schemes?*

By addressing these questions I intended to accomplish the following objectives;

(i) To develop an explanatory theory grounded in the data about the kinds of proof schemes held by the students.

(ii) To formulate a hypothesis (proposition) about how pre-service teachers' proof schemes emerge?

The idea was to conceptualise how the mathematical object of proof scheme develops. An outline of the data collection instruments is now presented. Three major sources of data for the study are: written tasks which are also known as task-based interviews, chalkboard demonstrations, and reflective interviews. Textual data elicited from these sources were of the forms: written responses by student teachers, students' utterances from interviews and chalkboard demonstrations. The task-based interviews were used as a technique for exploring students' proof schemes. The idea of task-based interviews was drawn from studies by (Housman & Porter, 2003; Stylianides & Stylianides, 2009; Varghese, 2009). For example, in Stylianides and Stylianides' (2009) study pre-service teachers were asked to construct proofs of given mathematical tasks after which they were then required to evaluate their constructions. The combined "construction-evaluation" exercise helped to establish poof conceptions among the students. Though some students produced empirical arguments as proofs of given statements they were able to point out that their arguments were invalid proofs. This study employed the same technique because of the strength just described.

Hence, in the current study students' evaluations of solutions were audio-taped during the reflective interviews.

4.5.1. Written tasks

Written tasks, alternatively referred to as task-based interviews were used as a setting for exploring the kinds of proof schemes held by undergraduate student teachers (Housman & Porter, 1997; Varghese, 2009). The proof tasks were prepared with a focus on addressing research question one: *what kinds of proof schemes characterise undergraduate student teachers' conceptualisations of mathematical proof?* There were two task sheets prepared for data collection.

One sheet of tasks was meant for Mid-instruction assessment data collection phase that occurred during week 6 of the teaching experiment. In order to prepare tasks that had the potential to illuminate students' thinking about mathematical proof, the selection of the task was informed by the need to engage students on tasks that were not routine exercises and for which students had no initial overall idea on how to find the solution (Mamona-Downs & Downs, 2013, p. 139). Hence, while students had pre-requisite concepts needed to resolve the tasks, I ensured that the tasks were novel to the students. For instance, the fourth task for Mid-instruction assessment data collection was: *Prove that the sequence defined by* $(u_n) = \frac{n^2-1}{2n^2+3}$ *converges.* Students had dealt with the definition of convergence of a real sequence and proved some similar tasks. However, I considered the task to be novel to the students because it was not among tutorial tasks assigned and it was not in prescribed textbooks for the Real Analysis course. Further, students' proof attempts when piloting the instruments also confirmed the fact that the task was indeed novel.

Another good measure of a problem employed in task selection was the idea of plurality of different directions that the proof tasks could be treated by the student teachers (Mamona-Downs & Downs, 2013, p.139). The wording of proof tasks was considered to be a critical factor in this regard to ensure that questions were open-ended enough in order to invoke different proof events among students. The specific wording for the task was: *Determine whether the following statement is true or false. For all real numbers* a *and* $b, a - b > 0 \Rightarrow a^2 - b^2 > 0$. *Justify your answer.* The same task had been used in pilot study and was used in the main study because of its potential to invoke different proof events among undergraduate student teachers. Similar wording was used in crafting tasks 2 and 3. For details of proof tasks for Mid-instruction assessment data collection phase refer to Appendices A and B. In this phase a realist process theory was employed to observe directly the causal processes and mechanisms that connect proof construction events (Bostic, 2016; Maxwell & Mittapalli, 2007). A realist process approach was used in this study to account for the students' formal praxis, that is, the students' established habitual practice regarding proof and proving activities in mathematics. The following definitions of mathematical proof informed the selection of tasks included in the task-based interviews

A mathematical proof is a connected sequence of assertions for or against a mathematical proposition used to either convert a conjecture into a mathematical fact or alternatively refute/disprove the conjecture (Selden & Selden, 2003; Stylianides & Stylianides, 2009). Hence, depending on whether the student is confronting a true or false mathematical proof task, there are two types of proof constructions (Lee, 2011). For a false mathematical proposition proof construction is defined as the search for counter examples to disprove or refute a conjecture. For a true mathematical proposition, proving is conceived as the search for arguments to validate a mathematical statement through deductive means where there is use of axioms, formal definitions and previously proven theorems (Sandefur, Mason, Stylianides & Watson, 2013). Accordingly a student' proof scheme is a cognitive scheme underlying the student's proving attempts. In this regard, inductive explorations and deductive inferential processes can be seen as two ends of a proof continuum between which students traverse in order to make logical conclusions about mathematical propositions. It is therefore, indicative of low mathematical proficiency for a student to search for deductive arguments in situations involving proof-by-counter example tasks, just as inductive explorations should not be elevated to the status of a proof for situations requiring proof by deductive argumentation.

The other sheet of written tasks was assigned during the End-of-instruction assessment data collection phase. This second and final phase of data collection took place during week 11 of the teaching experiment. This phase of data collection consisted of questions intended to tease out the casual links within proof schemes held by undergraduate student teachers (Maxwell, 2004; Yin, 2009). Each student attempted 9 proof construction tasks and the distribution of the tasks was as follows. Four tasks were involved in the Mid-instruction data collection phase and 5 tasks were attempted during the End-of-instruction data collection phase. The mathematical underpinnings of the notion of proof examined were decided a-priori by the researcher (Zammunier, 1998 in Furighetti & Morselli, 2011, p. 59). However the order in which students tackled the tasks was not prescribed. An assortment of tasks comprising true and false propositions was prepared. The tasks were drawn from concepts or topics in the Real Analysis course. Students' reasoning and proving of implication statements is crucial to the learning of mathematics. Tasks that provide opportunities for using counter-examples were also included in data collection sheets.

Tasks that require deductive justifications have received more attention in other studies than tasks that call for proof by refutation so I considered it necessary to include tasks that required proof by counter-argumentation in the written tasks to explore students' thinking of proof. Students were required to decide whether the given statements are true or false and justify their conclusions. Students were expected to use deductive-proof constructions for true statements and proof-by-counter examples for false statements. During lectures I introduced and explained the notion of structured derivations (Wallin & Manilla, 2009) as means of documenting students' thinking when proving. Students were encouraged to use structured derivations in the written tasks so as to document their thinking. The tasks were presented in written form on a work sheet.

4.5.2 Chalkboard demonstrations

An observation interview guide was used in recording field notes during presentation of proof attempts by the student teachers. The purpose of the guide was to ensure that I stayed focused on the most significant aspects of the study and avoid a detour to lesser issues (Yin, 2009). Therefore, I made constant reference to original goal of the inquiry, which was to account for the kinds of proof schemes by capturing data that assist in establishing causal links in students' schemes of argumentation. For an example, it was anticipated that data collected could provide evidence that could assist in describing mental events and processes that could then explain the tenacity of the empirical proof schemes or impasses experienced by students when dealing with tasks involving the axiomatic proof scheme. The observation guide was difficult to complete when a student was demonstrating so I focused on proof construction activities outside the lenses of video recording device. Details of modifications done the observation guide for chalkboard demonstrations are given in the next section.

4.6 Pilot study

4.6.1 Context of the pilot study

The pilot study was intended to validate the research instruments. In other words, the pilot study was carried out to pre-test, that is, to try out the research instruments in this case, the observation guide for the chalkboard demonstration, and proof tasks (task-based interviews) and trying out the reflective interview guide. The other major objective of doing the pilot study was to pre-test the research procedures, that is, to determine feasibility of the teaching experiment as a setting for collecting data needed to explore the kinds of proof schemes held by undergraduate student teachers and also to determine how these students develop their proof schemes. The focus here was on evaluating the methodological rigour of the anticipated data collection and analysis processes.

The Real Analysis course was taught under the Block Release learning mode during which the student teachers were taught over a period of two school vacations with each vacation stretching for 4 weeks. A school vacation constituted a block and two blocks of learning constituted a semester.

The study took place during second week of the second block of the semester. The second week was also considered strategic for assigning the tasks because students would have covered concepts required for them to resolve the proof tasks. Further, I considered it strategic to assign the proof tasks when pressure from Real Analysis and other courses had eased. Thus the decision to administer the proof tasks days before the end of block release teaching was made out of the realisation that doing so towards the end of the semester when students were about to sit for semester examinations would stifle the data collection process because of examination anxiety. Two proof tasks were assigned five days away from the end of second block teaching. A marking guide was devised to evaluate student teachers' proof attempts. The selection of proof tasks and the marking guides for the two tasks used in the pilot study is described next.

4.6.2 Task selection and marking guide

Task 1: *Determine whether the statement is true justifying your answer. For all real numbers a and b,* $a - b > 0, \Rightarrow a^2 - b^2 > 0$.

Students were required to determine whether the conditional statement is true or false and provide supporting evidence for their assertions that the statement is either true or false. The proof task was not in a sense suggested by Selden and Selden (2011, p. 676) a 'template' problem but on the 'surface' would tempt students to erroneously refer to algebraic and order properties of the real field $\mathbb{R}$ and yet students just had to find an appropriate counter example. For example, a = -4 and b = -6 could be used to refute the claim that: $a - b > 0 \Rightarrow a^2 - b^2 > 0$.

Task 2: *Define a sequence* (x_n) *inductively by* $x_1 = 1$ *and* $x_{n+1} = \frac{2x_n+3}{4}$. *Prove that the sequence is bounded and determine its limit.*

It is noted here that a student teacher's description of the solution process to a proof task such as the one described above can illuminate many of the causal processes involved. Some of these processes are mental rather than physical. The mental processes were treated as real observable phenomena according to the realist process approach employed in this case study. Proving processes were inferred from students' behaviour and speech during chalkboard demonstrations of their proof attempts. Task 2 was deemed to be appropriate on the basis of its envisioned potential to generate rich data (Charmaz, 2006; Maxwell, 2004). It was anticipated that task 2 would generate rich data because it provided student teachers with opportunities to bring together many proof related ideas learnt earlier. For example, task 2 would provoke students to combine ideas on mathematical induction with those about direct deduction methods of proving.

First, the proof task required students to apply the relation, $x_{n+1} > x_n$, for a monotone increasing sequence. The definition of a monotone increasing sequence would then be used together with the principle of mathematical induction to prove that the sequence is monotone increasing. To accomplish this, the student needed to establish that the result holds for initial value(s), say, $n = 1$ to get x_2 and $n = 2$, to generate the third term x_3. This is the base step of the method of proof by induction (Stylianides, Stylianides, & Phillippou, 2007). Next, students were expected to state the induction hypothesis, that is, $x_k > x_{k-1}$ an essential statement required to prove the implication statement. Third, students were then supposed to establish the induction thesis $P_k \Rightarrow P_{k+1}$ which would lead to:

$x_k - x_{k+1} = \frac{2x_{k-1}+3}{4} - \frac{2_k+3}{4}$.. (1)

$= \frac{2x_{k-1}-2x_k}{4}$ ....(2)

$x_k - x_{k+1} = \frac{1}{2}(x_{k-1} - x_k) < 0$...(3)

The implication statement is proved by striving to produce the induction hypothesis as shown in step 3. Because from the base step it has been shown that $x_1 < x_2$ and from the induction thesis it has been established that $x_k < x_{k+1}$ after making the assumption that

$x_k > x_{k-1}$, it can be concluded that the sequence (x_k) is monotone increasing.

To prove boundedness, the student teachers were supposed to capitalize on the relation:

$x_k < x_{k+1}$ … … … … … … … … … ….. (4).

Step (4) is in actual fact a consequence of Principle of Mathematical induction that has been used to establish that the sequence is a monotone increasing. Substituting for x_{k+1} in (4) gives;

$x_k - \frac{2_k+3}{4} < 0$...(5), simplifying gives,

$x_k < \frac{3}{2}$ ....(6).

Hence, the supremum of the set $\{x_k : k \in \mathbb{N}\}$ is $\frac{3}{2}$.

Finally, students were supposed to apply the convergence criterion of a monotone sequence: *A bounded monotone sequence converges*, in order to deduce that the limit of the sequence is $\frac{3}{2}$. A bounded monotone sequence converges to its least upper bound (supremum) or its greatest lower bound depending on whether it is increasing or decreasing. A bounded monotone increasing sequence converges to its least upper bound and a bounded monotone decreasing sequence converges to its greatest lower bound.

4.6.3 Lessons drawn from the pilot for the main study

The main purpose of the pilot study was to test the feasibility of instruments, methods and procedures for later use in the main study. So I conclude this section by discussing lessons drawn from the pilot study and how such lesson helped me to determine whether it was feasible to proceed to the main study.

One useful lesson for main study activities concerns the kind of tasks employed in the pilot study. Specifically, the use of non-directional proof tasks contributed in strengthening validity of data collected for the study and hence the validity of findings as well. For the instance, the formulation of the task 1: *Determine whether the following statement is true or false. For all real numbers a and $b, a - b > 0 \Rightarrow a^2 - b^2 > 0$. Justify your answers as much as possible.* The task is non-directional in the sense that it does not stipulate the type of mathematical resources participants had to deploy to resolve it (Wilkerson-Jerde & Wilensky, 2011). Consequently, it attracted a variety of thinking styles from the students such as use of order properties and specific examples. It was therefore adopted and its formulation influenced the wording of tasks used for the main study.

During the pilot study data collection was a one day event. It is possible that students might have developed conversational fatigue by switching from one data collection event to another, which is from written tasks to chalkboard demonstrations and then to the reflective interviews. Fatigue developed might have compromised the validity of data collected. To eliminate effects of conversational fatigue developed during data collection it was therefore considered strategic to collect data over a two day period for the main study. Collecting data in one day made implementation of validity checks such as member checking difficult because of conversational fatigue. Furthermore, implementing structured derivation mechanism was hindered by time constraints. Use of structured derivations and member checks had to be done during main study.

Emphasis in the pilot study was on assessing the feasibility of instruments, methods and procedures. The pilot study was seen as an adaptive trial design that allowed modifications to be made in preparation for the main study. The pilot afforded me the opportunity to check on the analytic procedures and provided a chance to evaluate their use in data analysis. In this regard directed content analysis was shelved in analysing textual data meant to address research question two: *how do undergraduate student teachers develop their proof schemes?* This modification of the initial analytic plan was necessitated by the realisation that by allowing use deductive codes directed content analysis was somewhat restrictive in terms of allowing the data to speak for themselves. In its place, summative content analysis, which is more inductive because it begins with actual words and terms in the data were used to give more room to the data to speak rather than relying on inferences on the basis of literature (Berg, 2009; Corbin & Strauss, 2008; Punch, 2005). In this sense, the pilot study provided with ideas that had not occurred to me before conducting it.

Another lesson drawn from the pilot study relates to the learning mode participants had enrolled for. During piloting the Block Release learning mode made prolonged engagement with students impossible. I used my sabbatical leave period to collect data from another state university that had a conventional learning mode for mathematics education students. That way, I ensured that there was intensive long term involvement with the participants for the whole semester during which the teaching experiment was conducted. Further, unlike with the Block Release learning mode where the experiment could proceed for two weeks and then shelved, with conventional learning mode in the study site for main study there was continuous engagement with students. Continuity ensured uninterrupted implementation of the research plan something that was not possible with Block Release mode of learning used for pilot purposes. It was also anticipated that prolonged engagement with students in the teaching experiment would contribute towards reducing the effects of "think aloud" protocols that were a feature of chalkboard demonstrations. Hence, use of multiple sources of data, that is, triangulation method could help differentiate genuine categories about the kinds of proof schemes held by students (Hellinink, Hutler & Bailey, 2013). Data from task-based interviews, chalkboard demonstrations and reflective interviews should therefore be compared for similarities and differences to obtain a full revealing picture about the kinds of proof schemes held and about the possible trajectories for the proof schemes

Another useful idea drawn from the pilot study relates to data collection instruments. The observation guide designed for pilot purposes had the format shown in Figure 3

Student teacher....................................

Task 2: *Define a sequence* (x_n) *inductively by* $x_1 = 1$ *and* $x_{n+1} = \frac{2x_n+3}{4}$. *Prove that:*

(i) the sequence is monotone increasing,

(ii) the sequence is bounded and hence determine its limit.

Proof scheme category: (i) External conviction

(ii) Empirical

(iii) Analytic proof scheme

Indicators of proof scheme	Category of construct	
	Technical handle	Conceptual insight
(i). Definition of monotone sequence		
(ii). Use of PMI to prove sequence is monotone increasing: use of empirical verifications. Use of arbitrary elements		
(iii). Boundedness: structural relationship (e.g, $x_n - x_{n+1} < 0$) to prove that sequence is bounded.		
(iv). Procedural determination of limit (e.g. use of formula) Inferring lub from structural relationship		
(v). Approach (counter argumentation or deductive)		

Other associated actions

………
………

Researcher Comments

………
………

Figure 3: Observation guide for chalkboard demonstrations by student teachers for pilot purposes

The guide shown in Figure 3 restricted data collection because it was heavily influenced by my preconceptions of proof (e.g., syntactic and semantic approaches to proof construction, notions of technical handles and conceptual insights). That way the guide suffered from some sort of threat to validity referred to as theory validity (Lewis, 2009). Lewis use the term theory validity to refer to influence of the researcher's preconceptions about the phenomenon being studied. Further, completing the observation guide for chalkboard demonstrations was a challenge. I could not cope with the speed of the presenters. To overcome this challenge, videotaping was critical and I had actually to play the videos repeatedly in order to fill in the observation guide. For the main study, the focus of the instrument was on students' actions outside the lens of the camera. To counter theory validity threat described earlier I modified the guide based on my pilot experiences. The observation guide for the main study had the following format shown in Figure 4.

Student teacher…………………………………..

Task……………………………………………………………….

Student's actions, proof behaviour, utterances

………………………………………………………………………..

Researcher comments

…………………………………………………………………………..

Figure 4: Observation guide for chalkboard demonstrations for main study

Consistent with directed content analysis technique used to analyse data the modified guide stayed open for actions and proof behaviours not covered by what I had called "indicators" so that those actions and proof behaviours could also be recorded.

Threats to validity, dependability issues- like "thinking aloud" verbal protocols by students and reactivity matters were given due attention in the main study using validity checklists and reliability measures discussed. For example, the chalkboard demonstrations and the written responses were compared for similarities and differences (Corbin & Strauss, 2008; Hellink, Hutler, & Bailey, 2013). Comparing these two sources of data served as a test-retest method of measuring the reliability of data collected (Lewis, 2009)

Another useful lesson from the pilot relate to handling of data during data analysis. Data pieces should not be treated in isolation when addressing the research questions. There should be a forward and backward movement within the data sources. The back and forth movement within transcriptions in the pilot study was facilitated by case study analytical skills employed in this study such as pattern matching logic (Yin, 2009). Pattern matching logic was then employed in the main study to map emerging categories to theoretical constructs in the analytic framework because of its capacity to illuminate causal links in students' proof schemes during the pilot studies.

Finally, the pilot study provided an opportunity for me to identify a possible line of research to pursue that had a potential to inject new ideas into the main study. The pilot enabled me to identify inconsistencies in students' proof attempts. Preliminary directed content analysis revealed that students displayed a tendency to use formal axiomatic reasoning in proof tasks that required counter-argumentation and vice versa. Those contradictory behavioural tendencies were followed up in the subsequent main study. In other words, main study involved efforts meant to establish causes of the student formal praxis. Overall, the outcome of the pilot study was to proceed to do main study with modifications of the observation guide as well as a consideration of feasibility matters discussed here.

4.7 Data generation procedure

4.7.1 Preparing for data collection

I begin this section by describing the logistical issues dealt with in preparing for both the Mid-instruction and End-of-instruction data collection activities. Immediately after settling down in the department of the university that was the site of my data collection, I started to make preparations for the data collection process. I began by formalising research assistantship services, including negotiating a fee for the technical services and payment modalities of the research assistant fee. The fee was staggered over the whole data collection period. I decided against a once off payment because of the reason that the research assistant would possibly lose the urge to offer elegant services once the money got finished.

Second, I then addressed issues related to entry into the research setting. The purpose of the research was discussed with the participants. During the first three weeks I discussed ethical issues with student teacher participants. Ethical matters were discussed during formal lectures for the Real Analysis course that I taught, both as a guest volunteer but formal course lecturer for their grade in the course, and as a researcher for my study. Each lecture was 2 hours long and I met with the students twice a week for 15 weeks. Hence, there was prolonged engagement with the participants and therefore data collected were more direct, open and less subject to inference (Creswell, 2009). In other words intensive long term involvement with participants allowed the researcher to generate rich data as described by Charmaz (2014) and Maxwell (2004).

Third, logistical issues also included rehearsing some data collection processes in the research venue with the research assistant. A rehearsal time table was agreed upon with research assistant. We met once every week in the Mathematics lecture room to assess feasibility issues such as checking if there was adequate furniture in the venue, furniture arrangement, cleanliness and ventilation matters in the research venue. Further, rehearsal time was also used to check whether data capturing devices were functioning properly and to ascertain whether support facilities such as electric plugs were functioning properly. Logistical issues also included rehearsing with data capturing devices (video camera and audio recording device) and interview guides and testing the devices and interview guides for timing and sequencing of interview questions (Hennink, Hutter & Bailey, 2013). The video camera was used to record chalkboard demonstrations of students' proof attempts and the audio recorder was used for audio tapping students' utterances of their proof experiences at various scholastic levels.

Finally, the preparations for data collection involved securing resource materials such as white board markers, erasers and final touches on matters related to functioning of data capturing devices, particularly ensuring that audio recording device had adequate storage space and confirming that video cameras had no faults. A few days before the actual day on which data collection could take place I checked level of preparedness of students who were also reminded of agreed times during which they were to meet with the researcher in the venue.

The overarching goal of the study was to explore students' mental constructs around the notion of mathematical proof through an examination of the kinds of proof schemes held by undergraduate student teachers and how proof schemes emerge among Zimbabwean undergraduate mathematics student teachers. In other words, the study intended to investigate students' basis of schemes of argumentation in validating mathematical statements. In pursuance of this goal the following main research question was raised: *In what terms do undergraduate mathematics student teachers think of mathematical proof?* Hence, the fundamental goal was to explore students' reasoning about mathematical proof and to trace how such reasoning evolves among undergraduate student teachers.

With respect to students' reasoning, that is, the kinds of proof schemes held by students data collection was meant to address the research question: *What kinds of proof schemes characterise undergraduate student teachers' conceptualisations of mathematical proof?* Data collected to answer this research question consisted of: students' written responses to tasks, chalkboard demonstrations by students of solution attempts and reflective interviews of students' proof efforts. I now discuss the rationale for using these instruments and the kinds of data captured by the instruments.

4.7.2 Written tasks

Responding to written tasks was the first activity of the data collection process. I started by distributing answer booklets and task sheets to students seated in the lecture room. Students were then reminded about the purpose of the research, ethical considerations and that there were no time restrictions for the activity. I explained that the written exercise was not an assessment test for the Real Analysis course they were taking and for which I was lecturing to them in parallel to the related research activity. There was also no assessment mark to be awarded for the research activity. Further, students were asked to document as much of their thoughts as possible using structured derivation format which is now discussed.

Documenting one's thinking in writing is efficient in explicating one's thinking while writing a proof and hence the decision to engage students on written tasks described under data collection instruments. Students were encouraged to respond to the proof tasks using a standard mechanism called structured derivations. Black and von Wright (1999) cited in Manilla and Wallin (2009, p. 6) describe structured derivation as a modification of Dijstra's notion of calculational proof style where Black and von Wright devised a mechanism for doing sub-derivations and for handling assumptions in proof constructions. I had demonstrated how structured derivations are used during parallel teaching of the Real Analysis course

Structured derivations were used to facilitate the data collection process during this study of kinds of students' proof schemes and how those proof schemes emerge. I illustrate this standard format by considering a situation in which the learner is confronted with the problem: *Solve the inequality* $x^2 > 3x$ for $x \in \mathbb{R}$. The example on use of structured derivations in proof construction was adapted from Manilla and Wallin (2009, p. 65).

Solution: $x^2 > 3x$

{Add $-3x$ to both sides}……………………………………………………(i)

$x^2 - 3x > 0$

{Factorize} ……………………………………………………………(ii)

$$x(x-3) > 0$$

{By assumption both x and $x-3$ are positive, hence by the order properties of $\mathbb{R}$,

$ab > 0$ implies that $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$} …………………(iii)

As illustrated above with structured derivations we start with a description of the problem and its assumptions. The solution is then arrived at by reducing the original term step by step (Manilla & Wallin, 2009). Each step in the sub-derivation comprises a relation and an explicit justification, inside brackets, for why each term has been transformed from the preceding step. The justifications shown inside brackets constituted data for this study of students' proof schemes. As each step in the derivation was justified, the final product contained documentation of the thinking the student was engaged in while proving. Therefore it was anticipated that the explicated thinking through use of structured derivations would illuminate the kind of proof scheme held by undergraduate students in terms of the level of utilisation of key ideas and other underpinnings of mathematical proof.

I illustrated the structured derivation format just described to the participants through use of examples using calculus content material the students are familiar with. The main reason behind the illustrations was to encourage students to document their thinking as they completed tasks on proofs. During data collection students were asked to justify as much as possible steps in their proof attempts in order to enable the researcher to determine proof events from students' attempts (Bostic, 2016).

Data collection took place in the Mathematics lecture room. The seating arrangement was such that it allowed students to respond to the tasks individually (Housman & Porter, 2003; Selden & Selden, 2003; Varghese, 2009). I used the time students were working on the proof tasks to take field notes about students' actions, bodily expressions and facial expressions. The student teachers worked without researcher assistance until the task was successfully completed or reached an impasse, that is, there were no time restrictions imposed. Students spent about 1 hour 15 minutes on written tasks during Mid-instruction data collection phase. For the End-of-instruction data collection phase, students took about 1 hour 35 minutes working on the written tasks. After completing the written tasks, I perused students' written attempts to decide on the follow up questions during chalkboard demonstrations and the reflective interviewing. To increase validity of data and hence credibility of findings about the kinds of proof schemes held by students it was necessary to use a multitude of sources of data (Creswell, 2014; Lewis, 2009). Hence, I gathered more data about students' thinking about mathematical proof from chalkboard demonstrations of students' proof attempts. I now describe the chalkboard demonstrations by student teachers as they worked on the proof tasks.

4.7.3 Chalkboard demonstrations

Task-based interviews were followed by chalkboard demonstrations by students. I availed chalkboard erasers and 5 new white board markers for use by the students to demonstrate their solutions. Students were not told in advance the tasks they would attempt on the chalkboard. Doing so would have given students an opportunity to use 10 minute short break allowed between written tasks and chalkboard demonstration to prepare proofs in advance. Further, students could use time during which other students would be demonstrating to prepare their proofs. So such a measure was meant to avoid mere regurgitation of already prepared solutions but to dig deep into their thinking as they were confronted with the proof tasks. Each student was asked to solve a task on the chalkboard.

The research assistant recorded videos of the students' demonstrations. The video camera recorded duration of each demonstration which was then also recorded by the researcher. I used the period when students were working on the tasks to record proof behaviours and actions outside the lens of the video camera. Video recordings of students' chalkboard demonstrations were then studied in order to determine questions to include in the reflective interviews on proof task attempts by students. This exercise involved playing each video recording several times to determine the range of issues raised by students and depth involved in each proof attempt (Hennink, Hutter & Bailey, 2013).

Mid-instruction and End-of-instruction data collection activities consisting of written tasks, chalkboard demonstrations and reflective interviews took place in two days. It was observed during the pilot study that going through all the three activities in one day was not feasible because participants were likely to develop conversational fatigue (Maxwell, 2004). Effects of such fatigue would lead them to what Corbin and Strauss (2008, p. 67) call a "rush" past diamonds in the rush. In other words, I was likely to miss some crucial data because conversational fatigue was likely to stifle data collection process. So the students answered the written tasks and performed chalkboard demonstrations during day one. Reflective interviews took place the following day and I now focus on how the reflective interviews were conducted.

4.7.4 Reflective Interviews on written responses and chalkboard demonstrations.

The proof construction exercise and the chalkboard demonstrations were followed by reflective interviewing where students were to justify specific actions taken. Students' articulations of specific actions and their beliefs serve as indicators of causal processes and mechanisms responsible for learning events (Housman & Porter, 1997; Maxwell, 2007). One of the key features of scientific realism is that mental events, beliefs are treated as real phenomena that are responsible for behaviour (Maxwell, 2007). This realist stance provided the basis for how I interpreted student teachers' actions and verbal and written articulations as manifestations of the students' understandings of their proof constructions.

The theme of the interview was on the nature of causal links within student teachers' proof schemes. On day two reflective interviews were conducted in order to collect strands that would be used to explore students' thoughts when they had engaged with the proof tasks through written responses and chalkboard demonstrations. The purpose was to approach the unknown point, that is, the kinds of proof schemes from more than two known points (Lewis, 2009). The known points in the context of this study were the written tasks and the chalkboard demonstrations. Therefore with respect to research question one, reflective interviews served as some form of triangulation measure that would strengthen the validity of inferences about the kinds of proof schemes held by undergraduate mathematics student teachers (Creswell, 2009; Lewis, 2009).

Hence, I engaged students with reflective interviews in order to validate students' thinking as they engaged with proof attempts. The in-depth reflective interviews involved all 10 BEd students on areas I had identified to have the potential to provide rich data, based on students' written responses and chalkboard demonstrations. The students' verbal reformulations of given proof concepts were considered as part of the students' evoked theorem and proof images that would in turn indicate the kinds of proof schemes held. The other goal of this study was to conceptualise how student teachers develop their proof schemes by addressing the research question: *how do undergraduate student teachers develop their proof schemes?* I now describe how reflective interviews were used to gather textual data that illuminated how students' proof experiences shaped how the mathematical object, (i.e., proof scheme) emerges among student teachers.

4.7.5 Reflective interviews on student proof experiences

With respect to research question two, the major theme of the interviewing sessions for both Mid-instruction and End-of-instruction data collection phases was on causal processes and mechanisms about how students' proof schemes emerge. Before conducting reflective interviews I sought the services of the technical assistant to check whether the audio recorder was functioning properly. The interviewing process started with questions meant to allow the interviewees to settle down and also to gain insights about their conceptions of mathematical proof. I then conducted individual interviews on students' experiences with mathematical proof at different scholastic levels by posing the questions such as: (i). describe your pre- A level, A level and undergraduate experiences with mathematical proof. (ii). what differences have you noted at different levels about your proof experiences? For further details of reflective interview guide refer to Appendix C.

Students produced narrative accounts of their proof experiences. The intent of the interviewing was to tease out from students' descriptions different trajectories through which their proof schemes come into being and the structure of the nature of the proof scheme, that is, its ontology (Lawson, 2009). The undergraduate students responded to the interview questions verbally and these were audio-taped. Verbal responses were intended to remove the strain of writing (Punch, 1998) and served as some form data triangulation.

Data collection was guided by the notions of theoretical sampling and theoretical saturation (Charmaz, 2006), meaning reflective interviews continued until no further theoretical insights could be detected from data. Hence, I played audio tapes later played to determine completeness and adequacy of data. Areas that needed clarification were identified for further reflective interview auditing which is now the focus of the next section.

4.7.6 Reflective interview audits for tasks and students' proof experiences

The purpose of the interview audits was to seek clarity on students' thoughts as they solved the proof tasks as well as getting clarifications on students' experiences with mathematical proof. For instance during piloting of research instrument, some inconsistency noted in students' formal rhetoric aspects was revealed where they showed a tendency to use axiomatic reasoning in proof situations requiring use of counter examples and vice versa. The contradictory behavioural tendency just described was further explored through interview audits to develop an account such proof behaviour.

Another purpose served by the reflective interview audits was to tease out reasons for the impasse reached by the student teachers when they engaged with the proof tasks in order to explicate students' thinking. Reflective interview questions crafted were in line with the research goal of trying to stipulate a set of causal links in students' proof schemes. Furthermore, reflective interview audits served as some form of member checking whereby I solicited feedback about data and my preliminary interpretations of those data from student teachers (Creswell, 2009; Lewis, 2009; Maxwell, 2004). I referred to the interpretations as being "preliminary" because they emanated from my initial readings of the data. Member checking was a crucial technique meant also to strengthen the validity of data elicited and hence ensuring credibility of the findings. Interview audits can be seen from the perspective of member checking since they provided me with an opportunity to correct errors and misinterpretations and to incorporate the students' perspectives and meanings in data interpretation.

Developing an understanding of a phenomenon from participants' own perspectives is a central requirement of the scientific realist philosophy that guided the study (Lewis, 2009; Maxwell, 2004). However, use of interview audits as some of member checking was employed while taking into account the fact that member checks are not always accurately produced because they could be constrained by, for instance, the lecturer/student relationship I shared with the participants. Hence in the next section, I discuss efforts taken to strengthen reliability and validity of the study.

4.8 Validity and reliability issues

In this section I discuss efforts intended to determine if proof tasks, chalkboard demonstrations and reflective interviews could provide the kind of information needed to give a revealing picture about the kinds of proof schemes held by undergraduate student teachers as well as how these proof schemes emerge. A number of threats to validity were addressed in this study.

First, this study of students' proof schemes employed verbal- protocol methods during the reflective interviewing phase. During such interviews students were asked to "think aloud" while validating mathematical statements. One of the major potential threats to validity was that with the "think aloud" verbal protocols students could only verbalise the conscious components of the proof processes and yet some processes involved in proof and proving may not be part of their conscious efforts (Selden & Selden, 2003, p. 5). It is therefore possible that some crucial information regarding how students would have completed the proof tasks might be left out during the reflective interviewing phase. This validity threat was countered through triangulation. In other words data from the three sources: written responses, chalkboard demonstrations and interview audits were compared.

Second, data might have contained some errors made during transcribing. Transcribing involves producing a written record of the interview. I checked for accuracy of transcripts to ensure that they do not contain mistakes (Corbin & Strauss, 2008; Creswell & Miller, 2000; Creswell, 2014). I read the transcript while playing the audio recorder several times to check for mistakes during transcribing (Hennink, Huttler & Bailey, 2013). Hence, to increase validity of data, there was back and forth movement within and between transcripts. For instance, I could identify a category in one transcript and then had to re-read earlier parts of that transcript for the same student in order to clarify the issues about that particular category. Alternatively, I could switch from Mid-instruction to End-of-instruction where similar words or phrases had been uttered by the same student. That way, I was able to distinguish between a passing mention of an issue from one which was a meaningful category that I could then explore further during the interview audits.

Third, another threat to validity is related to the notion of veridicality suggested by (Padayachee et al. 2011, p. 22). The requirement that students verbalise their thinking processes could have reduced the level of cognitive resources undergraduate mathematics education students devoted to the primary tasks, that is, solving problems related to theoretical underpinnings of mathematical proof (Lewis, 2009). Therefore, the data elicited through verbal protocols may not reflect with total accuracy the thinking processes students engaged in as they resolved the tasks on proof and proving. Another important resource pertinent to proving is the prover's command of language. For instance, regarding the characterisation of connected sets in R a description of the proof of the biconditional statement: *a subset of* $\mathbb{R}$ *is connected if and only if it is an interval*, may present some challenges to undergraduate student teachers because an outline of the theorem involves showing that there does not exist two open sets U and V that satisfy axioms for connectedness. To deal with effects of veridicality just described triangulation was employed to invoke different proof scheme states among students. Furthermore, data collection place over two days to reduce conversational fatigue (Maxwell, 2004).

Another threat to validity of data suggested by Shadish, Cook and Campbell (2002) is concerned with the question about whether data are forced to fit a theoretical orientation of the researcher. Hence, in order to resist the temptation to impose data on, for instance, proof scheme taxonomies, ideas about semantic and syntactic proof constructions, I tried as much as possible to be open minded and allowed data to speak for themselves. For example, in summative content analysis of students' proof experiences categories were inductively driven with in-vivo codes used to exemplify the inductive categories (Corbin & Strauss, 2008; Punch, 2005). In other words, inductive categories were derived using actual words and phrases from the data. That way I avoided forcing data to fit a given theory. Further, adhering to case study skills suggested by (Yin, 2009, pp. 68-72) that include asking good questions, being a good listener and avoiding being trapped by my grasp of issues to be investigated contributed significantly in minimizing the effects of theory validity described here. Thus, when studying undergraduate students' proof schemes, caution was exercised to ensure that students' conceptions are not contaminated by my ideologies or preconceptions of mathematical proof schemes.

The underlying assumption in data analysis was that those students' kinds of proof schemes could be illuminated through level of accessibility to key ideas and other underpinnings of mathematical proof in constructing proofs. The idea was to try to be unbiased in data collection by being mindful of the influence of my presence in the research setting more so in light of the tutor-student relationship I had with the participants. A pertinent question in this respect was: how much of what I was observing was being influenced by my presence in the research setting? The term reactivity is used to refer to the influence of the researcher's presence in the research setting, which for this current study was a teaching experiment. Maxwell (1996) cited by Lewis (2009) states that interviewees often have a tendency to react to the interviewer and not the situation being observed. This study was done alongside the usual teaching for the Real Analysis course which was to be examined at the end of the semester. It is possible that the desire to impress the instructor (researcher) might have influenced students during the interviewing process. Thus despite my assurances that students should not fear any retribution of any sort as a result of their participation in the study, students' motive to participate might have been out of the need to be close to their tutor. In other words, while student teachers were assured that participation was voluntary, the motive to participate in the study could have emanated from the need to impress their lecturer for the course. Intensive long term involvement in the research setting allowed me to develop mutual trust with the student teachers who then could even suggest possible changes to data collection schedule I had proposed.

An important data analysis activity that has far reaching consequences on reliability and validity of research findings is coding of data. Corbin and Strauss (2008) define coding as deriving and developing concepts from data. Regarding coding of data, I used a system that enabled me to keep track of which participants go with which set(s) of data. The strategy was useful when I took my interpretations of students' proof attempts and explanation(s) about how those proof schemes develop back to the students in order to check for validity of inferences the researcher had made or seek further clarifications and elaborations from the students (Padayachee et al., 2011, p. 22). This technique of assessing the validity of interpretations about data and inferences made from these data from research participants is called member checking (Cresswell, 2009, 2014). Member checking strengthened the validity of the study because the member checking strategy made it possible to consider the perspectives of students and meanings they attached to inferences I had drawn. This is a crucial factor in the realist process theory approach that informed the study.

Intensive long term engagement in the context of a teaching experiment allowed me to collect rich data that were more direct and less dependent on inference (Cohen, Manion, & Morrison, 2011; Maxwell, 2004). Intensive long term involvement of the researcher is one way of developing explanatory theory in qualitative research. The intent was to generate rich data (Charmaz, 2006; Maxwell, 2007; Yin, 2009). For this study, I met with undergraduate student teachers for 4 hours per week during the 15 week long semester during which the teaching experiment was conducted. Prolonged interaction with student teachers in the teaching experiment allowed me to generate rich data. Rich data often and erroneously referred to as thick description are data that are detailed and more revealing about the nature of causal links in students' proof schemes. Thick description on the other hand refers to descriptions of observable features or changes within a phenomenon without giving an account of reasons giving rise to such changes. Thus in thick descriptions there is no articulation about how and why such features or changes came about. The focus of the study was on generating rich data as opposed to thick descriptions. Soliciting rich data through prolonged engagement allowed me to identify the range and depth of issues involved in proof construction from the perspectives of the students. Further, prolonged contact with participants allowed me to distinguish substantive categories within students' proof schemes from a passing mention of a category that is typical of research designs in which data collection takes place over a short period of time.

Another area of focus for this section had to do with reliability concerns. Lewis (2009) defines reliability as a measure of whether a particular research instrument will yield the same data if applied repeatedly to the same participant. Two reliability measures were applied: test-retest method and the split half method. The two methods were deemed appropriate because they served as means of assessing the reliability of data from the three different sources. I describe each method of assessing reliability and illustrate how it was applied during data collection in this pilot study.

The test-retest method is a method of ensuring the reliability of data by determining whether previously gathered information from a respondent is accurate. The underlying idea is that once the information has been collected, the researcher can then check its accuracy by interjecting the information later into the conversation. Research question one: *What kinds of proof schemes characterise undergraduate student teachers' conceptualisations of mathematical proof?,* was addressed through students' proof attempts in written form from task-based interviews and through student teachers' articulations and 'workings' during chalkboard demonstrations. To assess their consistency the two sources were then compared for differences and similarities during data collection as some form of test-retest reliability measure because student chalkboard demonstration involved the same tasks they had solved in task-based interviews earlier on.

Another method is the split–half method for estimating reliability of data collected. The split- half method determines the reliability of data by soliciting several responses to the same question framed in different manners (Lewis, 2009). For example, regarding the reflective interviewing that sought to explore different proof scheme trajectories, the question: *How do you compare your A level and current proof experiences*? and the instruction: *Describe any differences you noted in your A level and current proof experiences,* though framed differently were both aimed at evoking differences in 'A'-level and undergraduate proof experiences.

The scientific realist process approach was used to conceptualise the kinds of proof schemes held by the students and how the proof schemes evolve. While proving events and processes are real observable phenomena, Dunn (1978) in Maxwell (2004) cautioned that there are no cheap ways to deep knowledge about people's lives. Hence, processes and mechanisms involved in proving were not easily observable but rather characterised by problems. Therefore the discussion about validity and trustworthiness of findings would be incomplete without examining potential problems associated with explanation building in the case study.

Explanation building process is iterative that requires comparing propositions developed in the study with those in the literature as one refines the emerging hypotheses from data (Yin, 2009, p. 141). A lot of analytic insight was needed during the iterative process so that the research could not drift away from the study of student teachers' proof schemes. Yin suggests that constant reference to purpose of the inquiry can help reduce the potential problem. Therefore I tried to stay focused on the goal of developing an explanation about the kinds of proof schemes held by students and to conceptualise how proof schemes emerge. These were the most significant goals of the study. Constant reference to these goals helped in avoiding detouring into less important issues.

Having noted the need to remain focused on the research goals, there was also need for me to attend to all evidence in the analysis of case study data (Yin, 2009, p. 141). Attending to all evidence implies that I should also account for disconfirming evidence. Yin suggests that rival explanation be defined and explained in the midst of data analysis. For example, while literature has pointed to the tenacity within the empirical proof scheme, possible contradictory proof behaviour deserved careful attention during data analysis for findings to retain credibility. Having examined problems that characterised explanation building in the case study, I still need to address the question: How transferable are the research findings to other similar contexts? Hence, transferability of case study findings becomes the focus of the next section.

The discussion on transferability of findings begins with a recap of the thesis title: *undergraduate mathematics students' conceptualisations of mathematical proof.* While generalisations are not necessarily the main objectives of many qualitative research projects (Punch, 2005, p. 146), findings from case can be transferable. Hence, results from this case study that embraced both exploratory and explanatory features can be more broadly applicable to other undergraduate students in Zimbabwe and other countries. Therefore, while there were only 10 undergraduate students involved in the study the question of transferability of results was addressed in the following manner.

It is possible for a researcher to produce transferable results from a case study depending on the degree of rigor in data analysis (Punch, 2005, p. 146). Hence, the purpose of a case study has a bearing on data analysis. Punch suggests that if a case study seeks to conceptualise, that is, to stipulate a set of causal links within a phenomenon studied then results can potentially be applicable to other similar contexts. A major goal of the study was to develop an explanation about the kinds of proof schemes held by the students in terms of their level of utilisation of technical handles and conceptual insights and other underpinnings of mathematical proof such as micro reasoning.

Another major goal of the study was to conceptualise how students' proof schemes emerge, that is, developing a hypothesis about how students develop their proof schemes. Further, as stated earlier the way data are analysed impacts on transferability of findings (Punch, 2005, p. 146). For results to be transferable data analysis needs to be conducted at a sufficiently high level of abstraction. The study focused on developing constructs, a process which raises data analysis above simple descriptions of what students do during proving. Consequently, I can claim with some confidence that the findings of this study can be used to represent students' thinking about mathematical proof in other institutions in Zimbabwe and other countries. Having taken steps to strengthen validity and hence credibility of the study, attention then shifts to preparing data for the subsequent data analysis process.

Table 4.1 : Summary of data gathering process

Phase	Activity	Instrument	Purpose
Mid-instruction data collection	Task-based interviewing (1 hour 15 minutes) Chalkboard demonstrations (1 hour 45minutes)	Task sheets Proof tasks Observation guide	To solicit textual data (written responses) for research question
	Reflective interviews on proof attempts (2hours 20 minutes) Reflective interview on students' proof experiences (2 hours)	Interview guide on proof attempts Interview guide on students' proof experiences	Textual data for research question 1 To address research question 2
	Reflective interview auditing on proof attempts (1 hour) Reflective interview audits on emerging themes proof (1 hour 10 minutes)	Reflective interview guide	Further exploration of students' kinds of proof schemes Further exploration of emergent trajectories of proof schemes
End-of-instruction assessment data collection	Task-based interviewing (1 hour 35 minutes) Chalkboard demonstrations (2 hours 40 minutes)	Written tasks Proof tasks Observation guide	To generate data meant for data for research question 1
	Reflective interviews on proof attempts (1hour 43 minutes) Reflective interview on students' proof experiences (1 hour 27 minutes)	Interview guide on proof attempts Interview guide on students' proof experiences	To solicit data intended for research question 1 To address research question 2
	Reflective interview auditing of proof attempts (1 hour 15minutes) interview audits on emerging themes (53 minutes)	Reflective interview guide Reflective interview audit guide	Further exploration of students' kinds of proof schemes Further exploration of emergent trajectories of proof schemes

4.9 Data analysis

4.9.1 Preparation for data analysis

Preparing data for analysis involved three main tasks: producing verbatim transcriptions of the interviews, data cleaning, and anonymising data. These tasks were intended to increase the credibility of data in order to allow me to immerse in valid data for the purpose of identifying and interpreting students' experiences with mathematical proof. A description of the three tasks is now presented.

Producing verbatim transcription involved making a written record of interview data. The purpose of the research was to develop a proposition about how the mathematical proof scheme evolves among student teachers. The focus of the interview transcriptions was on what was said and how it was said. I tried to produce a word-for-word replica of the spoken words (Hennink et al., 2013, p. 210). When transcribing of data, I paid attention to some aspects that would aid data interpretation such as speech fillers (e.g., eee you know), verbal gestures (e.g., umm aaah) and pauses. Verbal transcriptions were in this case informed by the multi-faceted scientific realist question: what works for whom, in what circumstances and how (Pawson & Tilley, 2004). Further, the verbal gestures, pauses were treated as real observable phenomena that were intrinsically relevant to the explanation of how proof schemes evolve among students.

Verbatim transcriptions also included emotions expressed by student teachers. Emotions expressed by student teachers were indicated in square brackets, […] while verbal gestures such as, umm aaah, were captured in parentheses (…). The verbal gestures and emotions were treated as real observable phenomena that are causally relevant to the explanation of how proof schemes come into being (Maxwell, 2004).

Each verbatim transcription captured words spoken by both the interviewer and interviewee, that is, the questions I posed and student teacher participant responses. In addition, verbatim transcriptions produced identified the speakers and differentiated words spoken by myself from those uttered by the students about their proof experiences. Furthermore, during the interview recordings I made efforts to achieve descriptive validity by capturing everything said in the student's exact words.

With regard to the data cleaning process it was necessary to check each verbatim transcription for accuracy and completeness. Some of the steps taken to guarantee accuracy of verbatim transcriptions include: listening to audio-tapped interview while reading the written record of the interview in order to identify any omissions and inaccuracies. After cleaning, each transcript produced was then labelled with a filename so that I could quickly identify the verbatim transcription whenever the need to do so arose during data analysis. Hence, it was necessary to create a folder named: Reflective interviews for Mid-instruction data collection. An extract of this folder that involved my conversation with a student called Taku had the following format shown in Figure 5.

Filename: Taku

Study site: Department of Science and Mathematics Education

Venue: Mathematics Lecture Room

Gender: Male

Interviewer: Researcher

Date:

Duration of interview:

Researcher…………………………………………….

Taku…………………………………………………

Figure 5: Mid-instruction reflective interview transcript for student teachers

The dotted space represents the recorded conversation between the student teacher and me. Similar folders of interview transcripts for the End-of-instruction and reflective interview audits were created. Finally, there was anonymizing of data. Anonymizing involved removing any identifiers from the transcript to preserve the anonymity of the student teachers. Students' names were removed and some pseudonyms were used instead. Once data were anonymized I was then ready to begin the process of data analysis which is the focus of the next section of this chapter.

4.9.2 Data analysis procedures.

The main goal of data analysis was to establish students' thinking around the notion of mathematical proof. The word "thinking" has dominated discussions in many sections of this thesis so far so I considered it necessary to begin this section by defining the word thinking. Sfard (2008) conceives thinking as some kind of intrapersonal communication wherein one argues, asks questions, formulates and waits for one's responses as well as informing oneself. In the context of this study mathematical thinking is used to denote the intrapersonal communication the student teachers engaged in as they argued, questioned the validity of mathematical statements when they engaged with proof tasks. The aim of this study was to determine the kinds of mathematical thinking involved in proving and how such thinking evolves.

I accordingly discuss data analysis procedures per research question. First, I discuss data analysis that addressed research question one: *What kinds of proof schemes characterise undergraduate student teachers' conceptualisations of mathematical proof?* The essential idea is that an explanation for the kinds of proof schemes held by students and a hypothesis about how proof schemes emerge were inductively developed (Charmaz, 2006, 2014; Punch, 2005, p. 243). In other words, theory was abstracted from data. Abstracting means concepts and propositions were inferred from data.

The logic of my analytic framework, that is, the main ideas that drove data analysis and interpretation are as follows. Proof can be conceived as part of discourse of practice in which we differentiate discourse with oneself from discourse with others (Sfard, 2008). Discourse with oneself is similar to Harel and Sowder's (1998, 2007) notion of ascertaining when one is trying to convince oneself about the validity of a mathematical statement. Discourse with others is similar to the notion of persuading which, according to Harel and Sowder, is when one strives to convince others about the truth of a mathematical statement. Mason (1985) cited in Raman (2003) proposes a pedagogical suggestion of creating a proof that bears some resemblance with Harel and Sowder's two notions of ascertaining and persuading in the following manner. Mason (1985) in Raman (2003) suggests that the proof construction process should involve convincing oneself, convincing a friend and finally convincing one's enemy. Thus discourse of practice regarding proof and proving can be construed as efforts to create self-conviction as well as trying to communicate the truth or falsehood of a conjecture with others. I argue here that the kind of discourse of practice is determined by the degree of accessibility to mathematical underpinnings of mathematical proof such as heuristic and procedural ideas which in turn would illuminate the kind of proof scheme held by an undergraduate student teacher.

Because of the internal and personal nature of a student's proof scheme the state and characterisation of proof events and mechanisms within the proof scheme could be inferred from students' verbal and written communication. Hence, a student's explanation of his or her proof construction attempts to a given task provided a glimpse of the kind of proof scheme(s) held by that student. Given the impossibility of directly observing the kinds of students' proof schemes and the corresponding key ideas employed in accomplishing given proof tasks I drew on Dahlberg and Housman's (1997) notion of a learning event and a closely related concept of a proof event by Bostic (2016).

A learning event is said to have occurred when a student communicates and applies a new understanding of a concept. A communication could be in the form of an utterance or in written form as students describe or justify the basis of their argumentation schemes when validating statements in completing proof tasks. An application involves using both heuristic and procedural ideas (key ideas) to solve proof tasks, re-explain proof steps during the reflective interview phase and working on the given tasks during the task-based interviews (Housman & Porter, 1997; Moore, 1994). A proof event is said to have occurred when a conjecture and a justification have been provided (Bostic, 2016).

I argue here that proof events are some special form of learning events in which underpinnings of a mathematical proof are applied to produce conjectures and their justifications. This study restricts discussion of a proof event to production of a justification of a given statement that is, to schemes of argumentation used to validate given statements because the study did not involve conjecturing. To reiterate, learning events and proof events are real entities that can be directly observed (Maxwell, 2004; Maxwell & Mittapalli, 2007, 2010).

This section therefore, focuses on how qualitative data, that is, text from written responses and transcripts of audiotapes and chalkboard demonstrations were interpreted using the realist process approach that drew upon Dahlberg and Housman's (1997) notion of a learning event and Bostic's (2016) idea of a proof event. Precisely, the notion of a proof event was used to determine the kinds of proof schemes held by undergraduate student teachers by allowing me to determine the extent to which key ideas of a mathematical proof were utilised in the task-based interviews in the manner observed. Hence, the student teachers' proof behaviours exhibited as the students engaged with proof tasks were some form of mappings to kinds of proof schemes held by the students. In other words, those observable proof events, that is, students' proof behaviours illuminated the kinds of proof schemes held by those student teacher informants.

Therefore the degree of utilisation or lack thereof key ideas of a mathematical proof assisted me gain insights on the structure of the proof schemes held by students and hence a measure of students' understanding of mathematical proof (Hanna & Mason, 2014; Raman, 2003). Further, in studying the kinds proof schemes held by students, a pertinent question is: How did the key ideas arise, that is, how did the key ideas come to students' minds? Did such ideas arise through generational memory, that is, as side effects or consequences of the sense making process during proving (Gowers, 2007, p. 40)? In generational memory key ideas are accessed without much intentional effort through processes such as evocation and recognition. Therefore body language, students' questions and comments during reflective interviews phase and chalkboard demonstrations could provide cues on whether access to key ideas would have been through generational memory or rote memory during proving.

Selden and Selden (1995) argue that students' inability to convert informally written mathematical statements into the language of calculus prevents them from recognising the logical structure or proof framework that enables them to determine the correctness of a given statement. That is, students are not able to validate the statement. Raman (2003) defines a key idea as the mapping of a heuristic idea (conceptual idea) into a formal proof with appropriate mathematical rigor (technical handle). I suggest that the process of converting informally written mathematical statements into the language of predicate calculus prior to the application of technical handles (procedural ideas) depends on the availability of relevant conceptual insights within the students' proof schemes.

From this discussion a crucial issue that informed the data analysis process had to do with recognising which underlying ideas are easily accessible to students within a proof scheme category. For instance, I needed to determine whether the identified proof scheme was dominated by procedural ideas or heuristic ideas and how those ideas were connected. Recalling that, ontology is the study of the nature of existence of objects and their relations, how then in light of this definition could the nature of relationships of objects be determined? Having discussed the focal issues regarding research question one now focus on how data were analysed to address issues raised.

Berg (2009) identifies three main approaches to the analysis of qualitative data namely: interpretative, social anthropological, and collaborative social research approach. This study used an interpretative approach, an orientation in which the researcher treats human action as text. Berg observes that a researcher with this theoretical bent will see human action as a collection of symbols expressing layers of meaning. For instance, it was anticipated that students' comments and facial expressions could provide clues on whether key ideas come to students' minds either through generational or rote memory. This study adopted a more general interpretative orientation whose aim was to organize and reduce data through identification of patterns, and analytic categories within data.

Which specific technique was used to unravel the set of students' ontological commitments? The specific method in line with nomothetic approach of knowledge generation is content analysis which Hsieh and Shannon (2005) define as a careful, detailed, systematic examination and interpretation of qualitative data intended to identify patterns, themes, biases and meanings. Hsieh and Shannon discuss 3 main techniques to the conduct of content analysis of qualitative data which differ depending on the degree of inductive inferences made during data analysis. The three content analysis techniques according to Hsieh and Shannon are: conventional, directed and summative content analysis.

Conventional content analysis involves using codes and categories that have been directly and inductively developed from the data themselves. The purpose is to develop theories and explanations grounded in the data. Consequently other methodologists often refer to it as Grounded Theory Analysis (GTA). Summative content analysis is very much similar to conventional content analysis described here since a researcher using summative content analysis develops codes and categories by beginning from words and phrases in the data themselves hence the term summative content analysis (Berg, 2009, p. 339).

Directed content analysis allows the researcher to immerse herself/himself in data guided by codes and categories derived from existing theories and explanations relevant to the research focus as well categories that can emerge from the data themselves (Hsieh & Shannon, 2005). To address research question one, directed content analysis was deemed appropriate. Directed content analysis allowed me to engage with data while being guided by theories that include proof scheme taxonomies (Balacheff, 1998, Stylianides, 2011) and mathematical underpinnings of the notion of proof and notion of key ideas by (Hanna & Mason, 2014; Raman, 2003). Directed content analysis is thus versatile and flexible as it also allows the analyst to stay open for inductive codes and categories that could possibly be inferred when data were fractured and hence its preference to the other content analysis techniques in addressing research question 1.

Directed content analysis of data from written responses, chalkboard demonstrations, and reflective interviews on tasks was accomplished in the following way. For each student and for the Mid-instruction and End-of-instruction assessment data collection phases a data matrix with the following column entries was constructed. Column 1 entries are proof tasks attempted by the student. In column 2, entries consist of written responses, utterances from reflective interviews on the tasks, students' actions, transcriptions from chalkboard demonstrations. Column 3 entries are profiles of students' proof attempts. These profiles are my own descriptions of the students' proof efforts. In the fourth column, I recorded descriptions of identified proof scheme elements and interpretations of students' proof attempts in terms of the mathematical proof concepts drawn from the analytic framework. Hence, for each student two data matrices: Mid-instruction and End-of-instruction assessment data matrices were constructed for the Real Analysis tasks attempted. For each student teacher, data matrix for Mid-instruction data collection phase has the following format;

Table 4.2: Mid-instruction assessment data matrix (e.g.,Tino)

Task	Student's response behaviour (written, oral, actions)	Profile of student's proving	Proof scheme elements
………	………………………………………………… …………………………………………………	…………………… ……………………	……………….

This matrix was then followed by the End-of-instruction assessment data matrix for the same student. Thus, for each student two matrices were composed giving a total of 20 data matrices for the 10 students involved in the study. The corresponding End-of-instruction assessment data matrix has the same format;

Table 4. 3: End-of-instruction assessment data matrix (e.g., Tino)

Task	Student's response behaviour (written, oral, actions)	Profile of student's proving	Proof scheme elements
………	………………………………………………… …………………………………………………	…………………… ……………………	……………….

The reason for having the End-of-instruction assessment data matrix immediately after the Mid-instruction assessment data matrix was to scrutinise and compare the student profiles of their proof behaviour for differences and similarities (Hellink, Butter & Bailey, 2013). The two matrices illustrated here represent the first level in data reduction of the data analysis process. This corresponds to level of discrete facts in the nomothetic view of knowledge (Punch,1998, 2005). Columns 3 and 4 entries represent the first level of the data reduction process. These matrices constitute the greater part of Chapter Five.

At the second level of the data reduction process composite profiles of student proof behaviours were constructed using fourth column entries of level 1 data matrices. Scientific realist process approach holds that mechanisms and processes connecting events in a phenomenon being studied are real observable entities. So the proof events and processes were treated as directly observable real entities even in single cases, e.g., individual students without requiring a comparison group (Maxwell, 2004; Pawson &Tilley, 2004). So this realist feature allowed me to observe proof schemes even from level 1 data matrices of individual students. Hence, in precise terms, proof scheme elements observed from the fourth column entries of level 1 data matrices were used to construct the composite profiles summary matrix that has the following format.

Table 4.4: Composite profiles of students' proof behaviours

Student teacher	Summary of proof behaviour s observed (proof scheme elements)
Taku	…………………………………………………………………………………………
………………	………………………………………………………………………………
Getrude	…………………………………………………………………………

For the 10 student participants, a matrix was then composed. Table 4.3 illustrates level 2 of data reduction of the analysis process (Miles & Huberman, 1994). In terms of Punch's (1998, 2005) nomothetic perspective of scientific knowledge generation, the summary of composite profiles of student proof behaviours represent level 2 of abstracting concepts from data at which there are empirical generalizations. Level 2 generalizations are so called because theory at this level is not yet refined. Then column 2 entries derived from directed content analysis of students' proof behaviours were once again mapped to constructs from the analytic framework as an effort to build an account of the empirical generalizations from level 2. In other words, discussion of composite student profiles of their proof behaviours was an attempt to build an explanatory theory for the kinds of proof schemes held by undergraduate students.

Another major goal of the study was to determine how students' thoughts about mathematical proof evolve. This goal was pursued by addressing the research question: *How do the undergraduate student teachers develop their proof schemes?* Data available to answer this question were from in-depth reflective interview transcriptions of students' descriptions of their proof experiences. So summative content analysis was used in this collective instrumental case study with an exploratory bent to develop a proposition about how proof schemes emerge among the students. Briefly, summative content analysis begins with phrases and words in the data themselves to build inductive categories (Hsieh & Shannon, 2005). This underlying idea (use of exact words and phrases spoken by participants) of summative content analysis was instrumental in establishing the criterion of selection in content analysis of data. So the objective analysis of messages conveyed in the transcriptions from audio tapes from in-depth reflective interviews and reflective interview audits were accomplished by first establishing a criterion of selection (Punch, 2005; Yin, 2009).

The criterion of selection refers to explicit rules which must be consistently applied and adhered to during the course of data analysis so that other researchers or readers scrutinising the same data get the same or comparable meaning, patterns and themes (Berg, 2009; Yin, 2009). The categories that emerge when developing the criterion of selection must be sufficient to cater for all relevant aspects and variations in the undergraduate student teachers' proof construction attempts (Marshall, 2006, p. 158). Further, phraseology is a crucial consideration in developing the selection of criterion in the sense that categories must retain as much as possible the exact words or phrases used by the undergraduate student teachers as opposed to an arbitrary naming or superficial naming of the students' statements (Hsieh & Shannon, 2005; Yin, 2009).

Along with Berg (2009), Hsieh and Shannon (2005) and Yin (2009) summative content analysis of data involved use of in-vivo codes, that is, use of actual words and phrases spoken by the student teachers. During summative content analysis the validity of words and phrases (i.e., categories) that constituted students' utterances of their experiences with mathematical proof was checked by identifying whether they were repeated across different interviews (Corbin & Strauss, 2008; Hsieh & Shannon, 2005 in Berg, 2009; Hennink et al., 2013). Hence, frequencies of such categories were then determined to differentiate codes from just a passing mention of an issue by students.

To develop inductive codes during summative content analysis of data, first, the data were read for overall content where I identified the range of issues raised in the interview transcripts and the depth of each narrative account of student teachers' experiences with proof as proposed by Hennink, Hutter and Bailey (2013). In this regard, specific words and phrases used by participants were noted and their frequencies recorded. Second, textual data were annotated by writing down explicit issues raised by students about their proof experiences. Annotating data involved posing questions and noting issues about categories I had to pursue further by way of reflective interview audits. Third, I paid attention to changes of topic in students' narrations of their proof experiences in order to identify potential categories. Hennink et al. suggest that a natural change in the topic may indicate the end of an issue and the beginning of another issue. Hence, one of the strategies employed during summative content analysis involved checking for topic changes in student teachers' utterances of their proof experiences. This helped to delineate categories in students' proof experiences. Having discussed strategies used to develop inductive categories I now discuss how textual data were used to develop a proposition about how students' thinking around mathematical proof evolved among undergraduate student teachers.

First, verbatim transcriptions were studied to determine students' conceptions of mathematical proof. Emerging categories were formed using the exact words spoken by students. This resulted in a n by 3 matrix where n, the number of rows was determined by the number of categories that emerged from the data. The column entries that were constructed are as follows: column 1 is made of the identified category, column 2 entries were exemplifications of data that belonged to that particular category, and the third column captured the frequencies of the categories that emerged to reveal a picture of dominant characteristics in students' proof experiences. Consequently, the data analysis matrix for research question two has the format:

Table 4.5: Mid-instruction reflective interview on for example ways to gain conviction in proofs

category	Example of student utterance	Frequency (f)
……………	…………………………..	……………..
…………….	…………………………..	………………..
n…………	…………………………….	……………….

Matrices of the same format were constructed for End-of-instruction reflective and the reflective interview audits of students' experiences with mathematical proof. Matrices were constructed on various issues raised in the interview guides for different scholastic levels such as pre-Advanced level, Advanced ('A') level and university experiences with mathematical proofs. In Zimbabwe, 'A'-level refers to fifth and sixth years of secondary school learning. Another matrix was formed from my efforts to account for inconsistency in student teachers' formal rhetoric behaviour where formal axiomatic reasoning was used in proof situations that required proof methods by refutation and vice versa. The inconsistent student behaviour was identified during analysis of pilot data.

Similar to research question one, the matrix illustrated here represents level 1 of the data reduction process during data analysis. This level of data analysis corresponds to level of discrete facts in form of students' actual utterances when we view the matrices from the nomothetic perspective of knowledge generation (Punch, 1998, 2005). These discrete facts were then raised to a slightly higher level of data abstraction through researcher's comments that accompanied these level 1 data analysis matrices.

Once again in a similar fashion to the way research question one was addressed I then moved to level 2 of the data reduction process where discrete facts that represented initial level data reduction were converted to empirical generalizations (Miles & Huberman, 1994; Punch, 1998, 2005). To accomplish the formation of empirical generalisations, level 1 data analysis matrices together with researcher comments were used to construct level 2 matrices. At this level of the reduction process, the scientific realist process approach was used to directly observe main features of students' proof experiences from level 1 data analysis matrices for Mid-instruction reflective, reflective interview audits and End-of-instruction reflective interviews.

The realist position that causal mechanisms and proving events are real entities that can directly be observed provided a basis for abstracting the level 2 empirical generalisations from discrete facts formed at level 1 data matrices (Berg, 2009; Maxwell, 2004). As a result of the efforts just described a level 2 data analysis matrix was constructed that has the following column entries. Column 1entries consisted of main aspects picked from researcher's examination of reflective interviews. The main features picked from the researcher's comments constituted the first level of data abstraction. Hence, main observations were used to form composite data matrices on the themes; inconsistent student proof behaviour, conceptions of mathematical proof, pre-university and university students' mathematical proof experiences. Thus the themes indicated constituted the rows of level 2 data analysis matrix in which summative content analysis guided by the realist idea that proof events and experiences are real entities that are directly observable was used to construct column 2 entries.

Column 3 contains the researcher's comments which were attempts to raise the empirical generalizations from level 2 to a higher level of abstraction in light of the nomothetic perspective of knowledge generation (Punch, 1998, 2005). To accomplish the data reduction process, main ideas inferred from level 2 data matrix were mapped to constructs from the analytic framework that include proof scheme taxonomies (Balacheff, 1998; Harel & Sowder, 1998, 2007), syntactic and semantic methods of proof (Weber & Alcock, 2012), modes of thoughts in proving theorems (Alcock, 2010). Consequently, a data analysis matrix of order n by 3 was constructed that has the following format:

Table 4.6: Students' experiences with mathematical proof from pre-university to university level

Aspect	Main observations	Researcher comments
• Students' conceptions of proof	………………………	…………………………
• Pre- A level experiences with	……………………………….	………………………..
• Proof		
• ……………….		
• etc		

The next level of data reduction in an attempt to develop a hypothesis about how students' thinking about mathematical proof evolves involved discussion of the results. I engaged in the discussion of the results with the aim of drawing a conclusion about how proof schemes emerged among undergraduate mathematics education students. In the discussion of the results, I wrote ideas about categories and their relationships that came to my mind during data analysis (Corbin & Strauss, 2008; Punch, 2005). Documenting ideas about relationships between categories was important because it helped in conceptual elaboration of how proof schemes emerge (Miles & Huberman, 1994, p. 172). It can be noted that in the process of searching for regularities or inconsistencies in students' proof schemes, concepts would be developed from data by raising empirical generalizations to higher levels of abstraction. Documenting ideas emanating from fractured data during analysis facilitated the crucial data abstraction process. A conclusion about how students' thinking around the concept of mathematical proof evolves was drawn. An account of the conclusion about the kinds of proof schemes held and how proof emerges emerge among undergraduate students was then provided.

Table 4.7 presents a summary of ideas discussed in this chapter that pertain to data collection, data analysis and conclusions drawn about how students think about mathematical proof and how such thinking evolves among student teachers.

Table 4.7: Summary for data collection and analysis procedures

Research question	Data source	Data analysis procedure	Research outcome
1. What kinds of proof schemes characterise undergraduate students' conceptualisation of mathematical proof?	Written responses Chalkboard demonstrations Follow up interviews on students' proof attempts	Directed content analysis	Explanatory theory about the kinds proof schemes held by students
2. How do the undergraduate student teachers' develop their proof schemes?	Reflective interviews Reflective interview audits	Summative content analysis	Proposition about how proof emergence of proof schemes among students

Analytic induction was then applied to address the main research question: *In what terms do undergraduate student teachers think about the concept of mathematical proof?* Analytic induction consists of a series of both inductive and deductive steps whereby inductively developed categories are verified by deductive means, which is, mapping them to literature (Punch, 1998, 2005). Hence, codes derived from data about students' schemes of argumentation on proof tasks attempted and codes developed from verbatim transcriptions of students' descriptions of their proof experiences were compared with existing literature for similarities and differences for the purpose of drawing an overall conclusion about the terms in which students think around the notion of proof and how such thinking evolved (Corbin & Strauss, 2008; Hennink et al., 2013; Miles, Huberman, & Saldana, 2014).

4.10 Realist Analytic Framework

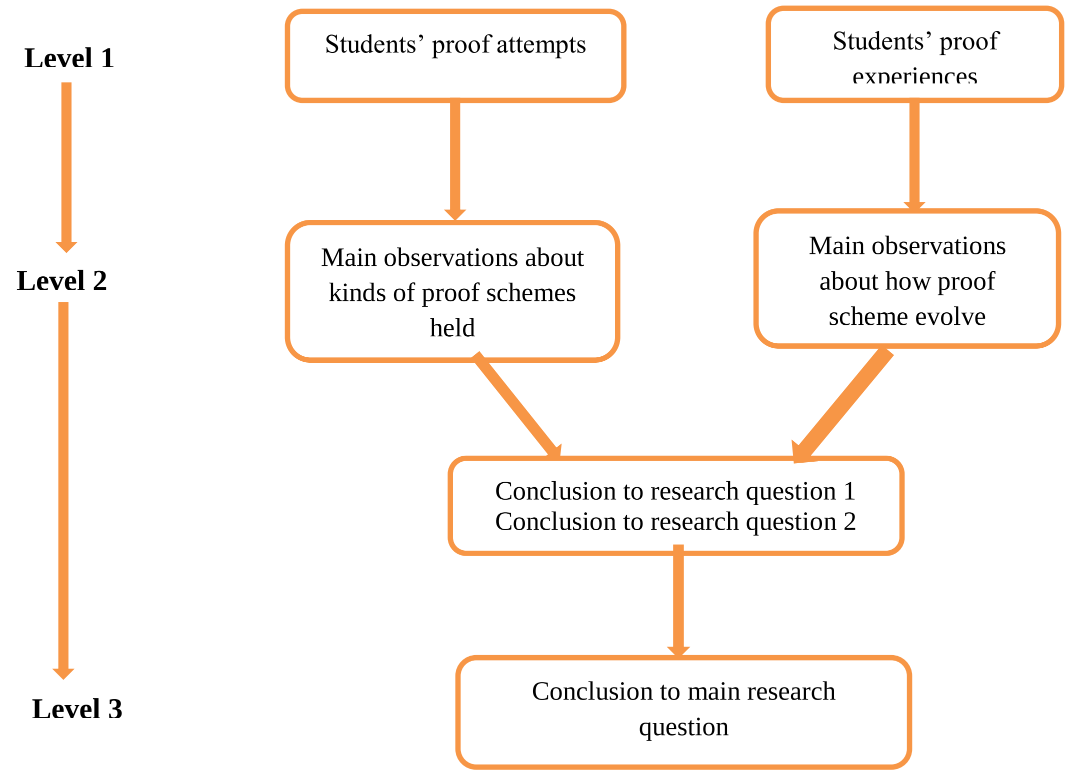

Figure 6: Realist process analytic framework

Scientific realism treats student teachers' proof schemes as real observable entities. To conceptualise the kinds of proof schemes held by students and how those proof schemes evolve a nomothetic view of knowledge generation was used. The nomothetic view holds that theory can be abstracted from discrete facts in the data, hence the approach has been considered to be appropriate for this study. The underlying idea of the nomothetic view is that theory building proceeds from discrete facts to explanatory theories. According to Punch (1998, 2005), there are three distinct levels of theory development. At the lowest level we have discrete facts in the form written responses from proof tasks and transcriptions of chalkboard demonstrations.

Discrete facts at level 1 of data reduction also consisted of verbatim transcriptions of students' proof experiences at various scholastic levels. As shown in Figure 6 directed content analysis was applied to these discrete facts (students' proof attempts) to generate themes and categories (empirical generalisations) tabulated in composite table as main observations for research question one. Similarly, summative content analysis data reduction technique was applied to verbatim transcriptions of students' descriptions of their proof experiences to obtain empirical generalisations recorded as main observations in the composite table of students' proof experiences. The composite tables formed level 2 of data the reduction process. In other words, at level 2 there are empirical generalisations comprising patterns and processes inferred from initial codes and categorical themes which are formed by grouping discrete facts together (Punch, 1998, 2005). Level 2 empirical generalizations were then mapped to theoretical constructs from literature such the notion of technical handles and conceptual insights, Manipulating-getting a sense of-articulation (MGA), cognitive analysis in argumentation, and syntactic and semantic approaches to composing proofs (CadawalladerOlsker, 2011; Raman, 2003; Sandefur et al. 2013). Harel and Sowder's (1998, 2007) taxonomy of proof schemes also informed the data reduction process. Main conclusions were drawn from Level 2 data reduction to reach level 3 — the highest level. At the highest level, we have theory, whose function is to explain the terms in which the student teachers think of the concept of mathematical proof (Berg, 2009; Punch, 1998, 2005).

4.11 Ethical considerations

Major ethical issues which were pertinent to this research study include: participants' interest, participants' informed consent in addition to being honest and ensuring participants' autonomy as suggested by Denscombe (2007) in Padayachee, Boshoff, Olivier and Harding (2011). First, regarding protecting participants' interest, issues pertaining to confidentiality and anonymity were considered. Mathematics student teachers were assured that during data coding and reporting, anonymity was guaranteed through use of pseudonyms (Gravetter & Forzano, 2009, p. 114). The second ethical consideration is participant's informed consent. The undergraduate students were assured that participation was voluntary and they were free to withdraw from the study without any form of retribution such as compromising their grades in the undergraduate Real Analysis course in which I provided tutoring services. Written consent was secured from each participant where each student confirmed voluntary participation by completing a form— see Appendix I for a sample of completed informed consent forms.

Third, I was honest and ensured that there was no betrayal (Creswell, 2009). According to Creswell betrayal occurs when participants understand one commitment but the investigator does other things from those agreed upon with participants. I adhered to the objectives of the study I had agreed on with students. For instance, assessment of Real Analysis and data collection for the study were separate activities and this distinction was made explicit to students. Hence, students' proof attempts to proof tasks were not used to determine their coursework scores for the Real Analysis course. The entire process was also overseen by research project supervisors.

Fourth, I made efforts to ensure autonomy of the participants (Creswell, 2014, p. 135). The undergraduate mathematics education students' rights were respected by ensuring that the relevance of research was explained to the students, that is to generate insights about students thinking about proof in order to inform the teaching and learning of proof at university level. Fifth, when interviewing students, I tried as much as possible to be sensitive to how students I probed and also to be sensitive to the way I handled students' responses to avoid embarrassing students (Creswell & Miller, 2000; Lewis, 2009). Further, interactions with students during interviewing were transparent and free of personal biases in order to develop a relationship of mutual trust with the participants.

Finally, after the data collection process, students were allowed access to interview and audio transcripts in order for them to inject contradictions, confirm or make any necessary changes deemed necessary from their point of view (Creswell, 2009). Member checking strategy allowed student teachers to verify interpretations made from data analysis. In addition, the technical assistant was not exploited in interviewing and other data collection processes other than providing technical services. So the technical assistant did not place noticeable influence on the participants' integrity and composure.

Chapter Five

Results

The analysis presented here is based on students' written responses to proof tasks and interview transcriptions based on students' chalkboard demonstrations of their attempts to proof tasks. To gain more clarity on the kinds of proof schemes held by students, reflective interviews on how students had tackled the proof tasks were conducted. During these follow up interviews student teachers were asked various questions with the aim of developing an understanding of the kinds of proof schemes held by undergraduate student teachers. The questions also aimed at establishing how students' thinking around the notion of mathematical proof evolves. Accordingly, data analysis is now presented per research question that was investigated.

5.1 Results: Research Question One

The first research question was: *what kinds of proof schemes characterise undergraduate student teachers' conceptualisations of mathematical proof?* The goal of the research was to establish a set of causal links within student teachers' proof schemes. In other words, the goal was to build an explanation that accounts for the undergraduate student teachers' set of proof schemes by focusing on what mechanisms and processes that characterise undergraduate student teachers' proof attempts. Tables 5.1-5.20 illustrate student teachers' responses to research question one.

Table 5.1: Mid-instruction assessment data matrix for Tino on Real Analysis proof tasks

Task	Student's response (written, oral, actions)	Profile of students' proving attempts	Proof scheme elements present
Describe whether the following statement is true or false. For all real numbers a and b, $a-b>0 \Rightarrow a^2-b^2>0$.	• Written response If $a-b>0 \Rightarrow a^2-b^2>0$. Multiply by a $\Rightarrow a^2>ab$…..(1) *and also multiply by b* $ab>b^2$…………(2) *Joining (1) and (2)* $a^2>ab>b^2$ $a^2>b^2$ by order properties $a^2>b^2$ ∴ *The statement is false*	Multiplying by a is a flawed process as it disregards the idea that a can be a negative number. Therefore the statement shown as (2) is not always true. The same argument applies to statement (1). Student joined statements (1) and (2) to get $a^2>b^2$. Student then concluded that the statement is false. The premises do not logically entail the conclusion drawn and therefore the argument is not sound. The student wrote: "$a^2>b^2$ by order properties", but specific order properties employed were not stated	Tino used formal deductive argumentation that failed to draw on all elements of the reference theory as shown by his disregard for cases when the expression $a>b$ is multiplied by negative real numbers. While the student claimed to use order axioms to prove the statement, he violated the order axioms. In spite of the flaw highlighted here, the argument built by the student shows that the proposition is true as shown by $a^2>b^2$ but the student concluded that: "The statement is false." In other words there was manipulation of symbols without getting the essence of underlying ideas (Hanna & Mason, 2014; Sandefur et al., 2013).

	• Follow up interview Researcher: […] You wrote that $a > b \Rightarrow ab > b^2$. […] Now is this necessarily true? Is the implication statement true? Tino: I overlooked one fact that if you are dealing with this inequality you should consider cases where $a < 0$ and $b < 0$ and cases where $a < 0$ and $b > 0$. I only considered one case then I only realizing later that I had made a mistake.	The flaws in the statements $ab > b^2$……..……..(1) and $a^2 > ab$……...(2) that were later joined to give $a^2 - b^2$>0 were later realized by the student during the follow up interview when he stated that he had a mistake by not considering cases in which a and b are negative real numbers.	Although the student confirmed that he had made a mistake by not considering negative real numbers, he did not question the implication of multiplying by negative real numbers. For instance, the student could not justify whether it would still be valid to use deductive reasoning to prove the statement. It can therefore be inferred that traces of symbolic proof scheme manifested during Tino's proof effort. The proof behaviour described above confirms the lingering effects of the symbolic proof scheme (Alcock, 2010; CadawalladerOlsker, 2011)
Determine whether the statement is true or false. If x is an integer, then $x^2 - x$ is an even integer. Justify your answer.	• Written response $x^2 - x = x(x-1) \Rightarrow$ *multiply two consecutive integers ,you get an even number*$x(x-1)$ *is even is true. If* $x = 2k(2k-1) = 4k^2 - 2k$ $= 2k(2k-1)$ *is even*	Factorising enabled Tino to notice that the expression $x^2 - x$ is a product of two consecutive integers x and $x - 1$ and the student then concluded that $x^2 - x$ is an even number if $x \in \mathbb{Z}$. Student also attempted to provide an alternative proof. The set from which the element k is drawn is not specified. It is also noted here that while the expression $2k(2k-1)$ shows that $x^2 - x$ is even, the student wrote the statement as $x = 2k(2k-1)$ instead of $x^2 - x = 2k(2k-1)$	The mathematical process of factorising gave Tino some insight into the underlying ideas pertinent to the statement to be proved. However, the alternative proof presented by Tino illustrates that the student lacked confidence in the single deductive statements such as "*multiply two consecutive integer, you get an even number.*" Therefore, Tino had to spice the formal deductive statement with symbol manipulation to reach conviction. Hence, it can be inferred that Tino had relative conviction in the argument he had produced (Weber & Mejia-Ramos, 2015).
Determine whether the statement is true or false. Justify your answer. For all real values of $x. f(x) \equiv 2x^2 + 7x - 4,$ *implies that* $f(x) > 0$.	*This is false.* *Proof* $2x^2 + 7x - 4 = 2x^2 + 8x - x - 4$ *=2x(x+4)-1(x+4)* *=(2x-1)(x+4)* *When x=-3 f(x)=18-21-4* *=-7 which <0*	The student started by asserting that the statement is false and then provided evidence for the decision. While factorising was done successfully it was however not used to show that the statement is false but it was just a dead end. Rather a counter example picked was substituted into the original expression $f(x) \equiv 2x^2 + 7x - 4$.	From the description of Tino's proving effort, it can be seen that his proof constructions were characterised processes that involved handling symbols (TH) without the student establishing a sense of the structural relationship (CI). The symbol manipulations were not ultimately utilized in drawing the conclusion. For instance, the factor form obtained through factorisation was not used in proving that the proposition is false. Instead, a counter example was used by the student to refute the claim that $\forall x \in \mathbb{R}$, $f(x) \equiv 2x^2 + 7x - 4$, implies that $f(x) > 0$. Tino lacked access to relevant conceptual insights (CI) and hence could not draw meaning from the solution constructed (Koichu, 2012; Sandefur et al., 2013)

Prove that the sequence defined by $(u_n) = \frac{n^2-1}{2n^2+3}$ converges.	• Written response $\lim_{n\to\infty}(u_n) = \frac{1}{2}$ $\quad \frac{1}{n^2} \to 0$ $\quad \frac{3}{n^2} \to 0$ *Proof* *Given $\varepsilon > 0$ we need to determine a natural number $N(\varepsilon)$ s.t. $n > N(\varepsilon)$ implies that $\lvert u_n - L \rvert < \varepsilon$* $\left\lvert \frac{n^2-1}{2n^2+3} - \frac{1}{2} \right\rvert < \varepsilon$ $\quad \left\lvert \frac{n^2-1-{}^3\!/_2}{2n^2+3} \right\rvert < \varepsilon$ $\left\lvert \frac{-2{}^1\!/_2}{2n^2+3} \right\rvert < \varepsilon$ $\frac{5}{2} < (2n^2+3)\varepsilon$ $\quad \frac{5}{2\varepsilon} < 2n^2+3$ $\frac{1}{2}\left(\frac{5}{2\varepsilon} - 3\right) < n^2$ $\quad \frac{5}{4\varepsilon} - \frac{3}{2} < n$ $n > \frac{5}{4\varepsilon} - \frac{3}{2}$ $\quad$ ∴ *The limit of u_n exists.*	The limit of the sequence $L = \frac{1}{2}$ was found by dividing each term by the dominant term n^2. Tino then stated the formal definition of a sequence correctly. This definition was applied to the task and algebraic manipulations were accurately performed up to the stage where the student wrote $\frac{1}{2}\left(\frac{5}{2\varepsilon} - 3\right) < n^2$, that led to $\frac{5}{4\varepsilon} - \frac{3}{2} < n$. Student then drew the conclusion that the sequence converges.	The proof attempt revealed that the student had a strong command of the hierarchical structure of the proof task (Selden & Selden, 2009) as well as the formal rhetoric aspect of the proof task as shown by executed behavioural components of the proof task in the form accurate algebraic manipulations. However a wrong deduction was made when student moved from $\frac{1}{2}\left(\frac{5}{2\varepsilon} - 3\right) < n^2$ to $\frac{5}{4\varepsilon} - \frac{3}{2} < n$ since $n \leq n^2$ for $n \in \aleph$. Finally the student concluded that the sequence converges without explaining how the algebraic manipulations illustrate that the natural number $N(\varepsilon)$ sought exists. In other words student did not get a sense of symbols handled (Raman, 2003; Sandefur et al., 2013)
	• Chalkboard demonstration [Looks at the task and proceeds as follows] {So the first thing is to determine whether there is a limit to that sequence, determine L}[student writes] 1^{st} *det L* [for determine L]{So we divide throughout by the dominant factor [referring to the dominant term], which is n^2so u_n} [Student writes] $u_n = \frac{1-\frac{1}{n^2}}{2+\frac{3}{n^2}}$ {Then the limit of this sequence as n goes to infinity is equal to}[Student writes] $\lim_{n\to\infty} u_n = \frac{1}{2}$ {so the limit is equal to $\frac{1}{2}$ so the limit exists, so the proof} [Student writes] Proof *Given $\varepsilon > 0$, we need to determine ∃ $N(\varepsilon)$ s.t. for $n > N(\varepsilon)$, $\lvert u_n$-$L \rvert < \varepsilon$.* {So we say}. [Writes the following while verbalizing it]. $\left\lvert \frac{n^2-1}{2n^2+3} - \frac{1}{2} \right\rvert < \varepsilon$ $\left\lvert \frac{2n^2-2-2n^2-3}{2(2n^2+3)} \right\rvert < \varepsilon$. {Common denominator. Uuu, this [referring to $2n^2+3$] into that [referring to the common denominator]. That's $2n^2-2$ minus this [referring to 2 into that [referring to common denominator], multiply -1 by this [referring to $2n^2+3$] we get } [Student writes] $\left\lvert \frac{2n^2-2-2n^2-3}{2(2n^2+3)} \right\rvert < \varepsilon$ $\left\lvert \frac{-5}{2(2n^2+3)} \right\rvert < \varepsilon$ $\frac{5}{2(2n^2+3)} < \varepsilon$ {Uuu, simplifying	Similar to the written response data source, the student started by determining the limit L of the sequence. Once again as was the case with written responses, student stated the formal definition of convergence of a sequence and then this definition was applied to the proof task correctly. During the chalkboard demonstration the error ($n \leq n^2$, $\forall\, n \in \aleph$) was not committed by the student. He stated correctly that: $\frac{1}{2}\left(\frac{5}{2\varepsilon} - 3\right) > n^2$ and raised each side of the inequality to the power $\frac{1}{2}$ that led to $n > \sqrt{\frac{1}{2}\left(\frac{5}{2\varepsilon} - 3\right)}$. Tino concluded that the limit exists once again could not justifying how the algebraic manipulations has led him to that conclusion.	Tino's proof effort demonstrated strong command of the formal rhetoric aspect of the proof task as he could articulate what the proving exercise sought to accomplish: "*Given $\varepsilon > 0$, we need to determine ∃ $N(\varepsilon)$ s.t. for $n > N(\varepsilon)$, $\lvert u_n$-$L \rvert < \varepsilon$.*" Not only was Tino able to articulate the behavioural knowledge of the proof, he could execute this articulated knowledge as shown by accurate algebraic manipulations that allowed him to determine the natural $N(\varepsilon)$. However, the student could not link the natural number obtained to the consequent statement. It can therefore be inferred that Tino had a weak command of the problem centred aspect of the proof task, that is, he had no sense of the structural relationship (Hanna & Mason, 2014).

	that} [referring to $\frac{5}{2(2n^2+3)} < \varepsilon$] $\frac{5}{2} < \varepsilon\ (2n^2+3)$ $\frac{5}{2\varepsilon} < (2n^2+3)$ {$\frac{5}{2\varepsilon}$ less than $2n^2+3$} $\frac{1}{2}\left(\frac{5}{2\varepsilon} - 3\right) > n^2${Taking roots both sides}[student writes] $n > \sqrt{\frac{1}{2}\left(\frac{5}{2\varepsilon} - 3\right)}${Then I got to this stage. So my conclusion is, the limit exists}		

Table 5.2: End-of-instruction assessment data matrix for Tino on Real Analysis proof tasks

Task	Student's response (written, oral, actions)	Profiles of students' proving attempt	Proof scheme elements present
Define a sequence (x_n) inductively by $x_1 = 1$ and $x_{n+1} = \frac{2x_n+3}{4}$. Prove that (x_n) is a bounded monotone sequence and hence determine its limit.	• Written response $x_1 = 1$ $x_2 = \frac{2x_1+3}{4} = \frac{2(1)+3}{4} = \frac{5}{4}$ $x_3 = x_{2+1} = \frac{2x_2+3}{4} = \frac{2\left(\frac{5}{4}\right)+3}{4} = 22$ $x_4 = x_{3+1} = \frac{2x_3+3}{4} = \frac{2(22)+3}{4} = \frac{47}{3}$ $x_1 < x_2 < x_3 < x_4 \ldots\ldots$ (x_n) is bounded below and not bounded above $\therefore$ (x_n) has no limit since it is not bounded above	The student teacher used specific examples to explore the behaviour of the sequence. The term x_2 was correctly determined. A wrong value ($x_3 = 22$) was used to find the fourth term x_4. Inductive explorations were used to infer that $x_1 < x_2 < x_3 < x_4$. The student then claimed that the sequence is not bounded, this claim could have been influenced by wrong empirical evaluations done. Tino did not state explicitly whether the sequence was monotone increasing or decreasing, Hence, empirical evaluations not linked to some demands of question. Student then concluded that the sequence has no limit.	Particular instantiations were used to explore the boundedness property of a sequence. Specific examples were used also to determine whether the sequence has no limit. In this case Tino concluded on the basis of wrong empirical evaluations that the sequence does not converge. It can therefore be noted from the above information that the student had fragile grasp of the fundamental limitation of the empirical proof scheme. Further, Tino exhibited a weak command of convergence criterion for bounded monotone sequences (Morselli, 2006)
Use the definition of an appropriate limit to prove that $\lim_{x\to 1}\left(\frac{x^3}{x-1} - \frac{1}{x-1}\right) = 3$	• Written response The limit of f as $x \to 1$ is 3 if given $\varepsilon > 0$ $\exists$ $\delta(\varepsilon) > 0$ s.t if $0 < \lvert x-1\rvert < \delta(\varepsilon)$ then $\lvert f(x) - 3\rvert < \varepsilon$, then we need to determine $\delta(\varepsilon) > 0$ $0 < \lvert x-1\rvert < \delta(\varepsilon)$ $\Rightarrow \left\lvert\left(\frac{x^3}{x-1} - \frac{1}{x-1}\right) - 3\right\rvert < \varepsilon$ $(x^3-1) = (x-1)(x^2+x+1)$ $\frac{x^3}{x-1} - \frac{1}{x-1} = \frac{x^3-1}{x-1} =$	The formal definition of the limit of a function $f(x)$ as $x \to x_o$ was correctly applied to the proof task as shown by correct substitution of the function $f(x)$ and the limit L into the expression, $\lvert f(x) - L\rvert < \epsilon$. The goal the proving efforts sought to accomplish was clearly specified: "need to determine $\delta(\varepsilon) > 0$." Algebraic manipulations were performed accurately including factorising the expression $\frac{x^3-1}{x-1}$. After factorising the student stated that	Deductive reasoning was employed correctly by the student. A strong command of the formal-rhetoric aspect (Selden & Selden, 2009) was demonstrated by the student. Accurate algebraic manipulations that led to the correct value of

	$\frac{(x-1)(x^2+x+1)}{x-1}$ ………………………….. $\lvert x^2+x+1-3\rvert<\varepsilon$ ………………………………………… $\lvert x+2\rvert\lvert x-1\rvert<\varepsilon$ set $\delta(\varepsilon)$=1 $\lvert x-1\rvert<1$ $-1<\lvert x-1\rvert<1$ $-1<x-1<1$ $0<x<2$ $\lvert 2+2\rvert\lvert x-1\rvert<\varepsilon$ $4\lvert x-1\rvert<\varepsilon$ $\lvert x-1\rvert<\frac{\varepsilon}{4}$ Set $\delta(\varepsilon)=\min\{1,\frac{\varepsilon}{4}\}$	"$\delta(\varepsilon)$= 1" instead of $\delta(\varepsilon)\leq 1$. A correct value of $\delta(\varepsilon)$ was obtained despite the wrong formulation $\delta(\varepsilon)$= 1. It is again noted that student did not explain how the value of $\delta(\varepsilon)$ illustrate that the function has limit as $x\to 1$.	$\delta(\varepsilon)>0$ being determined. As was noted with other tasks Tino did not explain how the value of $\delta(\varepsilon)>0$ determined illuminated that the function has limit $L=3$ as $x\to 1$. It can be inferred here that Tino lacked the relevant conceptual insight. His failure to connect the value of $\delta(\varepsilon)>0$ calculated reveals that he had weak conceptual understanding of the notion limit although his procedural knowledge was deep as illustrated by successful algebraic manipulations. Hence, according to Wilkerson-Jerde and Wilensky (2011) the student had no coherent connection of mathematical resources to enact on the proof task.
Use the definition of an appropriate limit to prove that $\lim_{x\to 1}\left(\frac{x^3}{x-1}-\frac{1}{x-1}\right)=3$	• Chalkboard demonstrations [student quietly writes the question on the chalkboard] $\lim_{x\to1}\left(\frac{x^3}{x-1}-\frac{1}{x-1}\right)=3$ $0<\lvert x-x_o\rvert<\delta(\varepsilon)$ $\lvert f(x)-L\rvert<\varepsilon$ [Student is urged to talk to which he responds] {I am trying to ,uuu, …[inaudible]. Ok, from the definition of limit of a function, aaa, (x) , sorry [student erases $f(x)$ that has been written} [student writes while verbalizing] *Given* $\varepsilon>0, \exists\delta(\varepsilon)>0$ *s.t. if* $\lvert x-x_o\rvert<\delta(\varepsilon)$ *then* $\lvert f(x)-L\rvert<\varepsilon$ {So x_o here is 1, our limit is 3. I want to determine ..}[student writes] *We want to determine* $\delta(\varepsilon)$ *s.t.if* $o<\lvert x-1\rvert<\delta(\varepsilon)$ *then* $\left\lvert\frac{x^3}{x-1}-\frac{1}{x-1}-3\right\rvert<\varepsilon$ {So from this part here, we say } [student writes] $\frac{x^3}{x-1}-\frac{1}{x-1}-3\left\lvert\frac{x^3-1}{x-1}-3\right\rvert<\varepsilon$ {If you simplify this part	The proving attempt started with moments of silence as student tried to recall the definition. He was even urged to talk and he mumbled trying to produce the definition and had also to erase the chalk board about twice. Once again the formal definition of limit of a function was correctly stated in terms of an arbitrary element, x_0. Student identified x_0 to be 1 and the limit L as 3 and these were substituted in the expression: $\lvert f(x)-L\rvert<\varepsilon$. Algebraic manipulations involving dividing x^3-1 by $x-1$ and factorizing x^2+x-2 were successfully handled. Efforts were then made to justify the need for the element 1by noting that "if we set $\delta(\varepsilon)$ to one" , a point not mentioned in the written response section. He then proceeded to note that $0<x<2$, from which he set $x=2$. This value was then substituted into the expression $\lvert(x+2)\rvert\lvert(x-1)\rvert<\varepsilon$ and simplified to $\lvert x-1\rvert<\frac{\varepsilon}{4}$. The final answer is stated as "Set $\delta(\varepsilon)=\frac{\varepsilon}{4}$", that is different from the one stated under the written response section. The element 1 whose existence the	Erasing the chalkboard and mumbling in an effort to reproduce the definition of limit of a function may point to the fact that the notion of limit was not strongly grasped by the student. Although the student could recall and apply the definition of limit of function to determine the size of $\delta(\varepsilon)>0$ a major weakness was that the student did show the relevance of the quantity obtained. Tino could not prove that the limit is 3 on the basis of answer obtained. In other words, Tino could not see how the piece of knowledge he had constructed resolved the problematic situation he was confronted with

	here. If you factorize this [referring to $x^3 - 1$], it becomes}$x^3 - 1 = (x - 1)(x^2 + x + 1)${So we substitute this part into that it becomes}……….. *Set* $\delta(\varepsilon) = \frac{\varepsilon}{4}$	student tried to justify earlier is not used in stating the final answer ($\delta(\varepsilon) = \min\{1, \frac{\varepsilon}{4}\}$) as expected. It was obser ved that $\delta(\varepsilon) > 0$ determined by the student was not used to explain the fact that the function $f(x)$ has a limit as $x \to x_o$.	(Koichu, 2012). In other words Tino engaged in symbolic manipulations without getting a sense of the underlying ideas.

Table 5. 3: Mid-instruction assessment data matrix for Tafa on Real Analysis proof tasks

Task	**Student's response (written, oral, actions)**	**Profiles of students' proving attempt**	**Proof scheme elements present**
Describe whether the following statement is true or false. For all real numbers a and $b, a - b > 0 \Rightarrow a^2 - b^2 > 0$.	• Written response $\forall$ $\mathbb{R}$ a and b $a - b > 0$ $a^2 - b^2$>0. If $a - b > 0 \Rightarrow a > b \Rightarrow a^2 > b^2 \Rightarrow a > b$. Required to prove that $a - b > 0 \Rightarrow a^2 - b^2$>0. $a^2 - b^2$>0 $(a - b)(a + b) > 0$ By order properties $a + b > 0$ and $a - b > 0$ $a > -b$ $a > b$ or $a + b < 0$ and $a - b < 0$ $a < -b$ $a < -b$ Setting $a = \frac{1}{2}$ $b = \frac{1}{3}$ $\frac{1}{2} - \frac{1}{3} > 0$ true $\Rightarrow \left(\frac{1}{2}\right)^2 - \left(\frac{1}{3}\right)^2 > 0$ true also $a = 5$ $b = 3$ $5 - 3 > 0$ true $\Rightarrow 5^2 - 3^2 > 0$ true $\left(-\frac{1}{2}\right) - \left(\frac{1}{4}\right) > 0$ true. $\frac{1}{4} - \frac{1}{16} >$ 0 true. From the given examples it is always true that for any chosen real number $a > b$ $a - b > 0 \Rightarrow a^2 - b^2 > 0$.	Tafa started by noting that if $a - b > 0 \Rightarrow a > b \Rightarrow a^2 > b^2 \Rightarrow a > b$. These statements are flawed since for instance, $-1 > -2$ but, $(-1)^2 < (-2)^2$. The symbol manipulations above were then disregarded without drawing meaning out of them. Student then considered the consequent statement $a^2 - b^2 > 0$ that he expressed as a difference of two squares: $(a - b)(a + b) > 0$. Order axioms were then applied to give $a + b > 0$, and $a - b > 0$ or $a + b < 0$ and $a - b < 0$. Use of arbitrary elements in building the argument was suddenly abandoned and the student switched to specific examples ($a = \frac{1}{2}$ and $b = \frac{1}{3}$) and also ($a = 5$ and $b = 3$). A wrong statement: "$-\frac{1}{2} - \left(\frac{1}{4}\right) > 0$, true" was stated by the student. Student teacher concluded on the basis of specific examples given that the given proposition is always true.	Tafa's proving attempts reveals the cyclic nature of his argument as can be seen from: $a > b \Rightarrow a^2 > b^2 \Rightarrow a > b$. In other words, there was no interplay between technical handles and conceptual knowledge. A simple counter such as the one illustrated under Tafa's proof profile could have helped the student realize limitations of squaring both sides done by the student. Student had a weak command of the proof framework as shown by arguing from the consequent statement; $(a - b)(a + b) > 0$. Order axioms correctly manipulated but their use in reaching the intended goal not illuminated. Student manipulated the objects without getting a sense of their relationships. Tafa's proof attempt revealed some ontological oscillations as he moved from symbol manipulations to use of specific examples. The decision to resort to particular instantiations revealed students' struggles with structural reasoning (Alcock, 2010; Hanna & Mason, 2014).
Describe whether the following statement is true or false. For all real numbers a and $b, a - b > 0 \Rightarrow a^2 -$	• Follow up interview Researcher:[…] Let me see how you tackled it [referring to the solution booklet] Tafa: [Laughing] Aaa, at first I thought of both axioms and eee... then I was stuck. That's where I was stuck like I can see kuti (*that*) after there then I started using actual numbers because there were closer to home than the axioms.	The researcher's goal was to elicit data from the student that would account for behavioural tendencies displayed when proving in particular the switch from use of axioms to instantiations. Tafa explained that specific examples were used	Tafa described use of specific examples as part of their culture. Use of particular instantiations was seen as an alternative method of proof after failing to make progress with symbolic manipulations; "first I thought of both axioms and eee... then I was stuck. That's where I was stuck like I can see that after

b^2>0.	Researcher: So in other words, people are more comfortable with numbers? Tafa: But it is like, it's coming from where we are coming from that we were taught to just use numbers even when we are looking at word problems. Yes but you will always change it to a mathematical statement and go down and obtain number solutions we are used to working with numbers.	when the student was stuck. Tafa stated that the examples were such that they satisfied conditions of the antecedent part, that is, $a - b > 0$. He described use of empirical verifications as part of their culture	there then I started using actual numbers because there were closer to home than the axioms." It can be inferred from Tafa's proof solution that use of particular instantiations was not done out of the realisation of their potential benefits such as providing counter argumentation cases, illuminating the mathematical property that can form the crux of the proof (lack of accessibility to relevant conceptual insight (Alcock & Weber, 2005; Sandefur et al., 2013).
Determine whether the statement is true or false. If x is an integer, then $x^2 - x$ is an integer. Justify your answer.	• Written response An integer is a number that is divisible by 2 without leaving a remainder. If x is an integer we want to show that $x^2 - x$ is an even number. Setting $x_1 = 2 \Rightarrow 4 - 2 = 2$ even $x_2 = 1 \Rightarrow 1 - 1 = 0$ which is even $x_3 = 3 \Rightarrow 9 - 3 = 6$ which is even $x_4 = 4 \Rightarrow 16 - 4 = 14$ which is even $x_5 = -2 \Rightarrow 4 — 2 = 6$ which is even $x_6 = -7 \Rightarrow 49 — 7 = 56$ which is even $x_7 = -1 \Rightarrow 1 — 1 = 2$ which is even $x_8 = 1000 \Rightarrow 1000000 - 100 = 9990000$ which is even. From the above examples it is clear that $x^2 - x$ is always even.	The student defined an integer as a number that can be divided exactly by 2. Tafa stated that the expression $x^2 - x$ is even and set out to provide evidence for the claim. Then Tafa used specific examples to determine whether $x^2 - x$ is even. Contrary to the definition of an even number given, the empirical tests by the student involve the numbers 3 ,-7 and -1 that are not exactly divisible by 2.	A weak command of proof framework (Selden & Selden, 2009) shown by the student stating the conclusion before he provided the premises. Student's numerical tests not guided by definition of an integer the student had stated as the specific examples also includes numbers not divisible by 2. Tafa had defined an integer as "a number that is divisible by 2 without leaving a remainder" It can therefore be inferred that Tafa lacked micro reasoning since his proof attempt involved integers outside the scope of the given mathematical statement.
Determine whether the statement is true or false. If x is an integer, then $x^2 - x$ is an even number. Justify your answer.	• Follow up interview Researcher: You wrote […] , an integer is a number divisible by 2 without leaving a remainder. Is this true […] a correct statement. Tafa: Aaaa, that's not. I wanted to say an even number.... Tafa: Why is not true? Researcher: Can you explain why? Tafa: Because there are other integers that can be divided by 2 and they leave a remainder.. Researcher: Ok, just making a follow up what you wrote here. $x = 2$ and you evaluated. In other words, why did you opt for specific examples? Tafa: It's a culture that is in us. You are taught maybe to talk about real numbers, whole numbers and very rarely do we talk about […] the numbers we are used to , it's very rare that when we were working that I realised that Oh, if I forget it, I will put	During the follow up interview I first queried the definition of an integer written by the student. Tafa explained that the definition he had written was for an even number instead of an integer in general. He explained why the definition as written could not apply to integers in general, "Because there are other integers that can be divided by 2 and they leave a remainder." Upon being asked why he had used specific examples Tafa reiterated that use of examples was a "culture in us." He pointed out that "we	Tafa clarified the definition of an integer by explaining that "Because there are other integers that can be divided by 2 and they leave a remainder." This clarification made on the definition might explain why specific examples used picked included odd numbers. The prominent use of specific examples in proving was also described as a "culture in us" and the formal axiomatic proof scheme referred as "the other" was not exploited as much as the empirical-numeric proof scheme. Hence, Tafa manipulated the mathematical object with a full awareness of the fundamental limitation of inductive arguments (Duval,

	a negative number.	were not very much involved with the other side.” The “other side” mentioned by Tafa presumptively refers to use of axioms and arbitrary elements to prove theorems.	2002; Morselli, 2006).
Determine whether the statement is true or false. Justify your answer. For all real values of $x. f(x) \equiv 2x^2 + 7x - 4, \Rightarrow f(x) > 0.$	• Written response $f(x) \equiv 2x^2 + 7x - 4 \Rightarrow f(x) \geq 0$ $2x^2 + 7x - 4 \geq 0$ $2x^2 + 8x + x - 4 \geq 0$ $2x(x + 4) - 1(2x + 4) \geq 0$ $(x + 4)(2x - 1) \geq 0$ $\Rightarrow x + 4 \geq 0 \quad 2x - 1 \geq 0$ OR $x + 4 \leq 0 \quad 2x - 1 \leq 0$ $\Rightarrow x \geq -4 \quad x \geq \frac{1}{2}$, $x \leq -4 \quad x \leq \frac{1}{2}$ [These solutions were illustrated on number lines and graph of $f(x)$ drawn]	First, the student factorized the quadratic expression $2x^2 + 7x - 4$ to get $(x + 4)(2x - 1) \geq 0$ Order axioms of the real field were applied to solve the inequality obtained after factorizing. Then Tafa’s graphical representation of the inequalities $x \geq -4$, $x \geq \frac{1}{2}$, the t graphical instantiations in the form of number lines led to the solution $x \geq \frac{1}{2}$. An argument similar to the one described was applied to the inequalities $x \leq -4$ and $x \leq \frac{1}{2}$ yielded the solution $x \leq -4$ Graphical illustrations were then used to refute the proposition that $f(x) \equiv 2 \; x^2 + 7x - 4$ implies that $\Rightarrow f(x) \geq 0 \; \forall \; x \in \mathbb{R}$	From the description of the proof attempt it can inferred that Tafa had a strong command of the formal rhetoric aspect of the proof as can be seen in accurate algebraic manipulations and strategic use of order axioms in solving the inequality $(x + 4)(2x - 1) \geq 0$ (Selden & Selden, 2009). Graphical instantiations used to refute the proposition revealed students’ grasp of the method of counter argumentation. It can therefore be inferred that Tafa deployed the correct mathematical resources for which he had a command of the problem centred part, that is, conceptual insight of the proof as solutions obtained helped in refuting the claim (Koichu, 2012; Sandefur et al., 2013).
Determine whether the statement is true or false. Justify your answer. For all real values of $x. f(x) \equiv 2x^2 + 7x - 4, \Rightarrow f(x) > 0.$	• Chalkboard demonstration {Task 3 says for all real values function of x is identical to } [Students writes] $f(x) \equiv 2\,x^2 + 7x - 4$ implies that $\Rightarrow f(x) \geq 0$. {Right, aaa, umm, lets assume 2 $x^2 + 7x - 4 \geq 0$, we solve this we get} [Student writes] $2\; x^2 + 7x - 4 \geq 0$ $2\; x^2 + 8x - x - 4 \geq 0$ $2x(x + 4) - 1(x + 4) \geq 0$ $(x + 4)(2x - 1) \geq 0$ *By order properties* [referring $ab > o$ implies either $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$] $x + 4 \geq 0$ *and* $2x - 1 \geq 0$ *or* $x + 4 \leq 0$ *and* $2x - 1 \leq 0$ $x \geq -4 \quad x \geq \frac{1}{2}$. {Right, this will give me $x \geq -4$; this $x \geq \frac{1}{2}$. We draw our number lines that will give us -4 here [pointing to the number line] {This gives us -4 here, 0 there, circle there, circle there} [referring to the endpoints]. [Student erases upon realizing that the number line	First, the student started by reading the question. Next, the student assumed that the statement is true. Method of factorization was applied in a similar manner to the working under the written response section. Order properties were mentioned and applied to solve the inequality $(x + 4)(2x - 1) \geq 0$. Similar to the written response section order axioms of the real field were used to obtain solutions for the inequality. Then the student argued by graphical means that in the interval $-4 \leq x \leq \frac{1}{2}$, the function $f(x) \equiv 2x^2 + 7x - 4$ is	From the description of the chalk board illustration it can be seen that the two sources had many common features such as accurate algebraic manipulations, and strategic use of order axioms of the real field and use of graphical instantiations to draw the conclusion that the proposition is false. One of the few distinctive features noted is that with chalkboard demonstrations Tafa started by assuming that the statement is true, “lets assume $2x^2 + 7x - 4 \geq 0$," but this was never referred to in the whole argument. This suggests that Tafa had not reflected on the whole argumentative process to determine if the sequence of assertions is logically consistent. Despite a somewhat vague formulation;

	illustration is wrong. Student draws another number line; {The intersection for this [pointing to the number line] $x \geq \frac{1}{2}$ {That's for this side [pointing to the $x \geq -4$ and $\geq \frac{1}{2}$], that means our solution is x is greater than or equal to $\frac{1}{2}$} [This is written on the board] $x \geq \frac{1}{2}$ {For this side [referring to $x+4 \leq 0$ and $2x-1 \leq 0$], we are going to have } [verbalizes and writes] $x \leq -4, x \leq \frac{1}{2}$ {Again we have our number line} [Student draws number line{This was now intersecting this side }[pointing to points less than -4 and student writes the solution] $x \leq -4$ {Our solution there is $x \leq -4$. Aaa, if we are going to represent this on a graph, our graph is going to be} [Student draws the graph][Student describes the graph drawn] {From this the other part is negative} [pointing to the part of the graph below the x-axis] {And from this, we can conclude that function of x is not identical to $f(x) \equiv 2x^2 + 7x - 4$ Implies that $f(x) \geq 0$ ∀ real values of x} Researcher: Can you explain why you were able to make that conclusion Tafa: Because for this graph [pointing to the graph], the other part (repeated) from -4 ,from -4, $-4 < x < \frac{1}{2}$, thus exclusive aaa, the function of x is less than 0}. From $-4 < x < \frac{1}{2}$, the $f(x) < 0$.	negative. Student then used this graphical argument to draw the conclusion: *And from this, we can conclude that function of x is not identical to $f(x) \equiv 2x^2 + 7x - 4$ Implies that $f(x) \geq 0$ ∀ real values of x*}. .	"is function of x is not identical to f(x) $\equiv 2x^2 + 7x - 4$ Implies that f(x) ≥ 0 ∀ real values of x", Tafa did leverage on the graphical instantiation by explaining that, "from -4, $-4 < x < \frac{1}{2}$, thus exclusive aaa, the function of x is less than 0." Hence, the student used the graph to refute statement. Tafa's proof attempt reveals he had acquired the relevant instrumental knowledge as shown by correct by correct algebraic manipulations that were well complemented by a good grasp of relevant conceptual insight. Hence, there was critical reasoning shown by Tafa (Alcock, 2010).
Prove that the sequence defined by $(u_n) = \frac{n^2-1}{2n^2+3}$ converges.	• Written task $(u_n) = \frac{n^2-1}{2n^2+3}$ $\lim_{n\to\infty}\left(\frac{1-\frac{1}{n^2}}{2+\frac{3}{n^2}} = \frac{1}{2}\right)$ u_n converges given $\varepsilon > 0$, ∃ a natural number $n > N(\varepsilon)$ such that $\lvert u_n - L\rvert < \varepsilon$ $\left\lvert \frac{n^2-1}{2n^2+3} - \frac{1}{2}\right\rvert < \varepsilon$ $-\varepsilon < \frac{2(n^2-1)-1(2n^2+3)}{2(2n^2+3)} < \varepsilon$ $\frac{2n^2-2-2n^2-3}{2(2n^2+3)} < \varepsilon$ $\frac{-5}{2(2n^2+3)} < \varepsilon$ ……………………………………… …………….. $\sqrt{\frac{-5-6\varepsilon}{4\varepsilon}} < \sqrt{n^2}$ $\sqrt{\frac{-5-6\varepsilon}{4\varepsilon}} < n$ ∴ since $n > N(\varepsilon)$ then the sequence defined by $(u_n) = \frac{n^2-1}{2n^2+3}$ converges to $l = 1/2$	Student divided each term by the dominant term, n^2, and evaluated the limit of the sequence (u_n) as $n \to \infty$. The limit L was found to be $\frac{1}{2}$ and student stated that the sequence converges. Tafa then stated correctly the formal definition of the convergence of a sequence. The sequence (u_n) and the limit $L = \frac{1}{2}$ were substituted into the expression $\lvert u_n - L\rvert < \varepsilon$. Student then applied the theorem: *If* $c \geq 0$ *then* $\lvert a\rvert \leq c \iff -c \leq a \leq c$ to get rid of the modulus symbol. Student then took the part $a \leq c$ and used it to	The conclusion that: the sequence converges was stated after finding the limit, $L = \frac{1}{2}$. According to Selden and Selden, 2009), Tafa's proof behaviour revealed he had an awareness of the hierarchical order shown by articulated goal of the proving effort: to determine if "∃ a natural number $n > N(\varepsilon)$ such that $\lvert u_n - L\rvert < \varepsilon$." However Tafa had weak command of formal rhetoric aspect which could be seen from lack of accuracy in algebraic manipulation that led to a complex solution $\sqrt{\frac{-5-6\varepsilon}{4\varepsilon}} < n$ being found. Hence, the conclusion drawn that (u_n) converges is not a logical consequence of the

		get $\frac{-5}{2(2n^2+3)} < \varepsilon$ that was then simplified to $\sqrt{\frac{-5-6\varepsilon}{4\varepsilon}} < n$. Student then made the claim that $n > N(\varepsilon)$ which is not a consequence of $\sqrt{\frac{-5-6\varepsilon}{4\varepsilon}} < n$.	argument built by Tafa because $-5-6\varepsilon < 0$ so the existence of the natural number, $N(\varepsilon)$, the student was searching for was not connected to the working. This lack of clarity of the structural relationship of the objects manipulated became the focus of the follow up interview.
Prove that the sequence defined by $(u_n) = \frac{n^2-1}{2n^2+3}$ *converges.*	• Follow up interview Researcher: How did your working here lead you to the conclusion that the natural number exists? Tafa: I thought maybe the denominator will end up being a negative and negative and negative will be positive. Researcher: But the denominator is 4ε and here you stated that ε is always greater 0. So is there any way this denominator will be negative. Tafa: No Researcher: What is causing this? Tafa: Eeee improper use of modulus Researcher: I mean, what is causing you to deduce this could be sometimes less than 0 yet you started with something that is greater than 0? Tafa: Hee,hee [Laughing]. That's not wrong. I think it's coming from improper use of the modulus because in this point, I was suppose to have eeee, uum, ignore the use of negative because I was taking the modulus.	The lack of clarity about the existence of the natural number $N(\varepsilon)$ was revealed during the follow up interview when Tafa explained that he thought that the denominator will end up being negative. When reminded that there was nowhere the denominator would be less than 0 because $\varepsilon > 0$. The explanation provided by Tafa centred on improper use of the modulus property.	The follow up interview revealed that procedural techniques were a prominent feature of student proof construction efforts ahead of sense making of notions pertinent to the proof tasks (Hanna & Mason, 2014). The point made is supported by Tafa's failure to justify the existence of the natural number, $N(\varepsilon)$. Further, Tafa's working resulted in a situation where $N(\varepsilon)$ being a complex number, a weird solution outside the reference theory of the proof task. This further reinforces the observations that Tafa engaged in algebraic manipulations without paying attention to their sense (Sandefur et al., 2013).

Table 5.4: End-of- instruction assessment data matrix for Tafa on Real Analysis proof tasks

Task	Student's response (written, oral, actions)	Profiles of students' proving	Proof scheme elements present
Prove that $f(x) = x^2 + 2x - 5$ *is uniformly continuous on* $[0, 3]$.	• Written response $f(x)$ is uniformly continuous on [0,3] if it is differentiable on $x, y \in \mathbb{R}$ such that $0 < \lvert f(x) - f(y)\rvert < \delta(\varepsilon)$ then $\left\lvert\frac{fx-fy}{x-y}\right\rvert < \delta\varepsilon \;\; f' = 2x + 2$	Student teacher wrote that $f(x)$ is uniformly continuous on the interval if it is differentiable on the same interval. The condition for uniform continuity is stated as follows: $0 < \lvert f(x) - f(y)\rvert < \delta(\varepsilon)$ rather than: $\lvert f(x) - f(y)\rvert < \varepsilon$. Student finally differentiated the function $f(x) = x^2 + 2x - 5$. Tafa's proof attempt surprised the researcher because the concept had been covered in the Fundamentals of Analysis course a few days before the end of instruction data collection phase.	Work presented by Tafa, shows a mix up of mathematical ideas. The concept of uniform continuity was confused with the idea of differentiation. There was complete *chaos* as the student engaged in an irrelevant process, that is, differentiating instead of applying the definition of uniform continuity. Tafa lacked micro reasoning as he worked outside the reference theory of the proof task. A weak command of the concept of uniform continuity shown also by statements such as: ": $0 < \lvert f(x) - f(y)\rvert < \delta(\varepsilon)$." Tafa's proving profile reveals that he had not established a coherent network mathematical resources and shown by failure to enact the right resources at the right time (Duffin & Simpson, 2000; Wilkerson-Jerde &

			Wilensky, 2011).
Use the definition of appropriate limit to prove that $lim_{x\to\infty} \frac{\sqrt{3x}}{x}$ $= \sqrt{3}$.	• Written response Let $\varepsilon > 0$ be given we need to determine $\delta(\varepsilon) > 0$ st $\delta(\varepsilon) \in \aleph$ such that $x, X \in \aleph$ exist that $x \geq X$ then $\|f(x) - L\| < \varepsilon$ $\left\| \frac{\sqrt{3x^2+4}}{x} - \sqrt{3} \right\| \leq \varepsilon$ $\left\| \frac{\sqrt{3x^2+4} - x\sqrt{3}}{x} .(\frac{\sqrt{3x^2}+4+\sqrt{3x^2}}{\sqrt{3x^2+4}+\sqrt{3x^2}} \right\| \leq \varepsilon$ …………………………… ……………………… $\left\| \frac{4}{x\sqrt{3x^2+4}+\sqrt{3x^2}} \right\| \leq \varepsilon$ but we note that $\frac{1}{3x^2+4} < \frac{1}{x^2}$ since $x^2 <$ $3x^2 + 4$ $\frac{4}{x\sqrt{3x^2+4}+\sqrt{3x^2}} <$ $\frac{4}{x\sqrt{x^2}}$………….$\frac{2\sqrt{\varepsilon}}{\varepsilon} \leq x$ $\therefore$ set $X = \frac{2\sqrt{\varepsilon}}{\varepsilon}$	Tafa's attempt of the formal definition of the limit of a function f as $x \to \infty$ contained glaring errors. First, the quantity $\delta(\varepsilon)$ is not an element of the set of natural numbers as stated by the student. In any case, the focus should have been on finding $X \in \mathbb{R}$. Also, the numbers x and X do not belong to the set of natural numbers. However the student showed a strong command of algebraic manipulations as shown by correct application of the identity $\sqrt{x} - \sqrt{y} = \frac{x-y}{\sqrt{x}+\sqrt{y}}$ in simplifying the expression $\left\| \frac{\sqrt{3x^2+4}}{x} - \sqrt{3} \right\| \leq \varepsilon$. Strong command of Algebraic manipulations was seen in the observation $x^2 < 3x^2 + 4$ that enabled the student to determine X in terms of ε chosen. Conclusion not stated.	While the student had set out to find $\delta(\varepsilon) > 0$ which was incorrectly stated as an element of natural numbers, the student ended up finding a number X in terms of ε chosen and X was also conceived in terms of a natural number. It can be noted from the description of Tafa's proof attempt that although correct algebraic manipulations were performed and enabled the student to determine the correct value of X, these algebraic manipulations are not connected to goal articulated earlier, that is, to determine $\delta(\varepsilon) > 0$. Therefore in terms of the construct, Tafa engaged in technical symbolic manipulations without establishing a sense of the underlying ideas (Sandefur et al., 2013)

Table 5.5: Mid-instruction assessment data matrix for Tendai on Real Analysis proof tasks

Task	**Student's response (written, oral, actions)**	**Profiles of students' proving**	**Proof scheme elements present**
Describe whether the following statement is true or false. *For all real numbers a and $b, a - b > 0 \Rightarrow a^2 - b^2 > 0$.*	• Written response **…………………………** **……** $(a - b)$(a+b) > 0 $a.a + a.b - ba - b^2 > 0$ $a^2 + 0 - b^2 > 0$ a^2- $b^2 > 0$	Student focused on the consequent statement: $a^2 - b^2 > 0$. The student used an alternative representation of the statement, the difference of two squares: $(a - b)$(a+b) for $a^2 - b^2$. Student expanded $(a - b)$(a+b) > 0 and got a^2- $b^2 > 0$. Conclusion was not stated.	Proof convention, that is, the proof framework was violated by the student who worked from the conclusion instead of inferring the conclusion from the premises (Selden & Selden, 2009). The premises presented by Tendai do not logically imply the conclusion and hence the argument produced by Tendai is not valid (Stylianides & Stylianides, 2009). Non-goal directed symbolic manipulations were performed for which the student had no essence of the underlying ideas.
Prove that the sequence defined by $(u_n) = \frac{n^2-1}{2n^2+3}$ converges	• Written response $\frac{\frac{n^2}{n^2} - \frac{1}{n^2}}{\frac{2n^2}{n^2} + \frac{3}{n^2}} = \frac{1}{2}$ $\therefore$ it converges because the limit is $\frac{1}{2}$ which	Tendai divided each term of the sequence by the dominant term, n^2 and evaluated the limit as $n \to \infty$ in the expression got as a result of dividing by the dominant term. Then Tendai concluded on the basis of the limit, $L = \frac{1}{2}$, that the sequence converges. The	Although Tendai could determine the limit $L = {}^{1}/_{2}$ of the sequence, her proof attempt revealed severe limitations about her knowledge of the concept of a sequence. First, the need to determine a natural number

	approach 0 $\|u_n - L\| > \varepsilon$ $\frac{1}{2} > \varepsilon$	statements $\|u_n - L\| > \varepsilon$ and $\frac{1}{2} > \varepsilon$ then suddenly sprang from nowhere without any explanation for their purpose by the student.	$N(\varepsilon)$ for some fixed $\varepsilon > 0$ not shown but rather a wrong statement "$\|u_n - L\| > \varepsilon$" instead of $\|u_n - L\| < \varepsilon$ was written. Second the inequality “ $\frac{1}{2} > \varepsilon$” was stated without explaining its purpose. Third, the link between $[u_n - L] > \varepsilon$ and $\frac{1}{2} > \varepsilon$ not explained. Overall, Tendai’s proof behaviour revealed a tendency to engage in symbolic manipulations for which the student had no essence of their meaning (Hanna & Mason, 2014).
For all real values of $x. f(x) \equiv 2x^2 + 7x - 4$, *implies that* $f(x) > 0$.	• Written response Suppose $f(x) = 2x^2 + 7x - 4$, implies that $f(x) \geq 0$. $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ $x = \frac{-7 \pm \sqrt{(7)^2-4(2)(-4)}}{2(2)}$ ……………………… $x = \frac{1}{2}$ or -4 critical condition $\therefore \frac{1}{2} \geq 0$	Tendai started by assuming that the statement was true. The general quadratic formula was then used to determine the zeros of the given function: $x = -4$ *or* $x = \frac{1}{2}$. Finally, the student wrote: “$= \frac{1}{2}$ or 4 critical condition.” No conclusion was given as to whether the supposition made earlier holds or should be refuted. This lack of clarity on the status of the proposition became the focus of the follow up interview.	As was also noted with the task on sequences Tendai’s written response to this task consists of disjointed statements. For instance the supposition: *Suppose* $f(x) = 2x^2 + 7x - 4$, *implies that* $f(x) \geq 0$, was not connected to the conclusion: $\therefore \frac{1}{2} \geq 0$. The statement “ $x = \frac{1}{2}$ or -4 critical condition” also not linked to the conclusion. The conclusion did not specify whether the proposition is true or false. The proof behaviour reveals that Tendai manipulated mathematical objects without establishing their essence (Weber & Mejia-Ramos, 2011).
For all real values of $x. f(x) \equiv 2x^2 + 7x - 4$, *implies that* $f(x) > 0$.	• Follow up interview Researcher: Here you have question 3 here you solved it as if it’s an equation [it was an inequality] apply quadratic formula and you got ½ or -4. How did it then lead to this answer [referring to $\therefore \frac{1}{2} \geq 0$]? Tendai: I prefer to use ½ because it is a positive. Researcher: Oh, since it was written greater than 0. So you prefer to take ½? Tendai: Because it is positive than -4. Researcher: But when you look at -4, you will	Probing on meaning of answers obtained by using the quadratic equation revealed that Tendai misconstrued the question. She took the condition: $f(x) > 0$ in the proof task to imply that only positive values from the domain of $f(x)$ were needed: “I prefer to use ½ because it is a positive.” Tendai’s misinterpretation of the proof task and hence her tendency to look for positive values of x was affirmed by the statement “Because it is positive than -4.” When she was reminded that values of x less than -4 would also give $f(x) > 0$, Tendai looked stuck and no further explanation could be elicited from her.	The follow up interview revealed that Tendai had difficulties in interpreting the question. Utterances such as “I prefer to use ½ because it is a positive,” reveal that despite procedural techniques engaged in, the student was clueless about the demands of the task as she thought her task was to find positive values of x using the general quadratic formula. It can therefore be inferred that Tendai had a wrong interpretation of the question and set out to find values of x that had no connection with the proof task. Tendai’s failure to interpret the question explains why she

	realise that the function will be positive also below -4. So why did you opt for the half? Tendai: [silent looks stuck]		was stuck when pressed to justify conclusion drawn. The proof behaviour demonstrates that she failed to see how the piece of knowledge she had constructed is a solution to the proof task at hand (Koichu, 2012).
Determine whether the statement is true or false. If x is an integer, then $x^2 - x$ is an integer. Justify your answer.	• Written response Let x be 3 $3^2 - 3 =$ 6 is even number	A single specific example was used to prove that the statement is true.	It can be seen from the single example used that Tendai did understand the limitation of empirical verifications.
Determine whether the statement is true or false. If x is an integer, then $x^2 - x$ is an integer. Justify your answer.	• Chalkboard demonstration {If the statement is true or false. If the integer, if x an integer then x^2 –x is even. Justify your answer. Then I said} [Student writes] *If x is an integer then x^2 –x is an even number*. {Then I said , let x be a 3} [Student writes] *Let $x = 3$* {Then we say x^2 minus 3} [writes] 3^2 -3 {We get say $9, 9-3$. Then my answer , then my answer becomes 6. So 3 is our integer then 6 is an even number….[inaudible], then is my integer} {Then let x be 2} [Student writes] *Let x be* 2 {Then I substitute} 2^2 -2 $4 - 2$ 2 {Then I also get an even number}.	Similar to the written response effort, Tendai used specific examples to evaluate the status of the statement given. However during the chalkboard demonstration one additional example: $x = 2$ was used. "Then I also got an even number". Tendai then concluded on the basis of two empirical verifications that: " […], if x an integer then x^2 –x is even."	Although Tendai used two specific examples, her efforts also revealed her limited understanding of the fundamental limitation of particular instantiations, that they cannot be used to represent general cases. Conclusion was provided first and then specific examples were then used to support the conclusion made: *If x is an integer then x^2 –x is an even number*. It can be seen that conventions of proof, proof framework (Selden & Selden, 2009). Tendai did not adhere to logical rules in proof making which stipulate that the premises should logically entail the conclusion.

Table 5.6: End-of-instruction assessment data matrix for Tendai on Real Analysis proof tasks

Task	**Student's response (written, oral, actions)**	**Profiles of students' proving**	**Proof scheme elements present**
A sequence (a_n) of real numbers is defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. Prove that (a_n) converges and find its limit.	• Written response $a_1 = \sqrt{2} = 1.41$ $a_2 = \sqrt{2 + \sqrt{2}} = 1.84$ $a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}} = 1.8328$ $a_4 = \sqrt{2 + \sqrt{2} + \sqrt{2}} \quad = 1.832$ P_1 and P_2 holds	Specific instantiations were used to explore the behaviour of the given sequence. The statement "P_1 and P_2 holds" that is in apparent reference to the base step of the proof method by mathematical induction is not	It can be noted that Tendai wrote the statement "P_1 and P_2 holds" without scrutinizing results of particular instantiations made. It can be seen that Tendai handled mathematical objects without grasping the

	$a_1 < a_2$ $a_2 < a_3$ (a_n) is a monotonic sequence is increasing and converge at 2 which is the limit $a_{n+1} < a_n$ $a_{n+1} - a_n < 0\sqrt{2 + a_n} - a_n$ Square both $\left(\sqrt{2 + a_n}\right)^2 - {a_n}^2 < 0$ $(2 + a_n) - {a_n}^2 < 0$ $-(a_n)^2 + a_n + 2 < 0$ …………………………………… $(a_n + 1)(2 - a_n) < 0$ $a_n + 1 < 0$ $2 - a_n < 0$ $a_n < -1$ $-a_n = -2$ $a_n = 2$ ∴ this implies that $a_1 < a_2$ $a_n = 2$ which is the limit because a_n converges	connected to the specific instantiations done by the student. For instance the statement that (a_n) is monotone increasing sequence is contrary to instantiation: $a_2 = 1.84$, $a_3 = 1.8328$. The student wrote that the sequence converges to 2 before she determined that the limit was indeed 2. The student wrote the statement: $a_{n+1} < a_n$ which is in stark contrast to the claim made earlier that (a_n) is monotone increasing sequence. The student then performed algebraic manipulations on the flawed statement: $(2 + a_n) - {a_n}^2 < 0$ that led to the factor form $(a_n + 1)(2 - a_n) < 0$. Order axioms were wrongly applied and yielded $a_n < -1$, $a_n = 2$. It should have been $a_n < -1$, and $2 - a_n > 0$ for this statement to be true. Tendai's proof attempt also revealed lack of consistency in use of symbols e.g., $2 - a_n > 0$ was later written as $a_n = 2$. The conclusion "∴ this implies that $a_1 < a_2$ ", which is not connected to the working just sprang from nowhere. Tendai finally wrote: "$a_n = 2$ which is the limit because a_n converges" The statement $a_n = 2$ is misleading as it gives the impression that all natural numbers will map to 2, that is sequence consists of constant term 2	essence of their structural relationships. For example, according to Tendai $a_2 > a_3$ but she wrote "$a_2 < a_3$". This statement might suggest that Tendai did not reflect on particular instantiations made earlier on to get a sense of their meaning. Similar to the previous proof effort the student started by stating the conclusion: "(a_n) is a monotonic sequence is increasing and converge at 2 which is the limit." Once again the limit is stated before the student determined it, another violation of the proof framework. Algebraic manipulations that ensued the declaration that the limit is 2 do not point to the fact the limit is 2. For example, Tendai wrote "$(a_n + 1 < 0,\ 2 - a_n < 0$" which should have led $a_n > 2$ and $a_n < -1$. The intersection of these is a null set (no solution). But Tendai ignored the inequality sign and wrote "$a_n = 2$. So Tendai manipulated the objects without getting their sense. Tendai wrote "∴ this implies that $a_1 < a_2$ $a_n = 2$ which is the limit because a_n conveges" which is flawed argument. These misleading statements reveal that Tendai did not reflect on meaning on mathematical processes engaged in but the need to prove that a_n converges regardless of mathematical legitimacy of processes leading to this goal was influencing the manipulations.

A sequence (a_n) of real numbers is defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. Prove that (a_n) converges and find its limit.	• Chalkboard demonstration [Student reads the question] {A sequence of real numbers is defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. Prove that (a_n) converges and find its limit. We have..} [student verbalizes and writes] $a_1 = \sqrt{2}$ $a_{n+1} = \sqrt{2 + a_n}$ {Then } $a_1 = \sqrt{2}$ {Which is equivalent to} $a_1 =$ 1.41{Then a_2 which is equivalent to 1.84} $a_2 = \sqrt{2 + \sqrt{2}} = 1.84$ {Then a_3} $a_3 = \sqrt{2 + \sqrt{2} + \sqrt{2}}$ = 1.832 {Then a_4} $a_4 = \sqrt{2 + \sqrt{2 + \sqrt{2} + \sqrt{2}}}$=1.832 ...{ Then we see that our sequence from a_2 to a_3 is decreasing, so we say.} [student writes] P_1 *and* P_2 *holds* {Then $n = k$} $n = k$ {Then $n + 1 = k + 1$} $n + 1 = k + 1${Then we see that} $a_1 < a_2 < a_3$ {Then we say} $a_{n+1} - a_n < 0$ {Then we subtract our a_{n+1} which is } $\sqrt{2 + a_n} - a_n < 0$ {Then we say } $\sqrt{2 + a_n} < a_n$ {The we have} $2 + a_n < a_n^2$ [student then erases the above statement and writes] $2 + a_n < a_n^2 < 0$ [erases again and writes]$2 + a_n - a_n^2 < 0$ {Then we say (slowly)} $- a_n^2 + a_n + 2 < 0${Then we factorize, we get} - $a_n(a_n$ +1) + 2(a_n +1)< 0 $(a_n + 1)(2 + a_n) < 0${Then we say} $a_{n+1} < 0$ *or* 2 + a_n < 0) {The we say} $a_n < -1$ *or* $a_n < -2$[student goes back to $(a_n + 1)(2 + a_n)$ and changes statement to] $a_n + 1 < 0$ *or* 2 - a_n)< 0 $a_n < -1$ *or* $a_n < 2$ [No justification adduced by student]{Then [silent], we have} $a_n = 1$ *or* 2 [student says] {Which implies that} [student then writes} $a_1 < a_2$[student concludes that] {For the sequence then the limit is 2}	After reading the question the student verbalizes the specific instantiations she had made from her written response efforts. P_1 *and* P_2 not defined and the claim that P_1 *and* P_2 hold not justified. The statement} $a_1 < a_2 < a_3$ is contrary to "that our sequence from a_2 to a_3 is decreasing." Further, $a_{n+1} - a_n < 0$ is confirms Tendai's claim "that our sequence from a_2 to a_3 is decreasing." The student erased the statement and immediately replaced it with $2 + a_n < a_n^2 < 0$. Tendai again erased this statement. Slowly and with some hesitation, she wrote $- a_n^2 + a_n + 2 < 0$. The expression was wrongly factorised to give $(a_n + 1)(2 + a_n) < 0$. As was also the case in the written response effort, order axioms were incorrectly applied to get $a_n < -1$ *or* $a_n < 2$ and the student suddenly changed $2 + a_n < 0$ to $2 - a_n < 0$ without justifying the change of sign. Then the student mentioned that "we have $a_n = 1$ *or* 2" and then deduced that $a_1 < a_2$. This was yet another awkward formulation that was not linked to what the student was engaged with. From the same statement:$a_n = 1$ *or* 2, the student concluded that the sequence converges to 2, but the student did not justify this claim.	Tendai's actions: erasing the chalkboard and writing slowly and with some hesitation, unjustified and undefined statements such as "P_1 *and* P_2 holds" point also to weak conceptual knowledge of the student in the area of sequences. These severe limitations in the student's knowledge were also manifested through contradictory claims made about the behaviour of the sequence e.g., the use of the relation $a_{n+1} - a_n < 0$, yet Tendai had asserted that the second term is greater than the first term. Such statements reveal inconsistencies in logic. TH manipulated without reflecting on underlying ideas. Tendai's chalkboard demonstration also revealed weak command of central ideas in Real Analysis e.g., incorrect application of order axioms of the real field to solve inequality formed. Perhaps influenced by the claim made earlier in the written response section that the limit of the sequence prior to the calculation is 2. Tendai had produced some awkward formulations that led her to conclude that the sequence has limit 2. It can be inferred that Tendai's procedural knowledge and conceptual knowledge did not show a connection in Tendai's proof attempts yet there should be some interplay between procedural and conceptual ideas during proving (Raman, 2003; Weber & Alcock, 2004).
Use the definition of appropriate limit to prove that $lim_{x \to \infty} \frac{\sqrt{3x^2 + 4}}{x} = \sqrt{3}$.	• Written response Let $\varepsilon > 0$ be given, $\exists\, \delta(\varepsilon)$ st $\lvert fx - L \rvert < \varepsilon$	Upon fixing $\varepsilon > 0$ Tendai's goal should have been to determine $X \in \mathbb{R}$ for which $x > X \Rightarrow \lvert f(x) - L \rvert <$	Student had a weak command of knowledge of limit involving infinity. This interpretation can be supported by student's

		ε. Rather, Tendai stated that she intended to find $\delta(\varepsilon)$ such that $\lvert f(x) - L\rvert < \varepsilon$ which is not the focus of limit of f as $x \to \infty$.	stated goal: to find if " $\exists\, \delta(\varepsilon)$ st $\lvert fx - L\rvert < \varepsilon$." Student did not mention the fact that $\delta(\varepsilon) > 0$. Tendai stated the symbols without getting sense of what is involved in the proof. According to Raman (2003) Tendai had procedural fluency not well coordinated with conceptual understanding.
Use the definition of appropriate limit to prove that $\lim_{x\to 1}\left(\frac{x^3}{x-1} - \frac{1}{x-1}\right) = 3$	• Written response Let $\varepsilon > 0$ be given we want to produce $\delta(\varepsilon)$ st $0 < \lvert fx - L\rvert < \delta(\varepsilon)$ $\lvert x - 1\rvert < \delta(\varepsilon)$	The definition was not properly stated because it had the condition: "$0 < \lvert fx - L\rvert < \delta(\varepsilon)$". First it can noted that $\delta(\varepsilon)$ is a quantity associated with the domain of and not the range of the function f. Second, the condition $0 < \lvert fx - L\rvert < \delta(\varepsilon)$" point to a deleted neighbourhood of the the limit L, yet we should just have a mere neighbourhood of L	The flaws in the definitions stated point to the weak command of underlying ideas. Tendai thought of proving in terms of symbolism without reflecting on meaning of the symbols involved, no conceptual insight attained.

Table 5.7: Mid-instruction assessment data matrix for Cortney on Real Analysis proof tasks

Task	Student's response (written, oral, actions)	Profiles of students' proving	Proof scheme elements present
Determine whether the statement is true or false. If x is an integer, then $x^2 - x$ *is an even integer. Justify your answer.*	• Written responses Given x is an integer, $\{\ldots -1,0,2,\ldots\}$ $x^2 - x$, substituting -1 $(-1)^2 - -1$ $1 + 1 = 2$ which is even $(0)^2 - 0 = 0$ which is neither even nor odd $2^2 - 2 = 2$ which is even $5^2 - 5 = 20$ which is even $(-100)^2 - 100 = 900$ which is even $\therefore x^2 - x$ is an even holds for $x \in$ integers with 0 excluded	Specific values $(-1, 0, 2, 5)$ were substituted into the expression $x^2 - x$. These empirical verifications were used by Cortney to conclude that $x^2 - x$ is even for all integers that exclude 0. After stating that 0 is neither even nor odd Cortney kept on generating examples disregarding the fact that the question had specified that x is an integer.	The empirical proof scheme was exhibited as shown by Cortney's use of specific examples to evaluate the proposition. Cortney showed a weak command of the concept of a counter example in proving (Stylianides, 2011). The fact that the proposition failed for integers that yielded 0 should have led to the refutation of the proposition. This sort of proving behaviour reveals severe limitations in student's ability to use counter-argumentation. She continued generating specific examples despite the fact that a counter example had been found.
Determine whether the statement is true or false. *If x is an integer, then* $x^2 - x$ *is an integer.*	• Chalkboard demonstration [Student begins by reading the question] {If x is an integer, eee, $x^2 - x$ is it even, is the result an even number? So I considered x to be a set of integers.} [Student teacher then writes] *x = {-1,0, 1,2,...}*	Similar to the written response, particular instantiations were used to evaluate the statement. However, when Cortney substituted with -1, she got 0 instead of 2. She noted that 0 is neither even nor odd. Influenced by the fact 0 is neither odd nor even if x is an integer $x^2 - x$ is not even. In other words Cortney refuted the	The inductive proof scheme elements were revealed through use of specific examples by the student in refuting the claim (Alcock & Weber, 2005). Although the statement was refuted by the student her conception of the idea of a counter argumentation was still fragile as revealed by use of many

Justify your answer.	{And then substituting, substituting} [Student writes and verbalizes] *Substituting -1,* $x^2-\ x$*=* $(-1)^2$ *-1* *0(not even)* *Sub 1* x^2-x *=0 (not even)* …………… *Sub 2* 2^2*-2 = 2 (even)* {Substituting -1 into x^2-x, the result is $(-1)^2$ *-1* which is 0, and substituting 0-0 =0 and substituting 1, what do we have ? Aaa, are you sleeping? [referring to the class] What do we get, 1, 0 and [inaudible]. When I substituted 2, I got 2. So, aaa, at the substitutions when I substituted -1, I got 0 [….] I got 0 either and when I substituted 2 I got 2. So zero is neither even nor odd. [Student then considers results of substitutions and determined whether they are even or odd]{Well here [referring to 2^2 – 2 =2] we are getting an even and so the conclusion here was that considering integers $x^2-\ x$ doesn't give us an even number.}	assertion.	other specific examples when the case yielding 0 had been found. In other word the student should have realised that after getting 0 for $x^2-\ x$ then subsequent empirical evaluations were no longer necessary. Cortney should have refuted the statement on the basis of a single example generated (Alcock, 2010). Cortney's proof behaviour shows that procedural knowledge dominated her proof attempts while micro reasoning ability was low (Duval, 2002).
Describe whether the following statement is true or false. For all real numbers a and $b, a-b>0 \Rightarrow a^2-b^2$*>0.*	• Written response Consider two real numbers $a=3$ and $b=2$ $a-b>0$ substituting we have $3-2>0$ $1>0$ which holds Also $a^2-b^2>0$ 3^2-2^2 $5>0$ which is true. Consider also $a=-2$ and $b=-3$ $a-b>0$, substituting we have $(-2)^2-(-3)^2$ $-5>0$ which is false $\therefore a-b>0$, implies a^2-b^2>0 for all positive real numbers only.	Cortney considered examples. In the first case considered that is, $a=3$ and $b=2$, Cortney established that the statement is true but the same statement was false when she instantiated with $a=-2$ and $b=-3$. Overall, she concluded that the statement holds for positive real numbers only. This conclusion was reached on the basis of a single empirical evaluation involving $a=3$ and $b=2$.	Once again a fragile grasp of the idea of counter-example revealed in this empirical proof scheme. Weak command of counter-argumentation was revealed in the conclusion when Cortney concluded that the statement holds for positive numbers. Her micro reasoning was also weak as she did not capitalise on the term "For all real…" in the formulation of the proof task. Hence she should have refuted the statement upon realising that it fails for negative numbers evaluated into the expression.
Describe whether the following statement is true or false. For all real numbers a and	• Follow up interview Researcher:You ended using examples. Cortney: I ended up using examples because I didn't know anything about the (eee) order properties which I now understand. Researcher: I now want to	When I asked Cortney to justify use of examples, Cortney explained that she did not know anything about order properties. The student was also probed about her conclusion on the task. Probing was necessitated by apparent confusion shown by the student. Student was then asked to go through her written	Order axioms were indeed irrelevant in this task. Critical thinking, that is, use of counter-argumentation was strategic in this case and Cortney should have defended their use especially in light of the fact that she refuted the claim using the substitution $a=-2$ and $b=-3$ (Alcock,

$b, a-b >$ $0 \Rightarrow a^2 -$ $b^2 > 0$.	understand specific questions about question 1. What is your overall conclusion? Is the statement true or false? Cortney: I didn't indicate may I please go through it so that I can determine what I wanted to say [reads solution] $a-b>0$ implies $a^2-b^2>0$ for all positive real number only. Well, I think it is false [……]I said it was a holding for positive numbers only since I tried out with the negative I was getting a negative value.	solution so that she could determine how she had concluded. She then concluded that the proposition was false because " it applies somewhere and does not apply somewhere". By this statement, Cortney meant that the statement was holding for positive numbers only because when she tried it with negative integers she got $a^2-b^2<0$.	2010). In other words Cortney failed to get a sense of how pieces of knowledge generated were solutions of proof task. Once more, failure to articulate purpose of counter example revealed Cortney's fragile command of proof by refutation (Raman, 2003; Sandefur et al ., 2013).
Prove that the sequence defined by $(u_n) = \frac{n^2-1}{2n^2+3}$ converges.	• Written response …………………………… ………. $\frac{1-\frac{1}{n^2}}{2+\frac{3}{n^2}} = \frac{1}{2}$ $\therefore |u_n - L| < \varepsilon$ $-\varepsilon < |u_n - L| < \varepsilon$ …………………………… $\frac{-5}{4n^2+6} < \varepsilon$ ………. $n^2 < \frac{-5}{4\varepsilon}$	First, Cortney determined the limit $= \frac{1}{2}$. Immediately after finding the limit, the statement: "$\therefore |u_n - L| < \varepsilon$" just sprang from nowhere and was transformed to $-\varepsilon < |u_n - L| < \varepsilon$. Algebraic manipulations led to $\frac{-5}{4n^2+6} < \varepsilon$. Cortney did not apply the modulus that should have resulted in $\left|\frac{-5}{4n^2+6}\right| < \varepsilon$ that should have yielded $\frac{5}{4n^2+6} < \varepsilon$. Meaningless statements such as $n^2 < \frac{-5}{4\varepsilon}$ appear in Cortney's proving profile.	Student incorrectly applied the modulus property: If $c > 0$ then $|a| \le c \Leftrightarrow -c \le a \le c$. The modulus symbol should have vanished. This reveals lack of profound grasp of the modulus properties. The weak command of modulus properties forced Cortney to engage in meaningless algebraic manipulation. For instance expression such as $n^2 < \frac{-5}{4\varepsilon}$ could result in complex solutions that are irrelevant could have been avoided by correct application of modulus rule. It thus be noted from Cortney's proof attempt that she engaged in algebraic manipulations when her command of underlying ideas on sequences was weak (Inglis & Mejia-Ramos, 2009; Koichu, 2012; Raman, 2003)

Table 5.8: End-of-instruction assessment data matrix for Cortney on Real Analysis proof tasks

Task	**Student's response (written, oral, actions)**	**Profiles of students' proving**	**Proof scheme elements present**
Use the definition of appropriate limit to prove that *$lim_{x\to\infty} \frac{\sqrt{3x^2+4}}{x}$ $= \sqrt{3}$.*	• Written responses <u>Proof</u> Let $\varepsilon > 0$ be given, $\exists$ we need to determine $x \in \mathbb{R}$ such that if $x > X$ then $|f(x) - L| < \varepsilon \left|\frac{\sqrt{(3x^2+4)}^2 - (\sqrt{3})^2}{x^2}\right| < (\varepsilon)^2$ …………………………………… ………. $4 < \varepsilon^2 x^2$ $x > \frac{2}{\varepsilon}$ Set $X = \frac{2}{\varepsilon}$	Student started by stating the need to determine to establish $x \in \mathbb{R}$ instead of $X \in \mathbb{R}$ that satisfies the condition that if $x > X$ then $|f(x) - L| < \varepsilon$. $f(x)$ and L were identified and substituted in the expression: $|f(x) - L| < \varepsilon$. Student then squared each term inside the modulus sign to get $\left|\frac{\sqrt{(3x^2+4)}^2 - (\sqrt{3})^2}{x^2}\right| < (\varepsilon)^2$ Wrong algebraic manipulations led to $4 <$	It can be noted from Cortney's proving efforts here that she engaged with algebraic manipulations without relating them ultimately to the goals set out to pursue. Student lacked good command of formal definitions of concepts she engaged with. For example, it was not clear to Cortney that she needed to determine a real number X for which

		$\varepsilon^2 x^2$. The algebraic operations in this case violated properties of modulus. While the student had set out to find $x \in \mathbb{R}$, she at the end wrote: "Set $X = \frac{2}{\varepsilon}$." The student was as a matter of fact trying to find $X \in \mathbb{R}$. Conclusion was not articulated, that is, although the student wrote: $X = \frac{2}{\varepsilon}$, she did not explain how this value proves that $\lim_{x\to\infty} f(x) = L$	$\|f(x) - L\| < \varepsilon$. Thus, there was lack of critical thinking (Alcock, 2010) as shown by squaring each term inside the modulus sign without questions being raised about the implications of such a move in light of properties of modulus. Hence, while Cortney managed to state that: "Set $= \frac{2}{\varepsilon}$", she did not realize how this piece of knowledge generated resolves the problem situation she engaged in (Koichu, 2012). Cortney's proving profile reveals mechanical symbolic manipulation without grasping the essence behind the symbolic manipulations (Inglis & Mejia-Ramos, 2011).
Use the definition of appropriate limit to prove that $lim_{x\to\infty} \frac{\sqrt{3x^2+4}}{x} = \sqrt{3}$	• Chalkboard demonstration So I am going to start up with the definition of an appropriate limit} {student writes and verbalizes the following] *Let* $\varepsilon > 0$ *be given, we need to determine* $x \in \mathbb{R}$ s.t. *if* $x > X$ *then* {what} [student writes] $\|f(x) - f(y)\| < \varepsilon$ [….] [referring to [$\frac{\sqrt{3x^2+4}}{x}$] minus this [referring to $\sqrt{3}$], must be less than ε. So, eee, umm} [student writes] $\|\frac{\sqrt{3x^2+4}}{x} <$ [Student immediately erases the above expression and says]{Ok, we will square here} [student writes] $\left\|\frac{(\sqrt{3x^2+4})^2}{x^2} - (\sqrt{3})^2\right\| < \varepsilon^2$. {And this is go to yield } [….]{It means , I will get ,umm, big X. Now, Iam going to set $X = \sqrt{\frac{4}{\varepsilon}}$. I think Iam done}.	Cortney stated that she was going to start by giving the definition of the limit of the function f as $x \to \infty$. Similar to written response effort, Cortney described that for $\varepsilon > 0$ chosen her goal was to determine $x \in \mathbb{R}$ s.t. *if* $x > X$ *then* {what} [student writes] $\|f(x) - f(y)\| < \varepsilon$. Another flaw can be identified here in the definition where in place of the limit L the student now has $f(y)$. Student then identified $f(x)$ as $\frac{\sqrt{3x^2+4}}{x}$ and the limit L as $\sqrt{3}$. Student wrote $\frac{\sqrt{3x^2+4}}{x} < \varepsilon$. And immediately erased the expression. Then she referred to $\left\|\frac{\sqrt{3x^2+4}}{x} - \sqrt{3}.\right\| < \varepsilon$ squared each term inside the modulus sign to get $\left\|\frac{3x^2+4}{x^2} - 3\right\| < \varepsilon^2$ that finally led to $\sqrt{\frac{4}{\varepsilon}} < x$. She then concluded that "it means I will get my big X. Now Iam going to set $X = \sqrt{\frac{4}{\varepsilon}}$."	As was noted in the written response section, the argument is flawed in that squaring was done indiscriminately, disregarding properties of modulus. The student squared term by term not taking into account properties of the reference theory of the mathematical domain, thus affirming my earlier interpretation that Cortney held an external conviction symbolic proof scheme. Typical behaviour shown in line with the external conviction symbolic proof scheme where the expression $\|f(x) - f(y)\| < \varepsilon$ (TH) was stated instead of $\|f(x) - L\| < \varepsilon$ but the student pointed out that the limit is $\sqrt{3}$. It can therefore be inferred that there is lack of consistency in the definition articulated and algebraic manipulations engaged in by Cortney (Sandefur et al., 2013).
Prove that $f(x) = x^2$	• Written response Proof Let $\varepsilon > 0$ be given, $\exists$	Cortney's definition of uniform continuity is flawed	Cortney's efforts to determine $\delta(\varepsilon)$ were

$+ 2x - 5$ *is uniformly continuous on* $[0, 3]$.	$x, y \in \aleph$ such that if $0 < \|x - y\| < \delta(\varepsilon)$ then $\|f(x) - f(y)\| < \varepsilon$ …………………………… Set $\delta(\varepsilon) = \frac{\varepsilon}{6}$	in two ways. First the arbitrary elements x, y should be elements from the interval [0,3]and not natural numbers as indicated by Cortney. Second, the condition: "$0 < \|x - y\| < \delta(\varepsilon)$" implies that x and y should be distinct, not a requirement in uniform continuity	based on definitions which had flaws highlighted under description of her proof attempts. $\delta(\varepsilon)$ found by Cortney was also not stated precisely. She did not capture the essential condition that $\delta(\varepsilon) \leq 1$. It can be inferred also that the quantity determined was not used to prove that the function is uniformly continuous an indication that student had procedural knowledge of the concept of uniform continuity (du Toit, 2009).
A sequence (a_n) *of real numbers is defined by* $a_1 = \sqrt{2}$ *and* $a_{n+1} = \sqrt{2 + a_n}$. *Prove that* (a_n) *converges and find its limit.*	• Written responses Now we prove it's a monotone sequence. ………………………………………… inductively $a_1 = \sqrt{2} = 1.4142$ $a_2 = \sqrt{2 + 1.4142} = 1.84775$ ………………………………………………….. $a_4 = \sqrt{2 + 1.9615} = 1.99036$ From above, $a_1 < a_2 < a_3 < a_4 < a_n$ We conclude that it holds for $a_1, a_2, a_3, a_4, \ldots . a_n$ We now assume that it holds for $n = k$ in other words we saying $a_k > a_{k-1}$ We now want to prove if its true for $n = k + 1$ $a_{k+1} - a_k = \sqrt{2 + a_k} - \sqrt{2 + a_{k-1}}$ ……………………………………………………. $= \frac{a_k - a_{k-1}}{\sqrt{2 + a_k} + \sqrt{2 + a_{k-1}}}$ From the hypothesis from the induction $a_k > a_{k-1}$ ∴ we conclude that the sequence is a monotone increasing which converges to its lub. Now determining the limit which is the lub. $a_{k+1} - a_k > 0$ $\left(\sqrt{2 + a_k}\right)^2 - (a_k)^2 > 0^2$ ……………………………………. $(a_k + 1)(a_k - 2) < 0$ By order properties either $a_k + 1 > 0$ and $a_k - 2 < 0$ or $a_k + 1 < 0$ and $a_k - 2 > 0$ ………………………………… $\sqrt{2} < a_n < 2$ …its limit is 2	Particular instantiations ($a_1 = \sqrt{2}$, $a_2 = 1.84775$, $a_3 = 1.99036$) were used to investigate the behaviour of the sequence and the student wrote: $a_1 < a_2 < a_3 < a_4 < a_n$. In other words particular instantiations were used to infer that (a_n) is a monotone increasing sequence. Cortney then assumed that the fact that the sequence is a monotone increasing sequence holds for $n = k$. The induction hypothesis was correctly stated as $a_k > a_{k-1}$. The student then correctly proved the implication statement $P_k \Rightarrow P_{k+1}$ by proving that the statement holds for $n = k + 1$ that was established by performing algebraic manipulations on expressions for: $a_{k+1} - a_k$. The identity $\sqrt{x} - \sqrt{y} = \frac{x-y}{\sqrt{x}+\sqrt{y}}$ was correctly applied though implicitly without being stated. Induction hypothesis was correctly applied to reach the goal that the sequence is a monotone increasing sequence. This conclusion stated prematurely because she had to show that the sequence is bounded. However, the student went on to prove that (a_n) is bounded by noting that $a_{k+1} - a_k > 0$ which followed logically from the assertion that the sequence is monotone increasing that the	While the student's proof effort revealed sound knowledge about the formal rhetoric part of the proof (Selden & Selden, 2009), her argument showed a serious violation of the formal proof framework, that is the mathematical conventions involved in proving. The following evidence supports these observations. Conclusions were first stated and then premises were later on presented. This is in stark contrast to the idea that the premises should logically entail the conclusion for an argument to be valid (Stylianides & Stylianides, 2009). For instance, Cortney concluded that the sequence (a_n) is monotone increasing on the basis of specific examples. Then she went on to formulate the induction hypothesis and established the implication $P_k \Rightarrow P_{k+1}$. It was noted that Cortney stated the conclusion in the base step of the method of mathematical induction yet the conclusion should have stated when all the stages of proof by induction had been completed as they

		student had established. Order axioms of $\mathbb{R}$ were applied to this inequality to deduce that (a_n) is bounded and consequently has 2 as its supremum, which is the limit of (a_n)	form the basis of the conclusion. According to Curd (1982) the premises should logically entail the conclusion which was not the case in this example.

Table 5.9: Mid-instruction assessment data matrix for Bea on Real Analysis proof tasks

Task	Student's response (written, oral, actions)	Profiles of students' proving	Proof scheme elements present
Describe whether the following statement is true or false. For all real numbers a and b, $a - b > 0 \Rightarrow a^2 - b^2 > 0$.	• Written response $a - b > 0 \Rightarrow a > b$ $a^2 - b^2 > 0 \Rightarrow a^2 > b^2$ $\Rightarrow \sqrt{a^2} > \sqrt{b^2}$ $\therefore a^2 - b^2 > 0 \Rightarrow a > b$ Therefore the statement is true	Bea started by noting that $a - b > 0 \Rightarrow a > b$ which is a correct deduction. She then stated the conclusion [then part] of the proposition that she inferred would lead to $a^2 > b^2$. However, she then deduced that $\sqrt{a^2} > \sqrt{b^2}$, a wrong deduction. For instance if $a^2 = 4$ and $b^2 = 9$, the statement could lead to $-2 > 3$ which is a wrong assertion. Bea then stated that $\therefore\ a^2 - b^2 > 0 \Rightarrow a > b$ and finally concluded that the statement is true	From the description of the proof attempt by Bea it can be seen that she engaged with algebraic manipulations without paying due attention to the reference theory or mathematical domain concerned, e.g., the deduction $\sqrt{a^2} > \sqrt{b^2}$ (Duval, 2002). It can also be observed from the solution that Bea lacked grasp of the relation $\lvert x \rvert = \sqrt{x^2}$ as can be seen from $\sqrt{a^2} > \sqrt{b^2}$. Also Bea started by stating the conclusion instead of developing an argument that would lead to the conclusion. Hence Bea violated the proof framework (Selden & Selden, 2009).
Determine whether the statement is true or false. If x is an integer, then $x^2 - x$ is an even integer. Justify your answer.	• Written response Taking x to be an odd number e.g $n = 3$ $3^2 - 3 = 6$ (even). $n = 5$ $5^2 - 5 = 20$ $(even)$ $n = 7$ $7^2 - 7 = 42$. If n is an even number e.g $n = 2$ $2^2 - 2 = 2$ (even) $n = 4$ $4^2 - 4 = 12$ (even) $n = 6$ $6^2 - 6 = 30$ (even) […] the statement only holds for $x \neq 1$ and $x \neq 0$. Therefore since the statement does not hold for $x = 1$ and $x = 0$, the statement is false	Bea used specific examples to refute the statement. First, she used odd numbers (3, 5, 7) and noted that empirical evaluations of the expression $x^2 - x$ with these odd numbers gave even numbers. Next even numbers (2,4,6) were evaluated into the expression and the student obtained an even number in each case. Bea observed the statement does not hold for the cases $x = 1$ and $x = 0$ and consequently concluded that the statement was false. However no evidence was adduced by her to justify her claim that the statement is false when $x = 1$ and when $x = 0$	All specific examples used gave even numbers. Bea stated that the statement does not hold for the cases $x = 1$ and $x = 0$. This claim was not justified by Bea and her basis for drawing the conclusion that the statement is false was not articulated. Bea's proving profile reveals her weak command of the concept of a counterexample because she failed to justify how the particular instantiations rendered the statement invalid (Alcock, 2010). Therefore examples were used without drawing meaning from them. It can be inferred that Bea did not grasp how pieces of knowledge generated are solutions to given tasks given (Fukawa-Conelly, 2012; Koichu, 2012).

Determine whether the statement is true or false. If x is an integer, then $x^2 - x$ is an even integer. Justify your answer.	• Chalkboard demonstration [Student writes the statement while reading aloud what is being written] *If x is an integer, determine whether x^2 $-x$ is an even number*. {If x is an integer, determine whether x^2 $-x$ is an even number}[Student writes} *even* 2{Aaa, I started with x as an even number and I considered 2. If you substitute 2, if you substitute 2, its 2^2 which is …} [Student writes] *If* $x = 2$ {Its 2^2, we get 4 minus 2 which is equal to 2. If $x = 4$, its 4^2, which is $16 - 4$ we get 12} [Student writes] $x = 4, 12$ even {Then if x is 5, its 5^2 which is $25 - 5$, you get 20, which is an even number} 5 20 *even* [Student writes even against each of case for $x = 2,3,4,5$ and writes the following on the board] *But if* $x = 0$ *and* $x = 1$ {But if $x = o$ consider $x =$ 0 and $x = 0$, if $x = 0$, the answer is 0 and if $x = 1$, the answer is 0} 0, 0 [referring to answers obtained by inputting 1 and 0 into the expression x^2 $-x$] {And 0 is not an even number, therefore the statement x^2 $-x$ is an even number is false because 0 is not an even number} [Student writes the conclusion] x^2 $-x$ *is an even number is false because* 0 *is not an even number*.	Similar to the written response effort, specific examples were used. Empirical evaluations involving $x = 2, x = 4$ and $x = 5$ yielded even numbers. During this phase Bea was able to produce examples to support her claim that the statement is false by considering cases when $x = 1$ and when $x = 0$. She explained that empirical tests with these numbers gave the answer 0. Bea concluded on the basis of this counter argumentation that the conditional statement: if x is an integer then x^2 $-x$ is even is a false statement.	A similar argument to the one in written response section with the exception that for the chalkboard demonstration Bea justified the conclusion that the expression x^2 $-x$ is even is false by citing the integers 1 and 0 as counter examples.
Prove that the sequence defined by $(u_n) = \frac{n^2-1}{2n^2+3}$ converges.	• Written response $(u_n) = \frac{n^2-1}{2n^2+3}$ divide each term by n^2 $\lim_{n\to\infty} \frac{\frac{n^2}{n^2}-\frac{1}{n^2}}{\frac{2n^2}{n^2}+\frac{3}{n^2}} = \frac{1}{2}$ Therefore the sequence converges to $\frac{1}{2}$	Bea divided by the dominant term and established the limit $L = \frac{1}{2}$. Bea concluded that the sequence converges to $\frac{1}{2}$.	Bea focused on techniques for finding the limit rather the meaning of the concept of limit in terms of finding a natural number for which $\|u_n - L\| < \varepsilon$. In other words Bea's proving profile focused on instrumental techniques for determining the limit of a function as opposed to exploring conditions that establish that a sequence converges. Precisely, Bea should have focused on finding a natural number $N(\varepsilon)$ for which $\|u_n - L\| < \varepsilon \quad \forall\, n \in \aleph$. This limitation in student's

			grasp of definition of convergence of a sequence was the focus of the reflective interview on Bea's proof construction effort. Hence, Bea's proof attempt reveals lack of interplay between procedural and conceptual knowledge (du Toit, 2009).
	• Follow up interview Researcher: […] Anyway after dividing by the dominant term and taking limit as n approaches infinity, you got half. Does this prove that it converges to $\frac{1}{2}$? Bea: You are supposed to go further taking this [pointing to working], the modulus of this [referring to the sequence (u_n),] minus the limit. We say this must be less than ε. Researcher: What are you trying to find by applying modulus? Bea: To verify whether the sequence converges. Researcher: In other words, to prove whether it converges? Bea: Whether it converges. Researcher: Can you explain further? What are you doing here [referring to attempts to determine n in terms of ε in] Bea: But after simplifying maybe after simplifying here Researcher: Ehee Bea: And get the answer then multiply the denominator by ε and bring the ε here. Aaa, Iam not quite sure on how to conclude the statement[repeated]. Researcher: Now you were talking about dividing by ε. what are you trying to find when you are dividing by ε. Bea: n Researcher: What is n? Bea: Is it greater than, is it greater than ε? (laughs) Researcher: […] How do you define the convergence of a sequence? When do we say a sequence converges? Bea: [silent]. I am not well versed in that because [silent]. This is why..[apparently stuck]	The focus of the follow up exercise was to examine Bea's conceptions of the idea of convergence because her solution effort had not revealed much data from which to infer the kind of proof scheme held by the student. First, she admitted that her efforts were inadequate and described the need to go further with the proving process. When asked to clarify what she would be trying to establish by "applying modulus", Bea stated that one would be trying to "verify whether the sequence converges." When I probed Bea on what she was trying to achieve by expressing n in terms ε, Bea's explanation tended to be focused on instrumental ideas rather than the conceptual ideas driving those procedural efforts, e.g., "simplifying …and get the answer .. and multiply". She did not explain the goal she sought to accomplish by engaging in the procedural ideas and she even said "Iam not quite sure how to conclude". Further, when pressed about what n represents, once again her explanation had a procedural or instrumental bias e.g., "…is greater than ε?" In addition when asked to define convergence of a sequence she retorted: "I am not well versed in that."	While Bea demonstrated an awareness of the formal rhetoric aspect (Selden & Selden, 2009) she could not proceed and execute the plan. Selden and Selden say it is not important that a student articulates the behaviour but he or she must be able to act on it. Thus while Bea explained that there was need to proceed further and determine n in terms ε she could not execute this plan. Moreover, Bea was supposed to describe the need to find a natural number $N(\varepsilon)$ such that $\lvert u_n - L \rvert < \varepsilon$. Bea could not explain these underpinnings of convergence of a sequence but rather her explanation centred on procedural techniques whose purposes she could not justify . When asked about crucial ideas on sequences like convergence, Bea retorted: "I am not well versed in that…" Responses such as this reveals that Bea engaged with mathematical ideas when they have limited knowledge about their underlying ideas. e.g., Bea talked about "is greater than ε?", without having a clear picture of the quantity that had been determined. Such proof behaviour point to mechanical manipulation of symbols by the student without grasping the essence behind those manipulations (Duval, 2002; Koichu, 2002).

Table 5. 10: End-of-instruction assessment data matrix for Bea on Real Analysis

Task	Student's response (written, oral, actions)	Profiles of students' proving	Proof scheme elements present
A sequence (a_n) of real numbers is defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2+a_n}$. Prove that (a_n) converges and find its limit.	• Written response $a_1 = \sqrt{2}$ $a_{n+1} = \sqrt{2+a_n}$ A sequence (a_n) converges to a real number iff every subsequence of (a_n) converges to L. Proof	Bea just recalled one of the theorems covered during the Fundamental of Analysis Course. Its application in solving the problem at hand was not demonstrated. An impasse was reached as can be seen by the blank space that appears after student's declaration of intention to produce a proof.	Bea could not relate the theorem she stated to the task she was supposed to solve. She might have memorised the theorem without grasping how it can be applied in problem solving (Koichu, 2012). Bea lacked a coherent network of resources on proof and proving because student could not link the stated resource, that is , she could not relate the theorem to the task and hence the impasse experienced (Wilkerson-Jerde & Wilensky, 2011).
	• Chalkboard demonstration [Reads the question taking a bit of time, about 1 minute. Student verbalizes and writes] $a_1 = \sqrt{2}$ $a_{n+1} = \sqrt{2+a_n}$. [student then describes that] {A sequence converges to a real number L iff every subsequence of (a_n) converges to L.} [student writes] <u>Proof</u> *we have* $a_1 = \sqrt{2}$, then $a_2 = a_1 + 1$ $\sqrt{2+1}$. [Students erases and says] {Its $\sqrt{2+\sqrt{2}}$ and $a_3 =$ {This number [pointing to $\sqrt{2+\sqrt{2}}$] is less than $\sqrt{2}$. Then $a_3 = a_2 + 1 = \sqrt{2+\sqrt{2}+\sqrt{2}} < a_3$ [Student is silent for a while, then rubs what has been written and says] {I wanted to show that} a_1 $\sqrt{2}+\cdots$ {This number is eee} [writes] $\sqrt{2}+\cdots$. [student changes sign and writes] $\sqrt{2+\sqrt{2}+\sqrt{2}} > a_3$. [student says] {Then } $a_1 < a_2 < a_2$ {Then the next step is its proof by induction} [rubbing the board and analysing earlier statements} [Student says]	Bea read the question to be answered slowly. She then mentioned the theorem from the written response data source and then wrote: "Proof." This was then followed by specific instantiations that had no connection with the theorem stated. In carrying out the numerical tests, Bea erased the board on each attempt, that is, when she evaluated a_2 and a_3. During the empirical verifications, false claims were made. For instance Bea stated that: "This number [...]$\sqrt{2+\sqrt{2}}$] is less than $\sqrt{2}$." Also, meaningless statements were produced during the proving attempt: "$a_3 = a_2 + 1 = \sqrt{2+\sqrt{2}+\sqrt{2}} < a_3$". The notation: $a_2 + 1$ is not consistent with (a_n) given. Also when the statement cited above is followed through it would give $a_3 < a_3$ which is senseless. While the representation of a_3 is not true , the statement $\sqrt{2+\sqrt{2}+\sqrt{2}} > a_3$ is not consistent with what the student had written earlier: $a_3 = \sqrt{2+\sqrt{2}+\sqrt{2}}$. Bea then concluded that $a_1 < a_2 < a_2$. Just as before the formulation $a_2 < a_2$ is	Just as was the case with the written response a theorem was just mentioned and never brought to bear on the task at hand. Bea showed lack of confidence when she dealt with problem. Lack of confidence was shown by actions such as erasing what had been written, being silent for some moments during question attempt, reading the question slowly and changing signs and writing senseless statements such as: "$a_3 = a_2 + 1 = \sqrt{2+\sqrt{2}+\sqrt{2}} < a_3$". These actions and senseless utterances and lack of confidence point to Bea's fragile grasp of the structural relations between underlying mathematical ideas , that is, severe limitations in Bea's knowledge about concepts of monotone sequences. In her proving effort, Bea did not attend to the question demands, that is, she did not articulate her goals precisely which should have been to determine whether the sequence was monotone increasing or decreasing, exploring whether the sequence is bounded and finally finding the limit. The concepts examined: boundedness, limit and monotone increasing or

	{Proof by induction, but ndinenge ndakanwa zvokumberi [*I have forgotten the subsequent steps]*}	senseless and a justification for such a conclusion not also given. Student finally erased the chalkboard and gave up on the proving effort as implied by the statement: "I have forgotten the subsequent steps"	decreasing give rise an important criterion for convergence of a sequence which is stated in the form of a theorem. Bea's efforts did not show the interrelatedness in those concepts. Hence, she could not access the relevant theorem (Raman, 2003; Sandefur et al., 2013).
	• Follow up interview Researcher: You can move on to other steps you are familiar with Bea: Zvimwe zvacho ndakanganwa (*I have forgotten the steps)* Researcher: You may refer to your answer sheet if you want. Bea: Hapana pakatosara pari blank *(There is nothing written)* . I can't recall some of the steps that Iam supposed to follow	Follow up interview on how the student had engaged with the task confirmed the impasses experienced by the student when writing the tasks and when also presenting the tasks on the chalkboard. Bea mentioned that she could not remember steps she was supposed to follow. When I urged her to refer to her answer booklet on written responses, Bea stated that: "There is nothing written"	Severe limitations in Bea's grasp of the convergence criterion of bounded monotone sequence were confirmed during the reflective interviewing phase. She explained that the concepts were far out of her reach that she could not remember even the steps involved. These data reveal that Bea experienced challenges with the task to a point where she could not figure how to begin the proof construction process. Bea conceived proving in terms steps to be remembered and followed.
Use the definition of appropriate limit to prove that $lim_{x\to 1}\left(\frac{x^3}{x-1} - \frac{1}{x-1}\right) = 3$	• Written response Let $\varepsilon > 0$ be given. Required to find $x \in \mathbb{R}$ such that if $x > X, \|f(x) - L\| < \varepsilon$ $\left\|\frac{x^3}{x-1} - \frac{1}{x-1} - \frac{3}{1}\right\| < \varepsilon$ $\left\|\frac{x^3-1-3(x-1)}{x-1}\right\| < \varepsilon$ ………………………….. $\left\|\frac{x^3-3x+2}{x-1}\right\| < \varepsilon$ …………………………… $(x+2)(x-1) < \varepsilon$	Definition of limit of function f as $x \to x_0$ confused with the definition of limits involving infinity. The condition of a deleted neighbourhood not shown in the definition given. Student wanted to determine a real number x instead of establishing the existence of $\delta(\varepsilon) > 0$ that would make it possible to map points in the neighbourhood of 1 to the neighbourhood of the limit, 3. Student was then stuck after factorising.	Bea's woes with proof tasks continued also with this task on limits. Limits involving infinity were confused with the limit of a function as the function $f \to x_o$. Procedural aspects were handled successfully as shown by correct factorisation. Impasses experienced after factorising can be attributed to weak command of concepts involved in limits. Underlying ideas of limit of a function were not accessed. Therefore algebraic manipulations were done without in-depth understanding of underlying ideas and the goals such efforts sought to accomplish. These were persistent characteristics of Bea's proving profile (Harel & Rabin, 2010).

Table 5.11: Taku's Mid-instruction assessment data matrix on Real analysis proof tasks

Task	Student's response (written, oral, actions)	Profile of student' proving	Proof scheme elements present
Describe whether the following statement is true or false. For all real numbers a and $b, a-b>0 \Rightarrow a^2-b^2>0$.	• Written response $a-b>0 \Rightarrow a>b$ squaring both sides $\Rightarrow a^2>b^2 \Rightarrow a^2-b^2>0$. ∴ The statement is true	The statement $a-b>0 \Rightarrow a>b$ is true. However the process of squaring was done without taking into the effect of squaring negative real numbers e.g., $-5<-2$ but $25>4$. After squaring both sides Taku then concluded that the statement is true.	The description of Taku's proof attempt reveals that mathematical processes are sometimes used in an instrumental fashion by students, that is, without attention to reference theory of the proof task. For example Taku did not question implications of squaring elements of $\mathbb{R}$ (Alcock, 2010).
Determine whether the statement is true or false. If x is an integer, then x^2-x is an integer. Justify your answer.	• Written response $x^2-x=x(x-1)$……Plugging integers $2,-1,\quad 1,2$ -2(-2-1) $\quad 1(1-1)$ 6 $\quad$ 0 …………………… From the result obtained shows that it is true	Taku started by factorising the expression x^2-x. Integers were then plugged into the expression and Taku on the basis of these empirical validations concluded that the statement is true.	Student was convinced by just two examples that the expression x^2-x is even if x is an integer. This reveals that Taku had a weak grasp of the limitation empirical verifications as means of validating mathematical statements (Stylianides, 2011). Further, the fact that 0 is neither even nor odd was not understood as shown by; "$(1-1$ 0 The result obtained shows that it is true"
Prove that the sequence defined by $(u_n)=\frac{n^2-1}{2n^2+3}$ converges.	• Written response ………………………… …… ….. $\frac{1-\frac{1}{n^2}}{2+\frac{3}{n^2}}=\frac{1}{2}$ Given that there is $\varepsilon>0$ for a natural number $n>0$ s.t. $n>N(\varepsilon)$ there is a natural number $\varepsilon>N(\varepsilon)$ ……………….. $\left\|\frac{2n^2-2-2n^2-3}{2(2n^2+3)}\right\|<\varepsilon$ $\left\|\frac{-5}{2(2n^2+3)}\right\|<\varepsilon \Rightarrow \frac{5}{4n^2+6}<\varepsilon$ ………………………….. $n>\sqrt{\frac{5-6\varepsilon}{4\varepsilon}}$ which is $>N$ Therefore the sequence converges	First, Taku found the limit by dividing each term of the sequence by the dominant term. Next, Taku described the formal definition of convergence of a sequence. The definition lacked clarity as shown by "for a natural $n>0$" then there is $\varepsilon>0$. The definition as stated serves little purpose because if n is a natural number then it is by implication greater than 0. Despite flaws in the definition , Taku managed to perform correctly the algebraic manipulations and was able determine the size the natural number $N(\varepsilon)$.	Also, the definition reveals that the student has limited knowledge regarding the proof framework, conventions of doing. In this case $\varepsilon>0$ must be chosen first and it will determine the natural number $N(\varepsilon)$ that will determine convergence. Student had a superficial grasp of the underlying notion of convergence of a sequence. The formulation $\varepsilon>N(\varepsilon)$ reveals a severe limitation in the student's conception of $\varepsilon>0$. Although the natural number was found in terms of ε the student did not explain how it proves that the sequence converges. In other words, Taku did not explain how the piece of knowledge he constructed resolved the task he was faced with (Koichu, 2012).

Table 5.12: End-of-instruction assessment data matrix for Taku on Real Analysis proof tasks

Task	Student's response (written, oral, actions)	Profile of student' proving	Proof scheme elements present
Prove that $f(x) = x^2 + 2x - 5$ *isuniformly continuous on* $[0, 3]$.	• Written response Let $\varepsilon > 0$ be given and there exists $\delta(\varepsilon) > 0$ for $n > N$ $\lvert f(x) - f(y)\rvert < \varepsilon$ for x and y such that $\lvert x - y\rvert < \varepsilon$ $\lvert x^2 + 2x - 5 - y^{2+} + 2y - 5\rvert < \varepsilon$ $\lvert x^2 - y^2 + 2x - 2y\rvert < \varepsilon$ $\lvert (x - y)(x + y) + 2(x - y)\rvert < \varepsilon$ Factorising $x - y$ $\lvert (x + y + 2)(x - y)\rvert < \varepsilon$ $\lvert (3 + 3 + 2)(x - y)\rvert < \varepsilon$ $\lvert 8(x - y)\rvert < \varepsilon$ $\lvert x - y\rvert < \frac{\varepsilon}{8}$ Set $\delta(\varepsilon) = \frac{\varepsilon}{8}$	Taku did not state the set from which the arbitrary elements x, y were selected, that is, he should have written that: $x, y \in [0,3]$. It can also be observed that the definition of uniform continuity by Taku included the condition $n > N$ and a wrong expression $\lvert x - y\rvert < \varepsilon$ instead of $\lvert x - y\rvert < \delta(\varepsilon)$. Algebraic manipulations done to determine $\delta(\varepsilon) > 0$ were correct.. However, the substitution: $x = y = 3$ was not justified. Although, the student managed to find an expression for $\delta(\varepsilon) > 0$, he did not explain how the value found showed that $f(x)$ is uniformly continuous on the given interval.	The flaws noted in the definition show that Taku had weak grasp of the concept of uniform continuity. For instance the statements: $\lvert x - y\rvert < \varepsilon$ and $\lvert f(x) - f(y)\rvert < \varepsilon$, show that the same quantity $\varepsilon > 0$ works for both the domain (set) and the range a serious misconception about the concept of uniform continuity. Taku thought of proof construction in terms of symbol manipulations but meaning of those was not understood. Algebraic manipulations meant to determine size of $\delta(\varepsilon) > 0$ were correctly done but student lacked micro reasoning (Duval, 2002) and carried out the mathematical processes outside the reference theory of the proof task
Prove that $f(x) = x^2 + 2x - 5$ *isuniformly continuous on* $[0, 3]$.	• Follow up interview Researcher: […] Can you describe uniform continuity Taku ? Taku: Right. Then we say given, let be given $\varepsilon > 0$ right then there is, then we are to find $\delta(\varepsilon)$ which is an element of the set of natural numbers, right in such a way that from zero up to say absolute values of $x - x_o$ is less than ε. Right then we are going to have $f(x) - f(y) - L$ being [silent] Researcher: So $f(x) - f(y) - L$. What are you saying on uniform continuity? Taku: Ok. I am say $f(x) - f(y)$ then we put them in absolute then there should be less that limit [referring to L]	Taku could not describe uniform continuity properly. For instance, Taku thought of $\delta(\varepsilon)$ as a natural number. There was a mix up of ideas as shown by Taku's claim that "absolute values of $x - x_o$ is less than ε", yet $\varepsilon > 0$ is a quantity associated with the range of the function, that is, $\lvert f(x) - f(y)\rvert < \varepsilon$. Uniform continuity was also confused with the concept of limit of a function as shown by the claim that $\lvert f(x) - f(y)\rvert < L$ where L denotes the limit of a function.	Taku had a fragile understanding of the concept of uniform continuity. First, Taku thought of $\delta(\varepsilon)$ as a natural number. He might have confused the definition of convergence of a sequence with uniform continuity. Second his weak understanding of uniform continuity was shown also by mixing up ideas of uniform continuity with the idea of limit of a function as $x \to x_o$.
A sequence (a_n) *of real numbers is defined by* $a_1 = \sqrt{2}$ *and* $a_{n+1} =$	• Written response …………………………………………………… $a_1 = \sqrt{2} = 1.414$. $a_2 = \sqrt{2 + \sqrt{2}}$ =1.848	Specific examples were used to explore the behaviour of the sequence. Empirical evaluations done by Taku lacked accuracy	While empirical verifications are good at unwrapping the underlying property that usually forms the crux of the proof they failed to serve that purpose

$\sqrt{2+a_n}$. *Prove that* (a_n) *converges and find its limit.*	$a_3 = \sqrt{2+\sqrt{2}} = 1.832$ $a_4 = 1.801$ $a_1 < a_2 > a_3 > a_4$ This is a monotone decreasing sequence which began increasing but ultimately decreased. Upper bound is 1.848 and lower bound is 1.414	e.g., $a_2 = 1.848 > a_3 = 1.832$, which is senseless statement because (a_n) is a monotone increasing sequence. Also, the student had $a_2 = \sqrt{2+\sqrt{2}}$ and $a_3 = \sqrt{2+\sqrt{2}}$, the same value for the two terms a_2 and a_3, but two different values were obtained as shown earlier. Inaccuracy in empirical evaluations led to a somewhat vague formulation: "monotone decreasing sequence which began increasing but ultimately decreased."	because Taku was not accurate (Alcock, 2010; Morselli, 2006) For instance, Taku wrote a somewhat vague statement: "This is a monotone decreasing sequence which began increasing but ultimately decreased. Upper bound is 1.848 and lower bound is 1.414"
Define a sequence (x_n) *inductively by* $x_1 = 1$ *and* $x_{n+1} = \frac{2x_n+3}{4}$. *Prove that* (x_n) *is a bounded monotone sequence is bounded and hence determine its limit..*	• Written response ……………………………………… ……………………... $x_1 = 1 \quad x_2 = \frac{2(1)+3}{4} = \frac{5}{4} = 1.25$ $x_3 = \frac{2\left(\frac{5}{4}\right)+3}{4} = \cdots = \frac{11}{8} = 1.375$ Since $x_1 < x_2 < x_3 = a_n < a_{n+1}$ So it is a monotone increasing sequence Proof by induction $P(1) < P(2) \Rightarrow a_n < a_{n+1}$ from above Assume it hold for $n = k$ We are to prove that $n = k+1$ it also hold $\frac{2x_{k-1}+3}{4} < \frac{2x_k+3}{4}$ ……………………………………… ………. $\frac{1}{4}\left[\frac{2x_k+3}{4} - \frac{2_{k-1}+3}{4}\right] > 0$ $2x_k + 3 > 2x_{k-1} + 3$ ∴ it is a monotone sequence $(x_n) = \{1, \frac{5}{4}, \frac{11}{8}, \ldots . \frac{2x_n+3}{4}$ $\lim_{x\to\infty} \frac{2x+3}{4} = \infty$ lub is ∞ glb is 1 It has lower limit is and upper limit is ∞	Specific examples were used: $x_2 = 1{,}25$, $x_3 = 1.375$. Taku then used these empirical evaluations to draw the conclusion: $x_1 < x_2 < x_3 = a_n < a_{n+1}$. It can be observed here that there was a mix up in notation. The task involved terms, x_i, of the sequence (x_n) and yet Taku introduced terms a_i of sequence (a_n). Also, senseless statements like $x_1 < x_2 < x_3 = a_n < a_{n+1}$were used to conclude that (x_n) is a monotone increasing sequence. Further, statements such as $P(1) < P(2) \Rightarrow a_n < a_{n+1}$ featured in Taku's solution effort without specifying the meaning of the quantities $P(1)$ and $P(2)$. Justification for the claim that $x_n < x_{n+1}$ not given. Taku also wrote $\lim_{x\to\infty} \frac{2x+3}{4} = \infty$. n that was given as subscript in the inductive definition of the sequence just vanished. Taku concluded that the least upper bound is infinity and infimum is 1.	Taku used empirical evaluations to draw the conclusion that the sequence (x_n) is monotone increasing. The student demonstrated lack of awareness of the limitation of empirical explorations in proving statements that require formal deductive reasoning (lack of relevant insihts. It can be seen from Taku's effort that proving is largely conceived in terms of symbol manipulation without drawing meaning from such symbols. The disregard of meaning of symbols was of an alarming level to a point where the student teacher specified terms as a_i for the sequence (x_n). Mathematically ambiguous and incorrect statements such as "$\lim_{x\to\infty} \frac{2x+3}{4} = \infty$ lub is ∞ glb is 1. It has lower limit is and upper limit is ∞." Mathematical induction was not done properly. For instance the induction hypothesis (lack of was not stated clearly and hence was not used to infer that the sequence is not monotone increasing Selden & Selden, 2009).
Define a sequence	• Chalkboard demonstration	Student found the first three terms, x_1, x_2 and	The chalkboard demonstration confirmed

(x_n) inductively by $x_1 = 1$ and $x_{n+1} = \frac{2x_n+3}{4}$. Prove that (x_n) is a bounded monotone sequence is bounded and hence determine its limit..	{Aaa, Iam going to work out number one just I like I have done in my write up. Right, according to that question it says define a sequence (x_n) inductively by x_1 and x_{n+1} as given. So I will write. You are given} [student writes] $x_1 = 1$, $x_{n+1} = \frac{2x_n+3}{4}$ {So now for x_2, we put the value of x_1 here [pointing to $\frac{2x_n+3}{4}$] which we are going to get something like} $\frac{2+3}{4} = \frac{5}{4}$. {Then x_3, we put our $\frac{5}{4}$ here} [students writes] $\frac{2\frac{5}{4}+3}{4} = \frac{\frac{5}{2}+3}{4} = \frac{11}{4}$ $x_1 = 1$ $x_2 = \frac{5}{4}$ $x_3 = \frac{11}{4}$ {Now after doing this, first, second and third, I would like to see how my sequence or how my sequence in numbers are going, the trend that they are following. Right, may somebody punch for me in the calculator, I want to change this into decimal} [Student writes down answers given by other members] $x_1 = 1$ $x_2 = 1.25$ $x_3 = 1.375$ {If we are to look here, we will see that that [student writes] $x_1 < x_2 < x_3$ {in essence , we can just say } $x_n < x_{n+1}$ {Then from this, we can easily see that the sequence is increasing. Now we have been asked to show that it is what, it is bounded [silent] and then it is a monotone sequence and aaa, we also asked to find the limit. Right after doing this maybe we need to do it by induction} [Student writes] *Prove by induction* {Now from the numbers, that we have, we can easily see that } [Student writes] $P_{(1)} < P_{(2)}$ {So we now assume, since $P_{(1)} < P_{(2)}$, so for the numbers, we said it holds. We assume, it is or it is going to hold for $n = k$} [Student writes] *Assume* $n = k${So what are we saying now, we want to prove, we want to get for $n = k + 1$} [Student writes] $n = k + 1$ {So I will [inaudible] come with my number, this one [referring to $x_{n+1} = \frac{2x_n+3}{4}$], that is where Iam going to put all these values and prove by induction. So our k is going to be } [student writes] $\frac{2x_{k-1}+3}{4}$ {And for our k+1, we are going to get} $\frac{2x_k+3}{4}$[Student the writes] $\frac{2x_{k-1}+3}{4} < \frac{2x_k+3}{4}${Now fro this [pause], it can easily be seen that a number that is less than k here will always give us a less the number than is the one that is k. So from this I can, I have proved …[inaudible]. Then the other part is saying hence determine the limit.	x_3 of the sequence. Then he noted that $x_1 < x_2 < x_3$ and he then claimed that in essence} $x_n < x_{n+1}$. Student then pointed out that he was going to prove by induction that the sequence (x_n) is bounded. An ambiguous statement $P(1) < P(2)$ immediately followed and the student concluded that from the numbers, referring here to the specific examples, the statement was true. Because the student had not correctly articulated the induction hypothesis he had difficulty in deducing that $x_k < x_{k+1}$. An impasse then ensued and the student admitted that he had not grasped the concepts well.	the tenacity of the empirical proof scheme. Taku calculated the second term and the third term of the sequence and then concluded basing only these two empirical evaluations that the sequence is monotone increasing. Taku then wrote "$x_1 < x_2 < x_3$ {in essence , we can just say } $x_n < x_{n+1}$ {Then from this, we can easily see that the sequence is increasing." An effort to prove by induction proved to be a challenge. Student did not explain meaning of "$P_{(1)} < P_{(2)}$" written and did not state the induction hypothesis clearly. Hence, technical handles were not strategically and readily accessed (Hanna & Mason, 2014; Raman, 2003). As a result Taku's efforts to prove that the sequence is bounded were not successful. It can be observed from Taku's solution that he was aware of the hierarchical order of the proof. In other words he knew the goal that he intended to accomplish through the method of mathematical induction but failed to coordinate different parts of the proof (Selden & Selden, 2009). For instance, he could not establish the implication statement because he had failed to state clearly the induction hypothesis. Hence, Taku was unable to prove by mathematical induction that the sequence is bounded. Therefore, he could not proceed to determine the limit of the sequence which depended on the fact (x_n) is a bounded monotone sequence, a fact Taku failed to establish. Hence, Taku reached an impasse in a similar fashion to impasses experienced by participants in a study by Varghese (2009).

	Maybe to get the limit the way I worked it out. [silent apparently stuck]. Unfortunately I did not do that part maybe one of the reasons could be when you started this topic I was away. I didn't grasp some of the concepts well. I have been the first to be asked to present. I am still trying to learn on how to go about it}		
Define a sequence (x_n) inductively by $x_1 = 1$ and $x_{n+1} = \frac{2x_n+3}{4}$. Prove that (x_n) is a bounded monotone sequence is bounded and hence determine its limit..	• Follow up interview Researcher: […] You have $\frac{\frac{2+x}{x}}{\frac{4}{x}} = \infty$ […] how did you conclude that the least upper bound is infinity? Taku: Maybe initially I had to divide by the highest power [referring to the dominant term] which was x and then as x approaches what infinity, aaaa, I ended up getting something like 0, 4 over [referring to divide by] 0 which then [inaudible] it becomes undefined. Researcher: So you took this one [referring to $4/x$] as a very small number dividing into…[interruption from student teacher] Taku: Yes, I took this [$4/x$] as a very small number dividing say this whole thing [referring to $\frac{2+x}{x}$] divided by 0 which becomes undefined. This is how I got it.	The notation $x_{n+1} = \frac{2x_n+3}{4}$ was not grasped by the student. Subscript notation x_n was treated as just the variable x. This led to wrong algebraic manipulations $\frac{\frac{2+x}{x}}{\frac{4}{x}} = \infty$. This working illustrates that there was a mix up of concepts here. The idea of limits involving infinity was confused with convergence of a real sequence	Taku had challenges in interpreting the inductive definition of the sequence (x_n). So the $(n+1)^{th}$ term of the sequence was interpreted as $\frac{2x+3}{4}$ There was lack of access to relevant technical handles. It can be noted student's reasoning about the proof task was not consistent with the reference theory. The student was thinking about a different concept to the one required by the proof task. So Taku engaged in irrelevant algebraic manipulations of limits involved as a result of failing to operate within the area of real sequences He lacked grasp of structural relationship (Hanna and Mason, 2014; Koichu, 2012) . He ended up mixing concepts and therefore could not succeed proving.

Table 5.13: Mid-instruction assessment data matrix for Debra on Real Analysis proof tasks

Task	**Student's response (written, oral, action)**	**Profiles students' proving**	**Proof scheme elements**
Determine whether the following statement is true or false. For all real numbers a and $b, a - b > 0 \Rightarrow a^2 - b^2 > 0$.	• Written response …………………………………… If $a - b > 0$ then $a > b$ and $a - b \in \mathbb{R}^+$ $a^2, b^2 \in \mathbb{R}^+$ If $a = -a$ then $(-a).(-a) = a^2 \in \mathbb{R}^+$. If $a = +a$ then $(+a)(+a) = a^2 \in \mathbb{R}^+$ …………………………………… Since $a^2, b^2 \in \mathbb{R}^+$ and $a - b > 0$ it is false that $a^2 - b^2$>0. Proof $a^2 = a.a,$ $b^2 = b.b$ given $a - b > 0 \Rightarrow a > b$ $a.a > b.a$ if $a = +a$ $a.a - b.a > 0$ … …(1) $a^2 > b.a$ If $a = -a$ $a.a < b.a$ $a^2 < b.a$………..(2) If $b = -b$ $ab < b^2$ …………..(3) NB $a = -a$ or $+a$, $b = b$ or $-b$	First, a correct observation was made that $a - b > 0 \Rightarrow a > b$ Next the student then considered cases where $a, b \in \mathbb{R}^+$ and $a, b \in \mathbb{R}^-$ described by Debra as follows. If $a = -a$, presumably referring to a as being a negative real number that then gave $(-a).(-a) = a^2 \in \mathbb{R}^+$. The student then made the claim that since $a^2, b^2 \in \mathbb{R}^+$ and $a - b > 0$ it is false that that $a^2 - b^2$>0. Debra used order axioms of a field to prove the claim that "it is false that $a^2 - b^2$>0." She	From Debra's proof profile Debra used the order property of $\mathbb{R}$ to refute the proposition that for all $a, b \in \mathbb{R}$, $a - b > 0 \Rightarrow a^2 - b^2 > 0$. The specific order property employed by Debra is: if $a > b$ and $c < 0$ then $ac < bc$. Hence, it can be deduced that Debra could access the strategic technical handle which correctly applied to refute the claim. Therefore although Debra's proof attempt contained some somewhat vague notation: "If $a = -a$ then $(-a).(-a) = a^2 \in \mathbb{R}^+$" and "If $b =$

	adding (2) and (3) $a^2 + ab < ba + b^2 \ldots . a^2 - b^2 <$ 0.Which is a contradiction to $a^2 - b^2 > 0$. $\therefore \ldots \forall\, a, b \in \mathbb{R}$. It is false that $a^2 - b^2 > 0$.	applied the concept, $a > b$ and $c < 0$ then $ac < bc$, to refute the statement that $a - b > 0 \Rightarrow$ $a^2 - b^2 > 0$.	$-b\ ab < b^2$," she used axiomatic reasoning correctly to refute the given proposition.
Prove that the sequence defined by $(u_n) = \frac{n^2-1}{2n^2+3}$ *converges.*	• Written response $(u_n) = \frac{n^2-1}{2n^2+3} = \cdots \frac{1-\frac{1}{n^2}}{2-\frac{3}{n^2}}$ As $n \to \infty$ u_n converges to $\frac{1}{2}$. Let $n \in \mathbb{R}$, then there exists $N(\varepsilon) \in \mathbb{R}$ if it converges. $\left\lvert\frac{1}{2} + L - \frac{1}{2}\right\rvert < \varepsilon$ $-N(\varepsilon) \le L \le N(\varepsilon)$	Debra divided the expression by the dominant term and evaluated the limit to be $\frac{1}{2}$. This stage was then followed by a wrong formulation of the definition of convergence of a sequence. Debra considered n to be a element of $\mathbb{R}$. She then claimed that if (u_n) converges then a real number $N(\varepsilon)$ exists. It can be noted from Debra's attempt that the quantity ε in the definition was never specified. An expression $\left\lvert\frac{1}{2} + L - \frac{1}{2}\right\rvert < \varepsilon$ just sprang from nowhere and its connection with $N(\varepsilon)$ and the element n was not articulated. Finally, another isolated bit of information $N(\varepsilon) \le L \le N(\varepsilon)$ was then written.	Debra's proving profile reveals severe limitations in her knowledge of the concept of a sequence. A sequence is a mapping from natural numbers to real numbers, yet Debra took the domain of real sequences to be $\mathbb{R}$. Further, statements such as: "$\left\lvert\frac{1}{2}+L-\frac{1}{2}\right\rvert<\varepsilon$", point to a fragile grasp of the notion of convergence of a sequence. Symbols just sprang from nowhere without an explanation provided for the purposes they served. Hence, symbols were handled in a mechanical way without Debra showing evidence of grasping their essence (no access to CI). No conclusions could be drawn from the "working" presented. Debra's proving profile is typical of the external conviction ritual proof scheme (Harel & Sowder, 1998; CadawalladerOlsker, 2011).
Determine whether the statement is true or false. Justify your answer. For all real values of $x. f(x) \equiv 2x^2 + 7x - 4$, *implies that* $f(x) >$ 0.	• Written response Factorising $f(x)$ $2x^2 + 7x - 4 = 2x^2 - x + 8x - 4$ $x(2x-1) + 4(2x-1) \ge 0$ $(2x-1)(x+4) \ge 0$ $2x - 1 \ge 0$ or $x + 4 \ge 0$ ………………………………… $x \ge \frac{1}{2}$ $x \ge -4$ [solutions illustrated on number line] $\therefore f(x)$ has a solution $x \ge \frac{1}{2}$	Debra factorized the quadratic expression and obtained $(2x-1)(x+4) \ge 0$. One case of the order axioms was applied to the factors and yielded $x \ge \frac{1}{2}$ $x \ge -4$. These two inequalities were then represented on the number line. She then concluded that (x) has a solution $x \ge \frac{1}{2}$. The conclusion stated did not address the consequent (then part) of the proposition.	Although Debra factorised the quadratic expression successfully, she showed a weak command of the order axioms. The case $a > 0$ and $b > 0$ leading to $ab > 0$ was used by Debra while the other case $a < 0$ and $b < 0$ was not used. The conclusion drawn did not show a link with the question. Debra did not determine whether the proposition is true or false but rather she looked for solutions to the inequality $f(x) > 0$. So failure to interpret the question correctly led to an irrelevant conclusion being drawn. It can therefore be inferred that Debra did not grasp how the piece of knowledge constructed resolved the problem confronted (Koichu, 2012).
Determi	• Chalkboard demonstration	Debra started by stating her	Debra's chalkboard

ne whether the statement is true or false. Justify your answer. For all real values of $x. f(x) \equiv 2x^2 + 7x - 4$, implies that $f(x) > 0$.	{So solving for f at}. [Student then writes while saying], {we are having }$2\,x^2 + 7x - 4 \geq 0$ {Then factoring the left hand side we have} [Then writes] $2\,x^2 - x + 8x - 4 \geq 0$ {Grouping in pairs and then factoring out common factors we have} [Student then writes] $x(2x-1) + 4(2x-1) \geq 0$ $(2x-1)(x+4) \geq 0$ [Student then explains]{So for a product to be positive, it means the two terms multiplying each other must be all positive or must be all negative, that is, they must be all greater than 0 or they must all be less than 0. So its either ….} [Student then writes the following while verbalizing what she writes] $2x - 1 \geq 0$ *and* $x + 4 \geq 0$ *or* $2x - 1 \leq 0$ *and* $x + 4 \leq 0$ {Then solving the inequalities} [Reads what is being written] $x \geq \frac{1}{2}$, $x \geq -4${Then representing the solution on a number line we have}[Student teacher draws number line describes the illustrations]{So we have $\geq \frac{1}{2}$, then we have $x \geq -4$. So our solution equals…} [Student then writes]$x \geq \frac{1}{2}$ [Student then considers $2x - 1 \leq 0$ *and* $x + 4 \leq 0$ and writes] {Then we have} $2x - 1 \leq 0,\ x + 4 \leq 0$ {Then representing the solution on a number line} [Student draws the number line] {We have $x \leq \frac{1}{2}$ and $x \leq -4$. So our solution two} [Student then writes]$x \leq -4$ [verbalized the answer]{Then combining the two solutions } {We have $x \geq \frac{1}{2}$, then $x \leq -4$. Since there is no intersection there [referring to the illustration] it means that it is not always that given that $f$at x identical to 2 $x^2 + 7x - 4, f$ at x is always greater or equal to 0 since we can see from this illustration [pointing to the number line], that between -4 and $\frac{1}{2}$, $f(x)$ is not greater or equal to zero}	goal "so solving for f". She then factorised the quadratic expression successfully. She then used order axioms to solve the inequality $(2x-1)(x+4) \geq 0$. Debra applied order axioms correctly to solve the inequality $2\,x^2 + 7x - 4 \geq 0$. The student illustrated the two solutions of the inequality on the number line. The graph was then used to refute the given proposition.	demonstration is different from her written response in two main ways. First, Debra considered both cases of the order axiom $ab > 0 \Rightarrow a > 0$ and $b > 0$ or $a < 0$ and $b < 0$. In the written response section she had just considered the case $a > 0$ and $b > 0$ and left out the other alternative $a < 0$ and $b < 0$. Second, the solutions obtained were related to the demands of the proof task. It can be seen from Debra's proving profile that her use of semantic and syntactic approaches to resolve this proof task helped to illuminate the important interplay between informal and formal mathematics. Hence, from Debra's proof profile it can be seen that knowledge of order properties were combined with particular instantiations in the form of number line illustrations to reach the correct conclusion that the statement is false (Raman. 2003;Weber & Alcock, 2004). Further, Debra's proof construction effort reveals that she had a strong command of the notion of a counter example and had grasped the essence of how the piece of knowledge she generated resolved the proof task (Harel & Sowder, 1998, 2007; Koichu, 2012).
Determine whether the statement is true or false.	• Written response If $\forall\, x \in$ integer then $x = -x$ if x is negative or $x = x$ if $x$is positive If $x = -x$ $x^2 - x = (-x)(-x) - (-x) = x^2 + x$ (1) If $x = x$ $x^2 - x = x.x - x = x^2 - x$----(2)From (1), let $x = -3$	The student considered cases when the variable x would be negative and cases x would be positive. These ideas were then expressed in a somewhat awkward manner: $x = -x$ if x is	Despite an awkward formulation introduced by the student: "$x = -x$ if x is negative or $x = x$ if x is positive", the particular instantiations made were consistent with the form of

If x is an integer, then $x^2 - x$ is an integer. Justify your answer.	$\therefore x^2 - x = (-3)(-3) - (-3)$ $= 9 + 3 = 12$ From (2) let $x = 3$ $x^2 - x = (3).(3) - (3)$ $= 9 - 3 = 6$ $\therefore$ if x is an integer then $x^2 - x$ is an even number.	negative or $x = x$ if $x$is positive.” This formulation then gave two distinct expressions for $x^2 - x$: $x^2 + x$ for $x < 0$ and $x^2 - x$ for $x > 0$. Specific examples consistent with the expressions stated, that is, a positive integer ($x = 3$) and a negative integer ($= -3$), were substituted into the two expressions. Even integers were obtained in each case. Basing of these two instantiations Debra concluded that the statement is true.	reasoning displayed. The student used two specific examples only and concluded that the statement is true. Such tendencies which are typical of the empirical proof scheme revealed that the student is not aware of the fundamental limitation of empirical evaluations that they cannot be elevated to the status of a proof for deductive proof tasks (Stylianides, 2011).

Table 5.14: End-of-instruction assessment matrix for Debra on Real Analysis proof tasks

Task	**Student’s response (written, oral, actions)**	**Profiles of students’ proving**	**Proof scheme elements present**
A sequence (a_n) of real numbers is defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2+a_n}$. Prove that (a_n) converges and find its limit.	• Written response Let $\varepsilon > 0$ be given, it is required to find $N(\varepsilon) \in \aleph$, s.t. $n > N(\varepsilon)$ if $(a_n) \to L$ then $\|a_n - L\| < \varepsilon$ $a_1 = \sqrt{2} = 1.4142$ $a_2 = \sqrt{2+\sqrt{2}} = 1.8478$ $a_3 = \sqrt{2+\sqrt{1.8478}} = 18328$ $a_4 = \sqrt{2+1.8328} = 1.8313$ $a_5 = \sqrt{2+\sqrt{1.8313}} = 1.83128$ $a_6 = \sqrt{2+\sqrt{1.83127}}$ [student writes and deletes] a_n is monotonic decreasing, that is $a_{n+1} < a_n$ $a_{n+1} - a_n < 0$ $\sqrt{2+a_n} - a_n < 0$ $2 + a_n < (a_n)^2$ …………………………………….. $a_n < 2$ or $a_n < -1$ [student draws number line]. So $-1 < a_n < 2$ $\|a_n - L\| < \varepsilon$ $-\varepsilon < a_n - L < \varepsilon$ $L - \varepsilon < a_n < L + \varepsilon$ $a_n \epsilon (L - \varepsilon,\ L + \varepsilon)$ $\therefore a_n$ converges and its limit is -1	Debra started by stating the formal definition of the convergence of a sequence, just as a dead end because it was never brought to bear on the problem. Rather after stating the definition she resorted to particular instantiations. However the empirical evaluations were not correctly done, e.g., $a_2 = 1.8478$, $a_3 = 1.8328$ $\Rightarrow a_2 > a_3$ which is not true for (a_n) , a monotone increasing sequence. These inaccurate empirical evaluations resulted in the wrong conclusion that (a_n) is a monotone decreasing sequence. The definition of a monotone decreasing sequence was applied to form the inequality: $2 + a_n < (a_n)^2$. Solution attempts to the inequality were not successful as the student got $a_n < 2$ or $a_n < -1$. Debra then wrote: $-1 < a_n < 2$. This is not a logical consequence of the solutions stated by the student. Springing from nowhere was the statement $\|a_n - L\| < \varepsilon$, which was correctly transformed to $a_n \epsilon (L - \varepsilon,\ L + \varepsilon)$. Debra finally concluded that (a_n) converges to -1.	Debra did not recognise that use of the formal definition could complicate matters when applied to this task. Student’s failure to access relevant conceptual insights that is, structural relationship of bounded monotone sequences might have forced Debra to shelve formal deductive reasoning and was shelved without providing any justification. Debra then started to use particular examples. It can be inferred that while Debra showed a preference for the axiomatic proof scheme she abandoned this approach and switched to use of examples. The shift from a higher level proof scheme to a lower level proof scheme might have been caused by the fact that Debra had not grasped the monotone convergence criterion Debra did not succeed in exploring properties of the sequence using the instantiations because of inaccuracy (Morselli, 2006). She then gave the wrong conclusion that the sequence was monotone decreasing. Wrong symbolic manipulations then followed which led the conclusion that: “$\therefore a_n$ converges and its limit is -1"

		which was not a logical consequence of the statement: "$-1 < a_n < 2$"
• Chalkboard demonstration So we are given that (a_n) is a sequence of real numbers then our first term is a_1} [student writes] $a_1 = \sqrt{2}$ $a_{n+1} = \sqrt{2 + a_n}$. {So the definition is [student reads from answer book], let $\varepsilon > 0$ be given} *Let* $\varepsilon > 0$ *be given* {Then it is required to $N(\varepsilon)$, which is an element of natural numbers such that $n > N(\varepsilon)$ then if $a_n \rightarrow$ L which is the limit then $a_n - L$ less than ε. So, first all we have to prove that a_n converges. So we will first of all find the terms of the sequence} [Student verbalizes and writes] $a_1 = \sqrt{2}$ = 1.4142 $a_2 = \sqrt{(2 + \sqrt{2})}$= 1.8478 $a_3 = \sqrt{(2 + \sqrt{1.8478})}$ = 1.8328 $a_4 = \sqrt{(2 + \sqrt{1.8328})}$ =1.8313 {Then we substitute a_{n+1} by $\sqrt{2 + a_n}$, so it becomes } [student writes] $\sqrt{2 + a_n} - a_n < 0$ $\sqrt{2 + a_n} < a_n$ {Then squaring both sides} $2 + a_n < a_n^2$ {Then taking terms on the LHS to the right we will have} $a_n^2 - a_n - 2 > 0$ {Then we solve for the LHS by factorization. We factorize the LHS} $(a_n)^2 - 2a_n + a_n - 2 > 0$ $a_n(a_n - 2) + 1(a_n - 2) > 0$ $(a_n + 1)(a_n - 2) > 0$ {If $(a_n + 1)(a_n - 2) > 0$, it means that these two expressions [referring to factors] are all positive or they are all negative, so} [student writes] $a_n + 1 > 0$ *and* $a_n - 2 > 0$ $a_n + 1 < 0$ *and* $a_n < 2$ $a_n > -1$ $a_n > 2${Then on this part we give}{So we have our solution} $a_n < -1$ *and* $a_n < 2$ *-1*$< a_n < 2$ {So the solution is the same *1*$< a_n < 2$ {So our question requires to prove that (a_n) converges. So as you have seen that it was	For the chalkboard demonstration, Debra preferred to use her write up to aid her presentation. As a result the chalkboard demonstration and the written responses shared many similar features. For instance the formal definition of convergence for sequence was correctly verbalised and was immediately abandoned. In other words the definition was just stated and never used to tackle the task. Rather, the student started using specific examples. Inaccurate empirical evaluations were used by Debra to infer that (a_n) is a monotone decreasing sequence. The inequality $a_{n+1} < a_n$ led to a wrong expression $2 + a_n < a_n^2$ that was factorised to give $(a_n + 1)(a_n - 2) > 0$. One notable distinction between the chalkboard illustrations and written responses was that with the latter order axioms of the real number field were applied to produce two solutions: $a_n < -1$ and $a_n > 2$. The solution $a_n < -1$ tells us that the sequence is bounded above by -1 which contradicts the fact that $a_1 = \sqrt{2}$, a sequence of positive terms. On the other hand the solution, $a_n > 2$, indicates that the sequence is not bounded above and hence has no supremum. Presumably, because of facts noted above the student disregarded the two solutions and finally wrote: *-1*$< a_n < 2$ Perhaps the solutions were neglected to allow Debra to squeeze the sequence (a_n) between -1 and 2.	The two data sources have many similar features because the student used her write up to do her presentation. So most of the inferences made under written response section apply to this section. However, some distinctive features were noticed. For instance, Debra claimed that "we will first of all find the terms of the sequence." The statement was made without realising that the set of natural numbers (domain of real sequences) is infinite. Debra stated her goal without reflecting on the definition of a sequence: a function with domain the set of natural numbers and range which is a subset of the real numbers. Another distinguishing feature between the written response and the chalkboard demonstration was that the student could not explain how the solution obtained resolved the proof task. Student did not get a sense of technical symbolic manipulations engaged in (Hanna & Mason, 2014). She might have been influenced by answers obtained in the previous attempt (written response). Debra wanted to squeeze the sequence between -1 and 2. Further, the conclusion drawn that (a_n) has limit -1 was made that once again was not a logical consequence of the working shown. It can be, therefore, inferred that the premises did not logically entail the conclusion drawn (Curd, 1992; Stylianides & Stylianides, 2009).

	decreasing, therefore it converges to -1. A sequence converges to its limit therefore the limit is -1}		
Prove that $f(x) = x^2 + 2x - 5$ *is uniformly continuous on* $[0, 3]$.	• Written work Let $\varepsilon > 0$ be given, it is required to find $\delta(\varepsilon) > 0$ such that $n, m \in [0,3]$ and $n, m > M \in \aleph$ $f(n) < \frac{\varepsilon}{2}$, $f(m) < \frac{\varepsilon}{2}$ $\lvert f(n) - f(m)\rvert < \varepsilon$ $\lvert f(n) - f(m)\rvert < f(n) + f < \varepsilon < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$	The first formulation of the definition was correct as shown by an awareness that $\varepsilon > 0$ had to be chosen first before $\delta(\varepsilon) > 0$ can be determined. Efforts to produce the definition turned into a total mess when the student wrote $n, m > M \in \aleph$, $f(n) < \frac{\varepsilon}{2}$, $f(m) < \frac{\varepsilon}{2}$. The condition $n, m > M \in \aleph$ was presumptively drawn from the area of Cauchy sequences and confused with uniform continuity. Debra then wrote: $\lvert f(n) - f(m)\rvert < \varepsilon$ and no conclusion was stated, there was symbol manipulation only. .	From the description of Debra's proof attempt it can be inferred that Debra confused the concept of Cauchy sequences with the notion of uniform continuity. The condition $\lvert a_n - a_m\rvert < \varepsilon$ for natural numbers $n, m > M$, where M is a positive integer was drawn from Cauchy sequences. The fact that no conclusion was drawn might suggest that that the student had no contact with underlying ideas in symbols she manipulated. In other words, Debra did not explain how the answer obtained showed that $f(x) = x^2 + 2x - 5$ is uniformly continuous on $[0, 3]$. Debra's proof is typical of the external conviction symbolic proof scheme (CadawalladerOlsker, 2011; Harel & Sowder, 1998)
Prove that $f(x) = x^2 + 2x - 5$ *is uniformly continuous on* $[0, 3]$.	• Follow up interview Researcher: […] describe uniform continuity. […] What is your definition of uniform continuity? Debra: Ehee, given $\varepsilon > 0$ Researcher: You can illustrate if you want Debra: [silently writing] Researcher: Talk Debra Debra: OK, let ε>0 they exist $\delta(\varepsilon)$ ok [silent apparently stuck] Researcher: Now you have forgotten? Debra: [laughs] Researcher: If I may take you back to what you wrote here. You say if I pick n and m that are Debra: Ehee, domain function	At first Debra struggled to state the definition of uniform continuity, was silent and apparently stuck during the follow up interview on the task. Debra agreed that she had forgotten ideas on uniform continuity and in an attempt to "just write something," she ended up mixing ideas on Cauchy sequences with those on uniform continuity. She attributed her difficulty with task to lack of practice that would allow one to differentiate concepts so that one can properly apply the	Moments of being stuck and silent confirm the inference made under the written response that the student actually confused concepts of uniform continuity and ideas on Cauchy sequences and therefore, statements such as "so I was just trying to write down something," were uttered. She attributed difficulties she faced to lack of practice. She argued that enough practice would allow one to distinguish and classify problems. Hence, Debra's problem solving abilities were weak (Fukawa-

	Researcher: Ehee and then n and m that greater than m which is a natural number. What were you trying to do here? Debra: Actually I had forgotten the concept on uniform continuity so I was just trying to write down something. So I think I was mixing concepts. Researcher: Yaa, because I thought here you were no thinking about Cauchy sequences […] yet you are dealing with functions […] what do you think is the cause of this where one ends up mixing concepts like you were mixing Cauchy sequences and uniform continuity? Debra: Yaa, I think it's lack of practice so that someone will be able to differentiate the concept so that you can properly know where to apply this and where to apply this other concept. Researcher: […] you said when you wrote the tasks you had forgotten but how do you prove now? Do you have some idea? Debra: Ok. You can prove by the magnitude of $n - m$ is less than $\delta(\varepsilon)$ Researcher:You can write that [referring to what the student teacher is narrating] Debra: The magnitude of m, may be $n - m < \delta(\varepsilon)$ where n and m are the elements of the domain function then at n minus f at m less than ε. Researcher: Where are you picking n and m from? Where are you getting them from? Debra: From the domain function Researcher: Ok. From the set. And then? Then we substitute where we have f at n and f at m. We will be given a function. So in this case we are given the function $f(x) = x^2 + 2x - 5$. So we substitute for x by n then we substitute x by m then we find the difference of the function [referring to $\|f(n) - f(m)\|$ Researcher: Alright. […]. And then this $\delta(\varepsilon)$ that would have been found, does it work for every element that is in this domain? Debra: Yaa, I think so. Researcher: Why do you say so? Debra: [Laughs] Because the difference must be small. Researcher: Is that the reason? Debra [Laughs, stuck]	concept. When asked to describe her current [at the time of interview] understanding of uniform continuity, Debra could state correctly the conditions $\|n - m\| < \delta(\varepsilon)$ and $\|f(n) - f(m)\| < \varepsilon$ but her description of where the elements n and m are picked from was not convincing. She referred to the set as "the domain function" instead of mentioning the specific interval [0,3]	Conelly, 2012; Mamona-Downs & Downs, 2013).

Use the definition of appropriate limit to prove that: $\lim_{x\to 1}\left(\frac{x^3}{x-1}-\frac{1}{x-1}\right)=3$	• Written response Let $\varepsilon>0$ be given it is required to determine $\delta(\varepsilon)>0$ such that $0<x-x_0<\delta(\varepsilon)$ implies $\lvert f(x)-L\rvert<\varepsilon$ then $0<x-1<\delta(\varepsilon)$ implies $\left\lvert\frac{x^3}{x-1}-\frac{1}{x-1}-3\right\rvert<\varepsilon$ ………………………………………… $\lvert x^2+x-2\rvert<\varepsilon$ $x^2+x-2\equiv(x-1)^2+b(x-1)$ …………………………………… $\Rightarrow b=3$ $\lvert(x-1)^2+3(x-1)\rvert<\varepsilon$ ………………………………………… ……Set $\delta(\varepsilon)=1\Rightarrow\cdots$. $\lvert(x-1)\rvert+3\lvert x-1\rvert<\varepsilon$ ………………………………………… $\lvert(x-1)\rvert<\frac{\varepsilon}{4}$ Set $\delta(\varepsilon)=\frac{\varepsilon}{4}$	Debra showed an awareness for the need to fix $\varepsilon>0$ in order to determine the size of $\delta(\varepsilon)>0$. However the condition $0<x-x_0<\delta(\varepsilon)$ that is true for a right hand limit of a function was written instead of $0<\lvert x-x_o\rvert<\delta(\varepsilon)$. Algebraic manipulations were accurate including successful factorisation of $\frac{x^3-1}{x-1}$ to get x^2+x+1. Debra realised the need to express x^2+x-2 in terms of $(x-1)$. A constant b was introduced to accomplish this goal. The constant was determined by comparing coefficients, one of the methods of solving identities. However a minor mistake was made by Debra when she wrote: Set $\delta(\varepsilon)=1$ instead of $\delta(\varepsilon)\le 1$. The condition set $\delta(\varepsilon)=1$ not utilized in stipulating the value of $\delta(\varepsilon)$ that was just stated as " Set $\delta(\varepsilon)=\frac{\varepsilon}{4}$" rather than $\delta(\varepsilon)=\min\left\{1,\ \frac{\varepsilon}{4}\right\}$. Conclusion was not stated.	The description of student's proof attempt reveals that Debra used the definition of a special limit, the right hand limit to represent the general case. Student demonstrated a good command of the formal rhetoric aspect of the proof (Selden & Selden, 2009). This was shown by successful algebraic and technical manipulations such as being able to solve the identity involving the constant introduced constant b. The value of $\delta(\varepsilon)>0$ determined by the student was not used to draw a conclusion. In other words, the student did not show an awareness of how the solutions obtained served as solutions to the proof task (Koichu, 2012).

Table 5.15: Mid-instruction assessment data matrix for Tina on Real Analysis proof tasks

Task	Student's response (written, oral, actions)	Profiles of students' proving	Proof scheme elements present
Determine whether the following statement is true or false. For all real numbers a and $b, a-b>0\Rightarrow a^2-b^2>0$.	• Written response We assume that $a-b>0$ does not imply $a^2-b^2>0$ If $a,b\in\mathbb{R}$ and $b<a$ and b is a positive real number -Since $a,b\in\mathbb{R}$ it means that $a-b\in\mathbb{R}$ that is, it is closed under the above operation. So $a-b\in\mathbb{R}$ where b is positive. $a^2-b^2>0$ is the same as $b^2<a^2$ by considering $b<a$, and taking b to be a negative integer, $b<a$ still holds and we know that all negative numbers are less than a, so is $-b$ Taking $b=-4$ and $a=2$, we have $-4<2$ and by squaring both sides of the inequality sign we have $(-4)^2<(2)^2$ which implies that $16<4$ which is not true so $(-4)^2<(2)^2$ $\rightarrow b^2>a^2$ so is not the same as $b^2>a^2$. $a-b>0$	Tina kicked off by assuming that the statement is false. An incomplete statement was then written: "If $a,b\in\mathbb{R}$ and $b<a$ and b is a positive real number". Closure property under addition was stated for $a,b\in\mathbb{R}$ but no evidence was adduced for insisting that $b\in\mathbb{R}^+$. Tina claimed that $b<a$ and if b$\in\mathbb{Z}^-$, $b<a$ still holds. Tina further claimed that all negative numbers are less than a. This claim would have been true if Tina had assumed earlier that $a\in\mathbb{R}^+$. These efforts to build an argument to	Tina's proof effort indicates that he intended to accomplish the proof by contradiction. The mode of argumentation that followed was in terms of arbitrary real numbers drawn from $\mathbb{R}$, unjustified claims were a feature of mode of reasoning, e.g., "So $a-b\in\mathbb{R}$ where b is positive." Hence, the student could not access relevant conceptual insights. From the opening remarks, one would have thought that the goal of the prover was to establish a contradiction because Tina started by negating the consequent statement. Lack of consistency in the argument might account for the impasse reached. Tina

	does not imply that $a^2 - b^2$, so it is false.	validate the proposition in terms of arbitrary elements a and b were then abandoned and Tina switched to use of the specific example, $b = -4$ and $a = 2$ that was then used to refute the assertion that $a - b > 0 \Rightarrow a^2 - b^2 > 0$. $\forall\, a, b \in \mathbb{R}$	then switched to use of a counter example and then asserted that the proposition is false. Hence, the impasses experienced and ontological oscillations from deductive to empirical-numeric proof schemes show Tina's discomfort with this proof task (Harel & Sowder, 1998, 2007; Varghese, 2009).
Determine whether the following statement is true or false. For all real numbers a and b, $a - b > 0 \Rightarrow a^2 - b^2 > 0$.	• Chalkboard demonstration *$a - b$ implies a^2 -$b^2 > 0$.* {$a - b > 0$ also} *$a - b > o$ implies a^2 -$b^2 > 0$.* {So we are supposed to state whether it is true or false that *$a - b > o$ implies a^2 -$b^2 > 0$*. Now we assume that , eee, $a - b > 0$ does not, does not imply a^2 -$b^2 > 0$.}[Student writes] *We assume that $a - b > o$ does not imply a^2 -$b^2 > 0$.*{If a and b are real numbers and we know that $-b > 0$, this means that, aaa, $b < a$ since the subtraction of a smaller number from a bigger number gives a number which is greater than 0}*-if $a, b \in \mathbb{R}$, and* $a - b > 0$ {Now, aaa, moving on with our statement we can see that if $a - b > 0$ then $a - b$ should be a real number since real numbers are closed under the operation of subtraction}[Student teacher writes] *If* $a - b > 0$ then $a - b \in \mathbb{R}$ {We want to consider a^2 -$b^2 > 0$. This means that $b^2 < a^2$ since we are subtracting a smaller number from a bigger number, we get a number which is greater than 0} a^2 -$b^2 > 0$ $b^2 < a^2$ {Now from this statement, we can get back to our first statement (slowly) which was $b < a$}[Student writes] *we know that* $b < a${ $b < a$ which means, any number which is less than a which could be a negative as well as a positive depending on, aaa, which is chosen then} [pointing to $b < a$ written on the chalkboard]. {Now take for example a number which is less than a, considering that a is positive and b can be positive or negative}[student writes] *Considering that a is positive and b can be positive or negative* {Now we have $b < a$, lets take for example $-4 < 2$. Lets say 2 representing a}	Similar to the argument in the written response section, Tina started by assuming that the statement is false. The closure property of elements of $\mathbb{R}$ under addition was also stated and the student claimed that $a - b \in \mathbb{R}$ by the closure property stated above. As was the case with the written effort, attempts to build deductive arguments were shelved and student used specific examples to refute the proposition.	The chalkboard demonstration and the written response were similar in many respects and consequently inferences drawn about the written response also apply here. However, while an appropriate counter was used to refute the claim, it's doubtful if Tina had a strong command of the method of proof by counter argumentation. If he had a good command of counter argumentation it was therefore needless for him to repeat the part where he tried to use formal deductive reasoning once he had generated a counter example. Hence, Tina had relative conviction in the argument he had produced (Weber & Mejia-Ramos, 2015).

	$b < a$ $-4 < 2$ {That is this statement is true that $-4 < 2$ } [Student verbalizes and writes] *If we square both sides of the inequality sign we get* $b^2 < a^2$ {But if we substitute our numbers there [referring to $b^2 < a^2$], we get something like } $16 < 4$ {which is not true, aaa, that b [apparently stuck], which is not true that a^2 -$b^2 > 0$. Because if we take these numbers [pointing to $16 < 4$], aaa, that -4 and 2 we substitute then [pointing to a^2 -$b^2 > 0$] The number we get is greater than 0 but because $a < b$, its like if you want to get a positive you have to interchange this statement [pointing to $a < b$]}{Which means that a is now less than b but initially it was greater than b, So we can say therefore} [student writes] $\therefore$ $a - b > 0$ *does not imply* a^2 -$b^2 > 0$ *and it is false* [student faces class and says] {I think , aaa, and its all I can do}		
For all real values of $x. f(x) \equiv 2x^2 + 7x - 4$, *implies that* $f(x) > 0$.	• Written task $f(x) = 2x^2 + 7x - 4$ $f(0) = -4$ $f(1) = 5$ $f(2) = 18$ $f(-1) = -9$ $f(-2) = -10$ $f(-3) = -7$ This implies that $f(x) < 0$ for values of $x < 0$ So is ≥ 0 for $x > 0$ values	Specific instantiations: $f(1), f(2), f(-1), f(-2)$ and $f(-3)$ were used to establish that for some values, $f(x) \geq 0$ while for other values $f(x) < 0$. However the conclusion was not well articulated. It does reveal clearly whether the proposition was refuted or turned a mathematical fact by the student	Instantiations used by Tina revealed that there are some values of x for which the function $f(x) < 0$. This observation should have made the student realise that the statement is false. Tina presented many counter examples. The moment she got "$f(-1) = -9$." She should have stopped generating the examples the moment she got a single specific example that would give $f(x) < 0$. It can be inferred that Tina had a weak command of the notion of counter- argumentation. Tina did not establish essence behind the proving effort (Sandefur et al., 2013).
Determine whether the statement is true or false. Justify your answer. *For all real values of* $x. f(x)$	• Follow up interview Researcher: […] So what is your overall conclusion about the statement. Tina: Haaa Researcher: It is greater than 0 for some values and less than 0 for other values. So what is your overall conclusion? Tina: Aaaa. The overall conclusion there might be, there might be just from –∞ to +∞ because if are check for values which are less than 0, aaa!	The focus on the interview was on understanding from the student whether his efforts had led to the rejection of the statement or not. When the researcher sought clarification on the conclusion Tina's response showed he engaged with the task without understanding what the task required him to do. For example statements	From the description of Tina's proof attempt it can be noted that Tina engaged with the task without interpreting the question correctly, that is, he did not get a sense . Responses which showed that the student was clueless about what the question demanded included: *Aaaa. The overall conclusion there might be, there might be just from –∞ to +∞ because if are check for values which are less than 0, aaa.* Even after

$\equiv$ $2x^2 + 7x - 4$, *implies that* $f(x) > 0$	Researcher: You said for some values it is greater than 0 and for others it is less than 0. This is what you wrote here. Tina: Yes. So this conclusion is based on what I have, what I had worked above. I could see the trend of the numbers. Researcher: And in your working here, you established that it is greater than 0 for a some values of x and less than 0 for other values ? Tina: Ok. Actually I was referring to the output there. For all values that are greater than 0, I took them as the domain, those numbers which I could substitute in but the output varied, they range from minus…,minus Researcher: For instance here [referring student's proof attempt] minus when they are negative, and then you have positive, positive there positive. Tina: Yes. So I could conclude if I substitute negative numbers you could get negative numbers also up to 0, greater than 0. But for numbers greater than 0, actually they were positive as outputs. Researcher: I get you but the statement had said: Prove that if $f(x) = 2x^2 + 7x$ then this implies $f(x) > 0$ for all x. What is your overall conclusion after doing this? Tina: [Coughs] Researcher: Are you accepting this statement or you're saying it's wrong? Tina: So it's a wrong statement because some of the real numbers are negative but we are not getting numbers greater than 0.	like: "The overall conclusion there might be from $-\infty$ to ∞. Efforts to redirect the interviewee elicited responses such as "I was referring to the output there…the output varied, they range from minus…minus." It took the researcher huge efforts in probing Tina to arrive at the conclusion that the statement is false but still he could not provide convincing evidence to justify conclusion reached.	probing student's responses such as: *So it's a wrong statement because some of the real numbers are negative but we are not getting numbers greater than* 0, reveal that Tina did not understand the demands of the proof task as he could not discern that the focus of the task was not on negative real numbers only but rather on finding single real number x for which $f(x) < 0$. Overall, it can be inferred that Tina engaged in formal deductive reasoning and instantiation without having a clear focal goal being pursued (Fukawa-Conelly, 2012; Koichu, 2012).
Prove that the sequence defined by $(u_n) = \frac{n^2-1}{2n^2+3}$ *converges.*	• Written response $(u_n) = \frac{n^2-1}{2n^2+3}$ $\frac{1-\frac{1}{n^2}}{2+\frac{3}{n^2}}$ converges $\varepsilon < \frac{3}{n^2}$ and $\varepsilon < \frac{1}{n^2}$	The student found the correct value of the limit by dividing each term by the dominant term n^2. However the student did not show an awareness for the need to find a natural number dependent on ε for which the condition $\left\|u_n - \frac{1}{2}\right\| < \varepsilon$ is satisfied. The symbol ε was not explained, that is, what it is and the purpose it was supposed to serve in resolving the task. Once again Tina did not provide an explicit statement for	The student wrote expressions without providing meaning to those expressions (no essence of conceptual insight involved). For instance Tina wrote: "$\varepsilon < \frac{3}{n^2}$ and $\varepsilon < \frac{1}{n^2}$." The purpose of symbol ε not stated. It can thus be inferred that Tina thought of proof in terms of symbolic manipulations which is a typical case of the external conviction symbolic proof scheme as relevant technical handles and conceptual insights were not accessed (Harel & Rabin, 2010).

		the conclusion.	
	• Follow up interview Researcher: […] how does it lead to $\varepsilon < \frac{3}{n^2}$? Where is it coming from? How did you get to this stage here? What were you trying to find? Tina: To tell you the truth, I knew nothing, […] but I know when you are testing for convergence it has to do with ε greater than something actually I had not grasped the formula, it's like that $\lvert a_n - L\rvert$ should be less than ε. That's what I was trying to apply but actually I had not grasped the [inaudible]. So what I wrote here was just a matter of writing, I did not know. Actually I was writing for the sake of answering the question	When asked to explain how he got to the stage where he had an expression for n in terms of ε and what is he was trying to find, Tina's explanation revealed he had limited knowledge about the convergence of a sequence. He stated that he knew nothing except that it involved "with ε greater than something …"	Responses such as these show that Tina was not aware of the underlying ideas of the convergence of a sequence of real numbers but "was actually writing for the sake answering the question", presumptively referring here to the idea just putting down something with having contact with its meaning, some sort of "ritual" undertaking so that one is seen to have done something. This interpretation is further supported by the idea that Tina could not even explain what the quantity $\varepsilon > 0$ meant and its significance or purpose.
Determine whether the statement is true or false. If x is an integer, then $x^2 - x$ is an integer. Justify your answer.	• Written response Suppose $x \in \mathbb{Z}$ and $x^2 \in \mathbb{Z}$ and considering that x is an odd number, -we want to take a, b and c as odd numbers a^2, b^2 and c^2 are still odd numbers because if we square an odd number we still get an odd number. Considering $a^2 - a = \alpha$ where α is divisible by 2, so $a(a-1) = \alpha$ $a = \frac{\alpha}{a-1}$ # Since we introduced α as divisible by 2 it means it is even # Since we know that if 1 is subtracted from an odd number we get an even number, it means $a-1$ is even. An even number can be written as $x = 2\alpha m$, where $x - 1 = 2\alpha$ which is even and every even number is divisible by 2	Tina considered cases when a, b, c are odd numbers and then asserted that their squares (a^2, b^2, c^2) are also be odd numbers. Tina then considered the expression $a^2 - a = \alpha$ where $\alpha = 2k, k \in \mathbb{Z}$ as specified by Tina. Tina indicated that the number α is even and $a - 1$ is also even. Hence according to Tina's argument $a\ (odd) = \frac{\alpha}{a-1} = \frac{even}{even}$.	Tina 's argument is flawed because it excludes other odd numbers that can be obtained by dividing numbers that are not even. An appropriate counter- argumentation could be $7 = \frac{21}{3} = \frac{\text{odd}}{\text{odd}}$ which contradicts Tina's argument that $odd = \frac{even}{even}$. It can be also be noted that Tina did not connect the premises with consequent statement. Therefore Tina's effort to use deductive argumentation was to no avail. Tina did not reflect on symbol manipulations done (Koichu, 2012; Stylianides, 2011).
Determine whether the statement is true or false. If x is an integer, then $x^2 - x$ is an integer. Justify your answer.	• Follow up interview Researcher: How did you show that a^2 and $b^2$are odd? Tina: Ok, ok, it's like aaaa, iii, I missed a certain statement there I could use the actual numbers like you take 3 for example, 3^2 you get a 9 which is odd. If you square 7 then you get 49	When the interviewee was asked to justify that the square of odd numbers is also an odd number, Tina picked specific examples 3 and 7 and used to show that the square of an odd number is also odd.	Tina switched from use of arbitrary real numbers to particular instantiations in order to show that the square that of an odd number is also odd. The switch from a higher level proof scheme to lower level scheme revealed that Tina faced difficulties with axiomatic reasoning and had to find comfort in use of specific examples. Hence, ontological oscillations were forced by impasse reached with axiomatic reasoning (Harel & Sowder, 1998, 2007; CadawalladerOlsker, 2011).

Table 5.16: Tina's End-of-instruction assessment data matrix on Real Analysis proof tasks

Task	Student's response (written, oral, actions)	Profiles of students' proving	Proof scheme elements present
A sequence (a_n) of real numbers is defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. Prove that (a_n) converges and find its limit.	• Written response It is a monotonic sequence which is increasing since $a_{n+1} - a_n = \sqrt{2 + a_n} - a_n =$ $2 + 2a_n\sqrt{2 + a_n} - (a_n)^2$ $a_1 = \sqrt{2}$ $a_2 = 2$ $a_3 = 2$ and $a_2 = \sqrt{2 + \sqrt{2}} = 2$ $a_3 = \sqrt{2 + a_2} = \sqrt{4\sqrt{2}} = 2$ $a_n = \sqrt{2 + a_{n-1}} = 2$ the sequence converges to 2 and the limit is 2 Alternatively $a_n - L < \varepsilon$ $\sqrt{2 + a_{n-1}} - L < \varepsilon$ $\sqrt{2 + a_{n-1}} < L + \varepsilon$ $2 + a_{n-1} < (L + \varepsilon)^2$ …………………………………… $L^2 + 2\varepsilon L = 2 + a_{n-1} - \varepsilon^2$ $\left(L + \frac{2\varepsilon}{2}\right)^2 = a_{n-1} + 2 - \varepsilon^2$ …………………………………… ……….. $L + \varepsilon = (a_{n-1} + 2 - \varepsilon^2)^{\frac{1}{2}}$ Using the binomial expansion $L = -\varepsilon + \left[\frac{1}{2}a_{n-1} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)(2-\varepsilon^2)}{2!} + \cdots\right] \quad -\varepsilon\left[\frac{1}{2}a_{n-1} + -\frac{1}{8}(2 - \varepsilon^2) + \cdots \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)(1-2)a_{n-1}}{3!} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)a_{n-1}}{4!}\right]$	The student first sought to establish that (a_n) is a monotone increasing sequence. The argument presented to support the claim consisted of a mix of particular instantiations and symbolic manipulations. For the particular instantiations Tina used $a_1 = \sqrt{2}$, $a_2 = 2$, $a_3 = 2$. False statements were a feature of this mix. For instance Tina wrote $a_2 = \sqrt{2 + \sqrt{2}} = 2$ and $a_n = \sqrt{2 + a_{n-1}} = 2$. On the basis of these incorrect empirical evaluations Tina concluded that the sequence (a_n) converges to 2. Another awkward symbolic formulation was presented as an alternative proof. From nowhere Tina stated that $a_n - L < \varepsilon$ $\sqrt{2 + a_{n-1}} - L < \varepsilon$ which led $2 + a_{n-1} < (L + \varepsilon)^2$. Tina made $L + \varepsilon$ the subject of the formula and claims to apply the Binomial Theorem were then made in an effort to determine L the limit of the sequence. The student teacher then reached an impasse. No conclusion was arrived at regarding whether the sequence (a_n) converges.	Tina had a weak command of the proof framework (Selden & Selden, 2009). The logical structure of a proof should be such that the premises should logically entail the conclusion. Rather, Tina started by asserting that the sequence is monotone increasing. However, according to his empirical evaluations "$a_2 = 2, a_3 = 2, \ldots\ldots a_n = 2.$" That is the second term in Tina's attempt was inaccurate. The particular instantiations revealed that the sequence (a_n) consisted of a constant term, which is a false assertion. An ontological oscillation was noticed here when the student switched to use of algebraic manipulations that involved somewhat complicated and lengthy expressions such as: "$L - \varepsilon\left[\frac{1}{2}a_{n-1}\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)(2-\varepsilon^2)}{2!} + \ldots\right] - \varepsilon\left[\frac{1}{2}a_{n-1} - \frac{1}{8}(2 - \varepsilon^2) + \cdots \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)(1-2)a_{n-1}}{3!} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)a_{n-1}}{4!}\right]$." The expression were overwhelming for the student and he could not find the limit as has been articulated. Once again the external conviction symbolic proof scheme was robust (CadawalladerOlsker, 2011).

A sequence (a_n) of real numbers is defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. Prove that (a_n) converges and find its limit.	• Chalkboard demonstration {A sequence (a_n) of real numbers is defined by} [Student writes] $a_1 = \sqrt{2}$ $a_{n+1} = \sqrt{2 + a_n}$ {Now we want to prove that (a_n) converges} [Students verbalizes and writes] a_n *converges and find its limit* {Now, aaa, since it is a sequence, we know that whenever we are given a sequence we know that it must be a monotone sequence but we don't know whether it is an increasing monotone or decreasing. So what we have to do to prove that is to subtract the subsequent from the first one. So we are saying} [student writes] $a_{n+1} - a_n = \sqrt{2 + a_n} - a_n$ {Our a_{n+1} is root 2 plus a_n. Then on the RHS we have to remove the square root sign. How do we remove it? We have to square this side [RHS], so that we get rid of the square root. So its} [student then writes] $a_{n+1} - a_n = (\sqrt{2 + a_n} - a_n)^2$ {If we square this side [RHS], we end up having}$a_{n+1} - a_n = 2 + a_n - 2a_n\sqrt{2 + a_n} + (a_n)^2$ {Now, aaa, we have to group the like terms such that if we take that $a_n{}^2$ as our first term we get} [student writes] $a_{n+1} - a_n = (a_n)^2 + a_n + 2 - 2a_n\sqrt{2 + a_n}$ {If we take a_2, we have something like} [student writes] $a_2 = \sqrt{2 + \sqrt{2}}$ {We can substitute here [$a_{n+1} - a_n$], we can take n for 1 then we have something like} $a_2 - a_1${Which must be equal to [silent]..} [student writes] $a_2 - a_1 = \sqrt{2 + \sqrt{2}} - \sqrt{2}$ {Since this [$\sqrt{2 + \sqrt{2}} - \sqrt{2}$], gives a positive number, it means that it converges, it must have a limit. So given} [student writes] $\varepsilon > 0 \; \exists$ {a natural number which must be greater than N} $n > N$ s.t. $a_n - L < \varepsilon$ {Now, for the value of a_n, we have } [student writes] $\frac{2x_{n-1} + 3}{4} - L < \varepsilon$ $\frac{2x_{n-1} + 3}{4} < L + \varepsilon$ {Then we are left with} $\frac{2x_{n-1} + 3}{4} < L + \varepsilon$ {Aaa, if we simplify that expression, we end up having } $2x$ {Aaa, actually, Iam mixing up. I am on number one instead of ..[student looks confused and embarrassed by the mix up.	Student started by stating his goal: to establish that (a_n) converges and then determine its limit. A false claim was made that any sequence given must be a monotone sequence. The student teacher then wrote: $a_{n+1} - a_n = \sqrt{2 + a_n} - a_n$. Tina then squared the right hand side and then wrote $a_{n+1} - a_n = 2 + a_n - 2a_n\sqrt{2 + a_n} + (a_n)^2$. Student then used specific examples. He started by noting that $a_2 = \sqrt{2 + \sqrt{2}}$. Student described that for $n = 1$, one would have $a_2 - a_1 = \sqrt{2 + \sqrt{2}} - \sqrt{2}$. Student then argued that since $a_2 - a_1 > 0$, it means the sequence converges. Student did not provide a justification for the claim that $a_2 - a_1 > 0$. These particular instantiations were abandoned and the student switched to the formal definition which was not correctly started because the student had $a_n - L < \varepsilon$ in the definition instead of $\lvert a_n - L \rvert < \varepsilon$. An awkward substitution $\frac{2x_{n-1} + 3}{4} - L < \varepsilon$ was made, yet the sequence involved terms in (a_n). This mix up was confirmed by the student who even felt embarrassed by this.	Student had a weak command of the hierarchical order (Selden & Selden, 2009). He was supposed to prove that the sequence was monotone increasing. Rather, he made a false claim that "we know that whenever we are given a sequence we know that it must be a monotone sequence but we don't whether it is an increasing monotone or decreasing." A sequence need not necessarily be monotone sequence. A single particular instantiation "$a_2 - a_1 > 0$", led to the conclusion that the sequence (a_n) converges. It can be noted that Tina's argument is flawed because he had not demonstrated that the sequence is a bounded monotone sequence. It can be deduced that the structural relationships of underlying ideas were not accessed (Hanna & Mason, 2014), referring here to the monotone convergence criterion. Student then shelved empirical explorations and then resorted to symbolic manipulations of the expression: "$a_{n+1} - a_n = (\sqrt{2 + a_n} - a_n)^2$". Symbolic manipulations were flawed, for instance Tina squared the right hand side only and the difficulties faced with this proof task were revealed more explicitly when he mixed up notation and wrote: "$\frac{2x_{n-1} + 3}{4} - L < \varepsilon$." He ultimately gave up the proof attempt. Therefore Tina's proof attempt revealed that he engaged with the task without clear goals about what he sought to accomplish Weak command of the proof framework was exhibited through flawed algebraic manipulations. Tina's proof attempt revealed that proving is conceived in terms of handling symbols but student lacked a grasp of exact goals he pursued. Hence, he failed to get essence of how answers obtained resolve the proof task at hand (Koichu, 2012).

Define a sequence (x_n) *inductively by* $x_1 = 1$ *and* $x_{n+1} = \frac{2x_n+3}{4}$. *Prove that* (x_n) *is a bounded monotone sequence is bounded and hence determine its limit.*	• Written response …………………………………… …………………… $x_1 = 1$ $x_2 = \frac{2x_1+3}{4} = \frac{2(1)+3}{4} = \frac{5}{4} = 1\frac{1}{2}$ For a monotonic increasing $(x_{n+1} - x_n) > 0$ $\left(\frac{2x_n+3}{4} - x_n\right) > 0$ $\frac{2x_n+3-4x_n}{4} > 0$ $\frac{3-x_n}{4} > 0$ As $n \to \infty$ $\frac{3-x_n}{4} < 0$, which means there is a contradiction, so x_n is bounded Given $\varepsilon > 0$, there exist a number which is a natural number n such that $n > N$, therefore $x_n - L < \varepsilon$ We know that $x_{n+1} = \frac{2x_n+3}{4}$, then $x_n = \frac{4x_{n+1}-3}{4}$ $x_1 = 1$ $x_2 = 1/2$ As $n \to \infty$ $x_n \to 0$, so 0 is the limit of x_n	A single example was used to infer that (x_n) is a monotone sequence. Tina compared x_1 and x_2 and concluded that on the basis of a single instantiation that $x_{n+1} > x_n$. The single empirical evaluation was wrong, x_2 was given as $1\frac{1}{2}$ instead of $\frac{5}{4}$. The definition of a monotone increasing sequence was applied correctly to the stage where the student got $\frac{3-x_n}{4} > 0$. What followed after getting to this stage was a complete *mess*. Student claimed that as $n \to \infty$, $\frac{3-x_n}{4} < 0$, and claimed to have established a contradiction which led to the conclusion that (x_n) is bounded. An awkward switch to the formal definition was done. The definition was not properly stated. Student wrote that there exists a natural number n, such that $n > N$ [...]. Tina's claim revealed that he had not grasped tha n is implied in the definition of a sequence and rather, we should strive to find a natural number N depending on $\varepsilon > 0$. The expression $x_n - L < \varepsilon$ should have been stated as $\lvert x_n - L \rvert < \varepsilon$ because it is possible to get terms of (x_n) less than L and such terms would make $\varepsilon < 0$ which is a senseless statement. Statements not linked to the definition of convergence of a sequence were written, e.g., $x_{n+1} = \frac{2x_n+3}{4}$ and x_n was made the subject of the formula and student got $x_n = \frac{4x_{n+1}-3}{4}$, a wrong expression. Specific examples were then written: $x_1 =$	A weak grasp of the notion of a counter example was shown because Tina used to conclude that the sequence is monotone is increasing that was not even accurate: "$x_2 = 1\frac{1}{2}$", instead of $\frac{5}{4}$. Tina's proof attempt can be described as chaotic. Although, he was able to apply the definition of a monotone increasing sequence the working degenerated into "mess" when he evaluated the limit $\frac{3-x_n}{4}$ as $n \to \infty$. As before the underlying idea not accessed. Tina claimed that to have established a contradiction, which is a false assertion. An ontological oscillation was noticed when the student then switched to use of the formal definition of convergence of a sequence. The fact that this definition was not correctly stated revealed that Tina had a weak command of the concepts he was using. For instance Tina claimed that "there exists a natural number n such that…" Rather, the focus should have been on determination of $N(\varepsilon)$. Further, the fact Tina's proof effort was loaded with many senseless statements, e.g., "$x_n = \frac{4x_{n+1}-3}{4}$" and false claims that a contradiction had been established pointed to serious challenges Tina faced in his proof effort. Tina had no grasp of fundamental ideas pertinent to proof and ended up mixing ideas as he switched from one approach to another. He could not draw from the formal structure of definition of convergence of a sequence (Duffin & Simpson, 2000; Lay, 2009)

		1 (given), $x_2 = \frac{1}{2}$, a wrong value different from the one stated earlier for the same question.	
Define a sequence (x_n) inductively by $x_1 = 1$ and $x_{n+1} = \frac{2x_n+3}{4}$. Prove that (x_n) is a bounded monotone sequence is bounded and hence determine its limit.	• Follow up interview Researcher: […] you wrote here as $n \to \infty$, $\frac{3x_n}{4} = 0$ *which is a contradiction*. I want to understand how you arrived at such a conclusion that this [referring to conclusion] is a contradiction. Tina: Umm, actually it was a matter of subtracting, eee, a number which is, which is before a certain number, like numbers which are arranged in ascending order. The number which is in front of the other then you subtract the previous number to check if it won't give a negative or a positive. So in this case I had to take x_{n+1} which is a higher number then I had to subtract x_n and the result was supposed to be greater than 0 if x_{n+1} is greater than x_n . Researcher: […] you collect like terms here [...] you have $2x_n - 4x_n$ Tina:Yes then we get a negative $2x_n$ Researcher: Yaa, […] It was supposed to negative $2x$ Tina: Ok, ok [agreeing] Researcher: But how did then reverse this sign [ ] Tina: Aaa, actually I was considering a sequence (x_n). Aaaa, if you subtract it from 3 then actually as $x_n \to \infty$ it, it gets larger so you will be subtracting a bigger number from 3 there. So it will end up being less than 0 as n approaches . Researcher: Oh, yaa, you are now subtracting a very huge number. Tina: Yes. As $x_n \to \infty$, that number there you will be subtracting something like a thousand from 3.	When the researcher sought clarification on how the contradiction was arrived at, the student stated that the contradiction was established by subtracting consecutive numbers which were arranged in ascending order to determine whether a positive or negative value could be obtained.	Probing revealed Tina's confusion as he failed to justify how as $n \to \infty$, $\frac{3x_n}{4} = 0$ pointed to a contradiction. Tina tried to explain this as "Umm, actually it was a matter of subtracting, eee, a number which is, which is before a certain number, like numbers which are arranged in ascending order. The number which is in front of the other then you subtract the previous number to check if it won't give a negative or a positive. So in this case I had to take x_{n+1} which is a higher number then I had to subtract x_n and the result was supposed to be greater than 0 if x_{n+1} is greater than x_n." This explanation is not relevant to the question posed to Tina, which had a focus on how a contradiction had been established. It can be, therefore, concluded from Tina's proof behaviour that he had no essence of objects manipulated (Hanna & Mason, 2014; Sandefur et al., 2013).
Prove that $f(x) = x^2 + 2x - 5$ *is uniformly continuous on* $[0, 3]$.	• Written responses $f(x) = x^2 + 2x - 5$ $f' = 2x + 2$	Tina differentiated the function $f(x)$ and drew the graph of $y = 2x + 2$ as an illustration of uniform continuity.	The decision to differentiate the function revealed that Tina had not developed the concept of uniform continuity. Further irrelevant instantiations were done, drawing the graph of the derivative of the function.
Prove that $f(x) = x^2 + 2x - 5$	• Follow up interview Researcher: […] When do we say a function f is continuous on a set?	Tina explained that a function is continuous on a set if a given range of	Tina's lack of contact with the concept of uniform continuity was also revealed during the

is uniformly continuous on $[0,3]$.	Tina: Aaa, it is continuous if, aaa, if a certain, let's say if a given range of range of numbers aaa, tend to increase within a certain range. Like I had to represent it graphically there [pointing to the solution]. Researcher: This was really going to be my next question to say you differentiated here, what made you differentiate the function. In other words, any relationship between differentiation and uniform continuity Tina: Aaaa, I just imagined that when we are saying uniform continuity, something that is uniform is just the same as gradient at any given point is just the same as we proceed Researcher: So that led you to differentiate Tina: To differentiate then prove that… [inaudible] Researcher: Why did you differentiate? Tina: Ok. I differentiated simply because I know gradient at any point is just the same, is uniform as we go along the line. So I just thought uniform continuity	values of the function tend to increase on the set. Student then referred to the graph of the linear function, $y = 2x + 2$. When probed about reasons that made him to differentiate $f(x)$ in order to prove that $f(x)$ is uniformly continuous, Tinashe explained that he thought that gradient at any point on a uniformly continuous function is just the same, "is uniform as we go along with the line."	reflective interviews about the task. First, he exhibited weak command of the concept of continuity by describing a continuous function as "if, aaa, if a certain, let's say if a given range of range of numbers aaa, tend to increase within a certain range." He then tried to present the same argument graphically. When pressed to justify why he had differentiated in order to establish that $f(x)$ was uniformly continuous Tina responded that it came from his imagination. In other words he had visualized uniform continuity as having the same gradient as reflected in his response: "Aaaa, I just imagined that when we are saying uniform continuity, something that is uniform is `1just the same as gradient at any given point is just the same as we proceed." His explanation revealed he had no idea of what uniform continuity involves as tried to solve the task.

Table 5.17: Tanya's Mid-instruction data matrix on Real Analysis proof tasks

Task	Student's response behaviour (written, oral, actions)	Profiles of students' proving	Proof scheme elements present
Describe whether the following statement is true or false. For all real numbers a and $b, a - b > 0 \Rightarrow a^2 - b^2 > 0$.	• Written response $a - b > 0 \Rightarrow a > b$ suppose $a = 5$ *and* $b = 3$ where a, b are integers then $a > b \Rightarrow 5 > 3$ *and* $5 - 3 > 0$ $2 > 0$ and $a^2 - b^2 > 0 \Rightarrow 5^2 - 3^2 > 0$ $25 - 9 > 0$ $16 > 0$ which is also true. $\therefore$ $a - b > 0 \Rightarrow a^2 - b^2 > 0$ is true	The substitution $a = 5$ and $b = 3$ was used to validate the proposition and Tanya concluded that on the basis of this single empirical verification that the statement is true.	Tanya's proof attempt revealed that she had weak grasp of the limitation of empirical verification and the notion of counter argumentation. She did not realize that the proposition may be false for just one single example she had not considered.
Determine whether the statement is true or false. If x is an integer, then $x^2 - x$ *is an integer. Justify your answer*	• Written response Proof by induction Let $x = 1$ then $1^2 - 1 = 0$ $\therefore$ the statement is false	A single empirical verification by the student led to the conclusion that the statement is false. No explanation was provided to justify why $x^2 - x$ is not an even number.	Similar to what has been noted above a single instantiation was used draw the conclusion that the statement is false. Tanya did not justify why 0 is not an even number. Tanya's proof behaviour points to the fact that she had a good grasp of proof method by refutation.
Determine	• Chalkboard	Tanya stated that she	Similar to the written response

whether the statement is true or false. If x is an integer, then x^2-x is an integer. Justify your answer	demonstrations [student writes] *Given x an integer then x^2 –x is even* {I am going to try to prove by induction} [Student writes] *Proof by induction* {So let x be equal to 1. Then we are saying } Let $x=1$, x^2 –x = $x(x-$1)=1(1-1) =0 {Then I conclude that since x^2 –x =0 which is not an even number, therefore } [Student writes] *Because x^2 –x =0 when x =1, then the statement is false*.	was going to prove the statement by induction. But what then followed after this declaration was a single numerical test where the value $x=1$ was evaluated into the expression x^2 –x. A justification for the claim that x^2 –x is not even was provided.	Tanya used a single specific example to evaluate the statement. While, Tanya had declared that she was going to prove by mathematical induction, she substituted 1 into the expression x^2 –x to get 0. Contrary to her claims that she was going to use induction the proof effort ended with this single instantiation. It can be noted that Tanya had not developed an understanding of the proof framework with respect to method of proof by induction because she confused it with use of examples (Selden & Selden, 2009). However, Tanya's proof attempt reveals that she had a good grasp of proof by use of a counter example.
Prove that the sequence defined by $(u_n)=\frac{n^2-1}{2n^2+3}$ converges	• Written response Let $\varepsilon>0$ be given s.t there exist an element $N\varepsilon>0$ $\|u_n-L\|<\varepsilon$ $\frac{n^2-1}{2n^2+3}-\frac{1}{2}<\varepsilon$ $\frac{2n^{2-}-2-2n^2-3}{2(2n^2+3)}\varepsilon$ $\left\|\frac{-5}{4n^2+6}\right\|<\varepsilon$ $\frac{5}{4n^2+6}<\varepsilon$ u_n converges	In her description of the formal definition of the convergence of a sequence, Tanya wrote "there exists an element $N\varepsilon>0$ without specifying the exact set from which is selected, which in particular case is the set of natural numbers. The expression for $\|u_n-L\|<\varepsilon$ was stated initially without the modulus symbol that was later on brought into the picture. Algebraic manipulations by the student led to $\frac{5}{4n^2+6}<\varepsilon$ from which Tanya drew the conclusion that (u_n) converges	It was not clear from Tanya's proof attempt that $N(\varepsilon)$ that the student wanted to determine is a natural number because there are many real numbers that are greater 0. Tanya was not consistent in her use of the modulus sign as shown by $\frac{n^2-1}{2n^2+3}-\frac{1}{2}<\varepsilon$. No evidence was adduced to support the claim that the expression $\frac{5}{4n^2+6}<\varepsilon$ shows that the sequence (u_n) converges. So Tanya handled symbols without justifying how the piece of knowledge constructed resolved the proof task (Koichu, 2009).
	• Follow up interview Researcher: […] What should be this $N(\varepsilon)$? Tanya: It should be $n>N(\varepsilon)$, it should be an element which is natural number. Researcher: Very good. So this whole thing is flawed, it's supposed to be […]. You illustrate. Tanya: Supposed to be n is greater (aah) such that there exists an element n, which is a natural number s.t. $n>N(\varepsilon)$. Researcher:[…] What were you trying to determine? Tanya: Determine if this natural	When the researcher queried the nature of $N(\varepsilon)$, Tanya explained that $N(\varepsilon)$ should be a natural number and described that her goal was to determine if $N(\varepsilon)$ existed. The researcher asked for clarification of how the conclusion was drawn. She responded that she wanted to find $N(\varepsilon)$. Tanya could not provide an explanation about how the conclusion was drawn and she kept silent	While, the description of Tanya's proof effort shows that she was aware that her goal was to find a natural number, $N(\varepsilon)$, her working did not show how the natural number could be determined from the expression: $\frac{5}{4n^2+6}<\varepsilon$. When pressed for a comment Tanya replied "We wanted to find $N(\varepsilon)>0$, that is greater." Tanya's utterance point to the fact the concept of convergence of a sequence was underdeveloped in Tanya's mind. It can therefore be inferred from Tanya's proof effort that

	number exists. Researcher: You wrote; $\frac{5}{4n^2+6} < \epsilon$, ∴ it converges. How did you conclude on the basis that u_n converges? Tanya: [silent] we wanted to find $N(\varepsilon)$ Researcher:What is the domain $N(\varepsilon)$. Where does it belong? Tanya: $N(\varepsilon)$. Natural numbers Researcher: Very good. It's a natural number. But on the basis of this [ ] how did you conclude that the natural number exists because you went to say therefore it converges. Tanya: [silent]. We wanted to find $N(\varepsilon) > 0$, that is greater. [became silent]	when pressed for an explanation.	she was dealing tasks for which she lacked profound grasp (Stylianides & Stylianides, 2009).
Decide whether the statement is true or false. For all real values of $x. f(x) \equiv 2x^2 + 7x - 4$, implies that $f(x) > 0$.	• Written response $f(x) \equiv 2x^2 + 7x - 4 \geq 0$ $(2x-1)(x+4) \geq 0$ either $2x - 1 \geq 0$ $x \geq \frac{1}{2}$ and $x + 4 \geq 0$ $x \geq -4$ [student draws number line and writes] Solution 1 $\Rightarrow x \geq \frac{1}{2}$ or $2x - 1 \leq 0 \Rightarrow x \leq \frac{1}{2}$ and $x + 4 \leq 0$ $x \leq -4$ [Student draws number line and writes] Solution 2 ⇒x≤ -4 Solution: $-\Rightarrow$ $f(x) \geq 0$ when $x \geq -4$ and when $x \geq \frac{1}{2}$ thus $f(x) \geq 0$ given $f(x) \equiv 2x^2 + 7x - 4$ is false	Tanya succeeded in factorizing $2x^2 + 7x - 4$. Order axioms: $ab > 0 \Rightarrow$ either $a < 0$ and $b < 0$ or $a > 0$ and $b > 0$, were applied to $(2x - 1)(x + 4) \geq 0$ to obtain $x \leq -4$ and $x \geq \frac{1}{2}$. Solutions were then illustrated on the number lines. The conclusion $f(x) \geq 0\ \forall\ x \in \mathbb{R}$ is false was drawn. However no justification for the conclusion was provided and hence, the focus of the follow up interview was on the basis upon which Tanya drew the conclusion that the statement is false	Tanya's proof attempt illustrates that she had a strong command of concepts pertinent to the task as was shown through correct application of order axioms and successful factorisation of the quadratic expression. These processes then led to the conclusion that the proposition is false. However, the conclusion lacked justification. The lack of clarity in the conclusion drawn became the focus of the follow up interview.
Decide whether the statement is true or false. For all real values of $x. f(x) \equiv 2x^2 + 7x - 4$, implies that $f(x) > 0$.	• Follow up interview Researcher: How does this solution [referring to lead to the conclusion that the statement is false? Tanya: Our statement was saying that for $f(x) > 0$ so if you look at our solution this [referring to number line illustration] empty part means our $f(x) > 0$. Researcher: So which method have you used here? Tanya:Counter-augmentation.	Concerning how the conclusion was drawn, Tanya used a graphical argument to explain that between -4 and $\frac{1}{2}$, $f(x) < 0$. She described this method of refuting the proposition as counter-argumentation.	A convincing explanation was given by Tanya that was aided by graphical instantiations in describing how counter examples could be selected from the interval $(-4, \frac{1}{2})$ and used to refute the proposition. Student had access to relevant conceptual insight. The proof behaviour showed by Tanya reveals that she had a strong grasp of the notion of a counter example.

Table 5. 18: End-of-instruction assessment data matrix for Tanya on Real Analysis proof tasks

Task	Student's response (written, oral, actions)	Profiles of students' proving	Proof scheme elements present
A sequence (a_n) of real numbers is defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$ Prove that (a_n) converges and find its limit.	$a_1 = \sqrt{2} = 1.414$ $a_2 = \sqrt{2 + 1.414} = 1.848$ $a_3 = \sqrt{2 + 1.848} = 1.962$ $a_1 < a_2 \Leftrightarrow a_2 > a_1$ and $a_2 < a_3$ Since $a_1 < a_2$ and $a_2 < a_3$ it holds for P_1 and P_2. Assume it holds for $n = k$ then it is true for P_k. We need to prove that it holds for $n = k + 1$ $a_{k+1} - a_k = \sqrt{2 + a_k} - \sqrt{2 + a_{k-1}}$ ………………… $= \frac{a_k - a_{k-1}}{\sqrt{2+a_k} + \sqrt{2+a_{k-1}}}$ $= \frac{1}{\sqrt{2+a_k} + \sqrt{2+a_{k-1}}}(a_k - a_{k-1})$ $\therefore (a_n)$ is a monotone increasing sequence then $a_{n+1} - a_n > 0$ $\sqrt{2 + a_n} - a_n > 0 \sqrt{2 + a_n} > a_n$ $2 + a_n > (a_n)^2$ $(a_n)^2 - a_n - 2 < 0 (a_n - 2)(a_n + 1) < 0$ $a_n - 2 < 0$ or $a_n + 1 < 0$ $a_n < 2$ or $a_n < -1$. Discard the value of (a_n) that is negative so $a_n < 2$. Thus (a_n) is a monotone increasing sequence that is bounded and converges to its least upper bound which is 2	Specific examples were used to determine the behaviour of the sequence. Student noted that $a_1 < a_2$ and $a_2 < a_3$ from the empirical evaluations. These numerical tests were used as base step cases for the Principle of Mathematical Induction. These examples enabled Tanya to infer that P_1 and P_2 hold. The induction hypothesis was just alluded to. The implication statement $P_k \Rightarrow P_{k+1}$ was proved and the student concluded that (a_n) is a monotone increasing sequence. Definition of monotone increasing sequence was then employed to prove that (a_n) is bounded. Thus, from $a_{n+1} - a_n > 0$, algebraic manipulations led to the factor form $(a_n - 2)(a_n + 1) < 0$. Order axioms were not correctly applied as student wrote: $a_n - 2 < 0$ or $a_n + 1 < 0$ instead of $a_n - 2 < 0$ and $a_n + 1 > 0$, or vice versa. Further, another wrong use of order properties is noted in use of word "or" to denote the intersection of two sets. However, Tanya still managed to state that $a_n < 2$ and then concluded that the sequence converges to 2, its least upper bound.	Tanya's proof attempt revealed that she had contact with ideas she was using during proving. First, the method of mathematical induction was correctly applied to prove that (a_n) is a monotone increasing sequence. Although the induction hypothesis was just alluded to, Tanya still managed to establish the implication statement $P_k \Rightarrow P_{k+1}$. It can be inferred that Tanya had a strong command of the hierarchical structure of the proof, that is, she was aware of what her proof efforts were meant to accomplish It can be deduced that Tanya had access to relevant key ideas (Raman, 2003). Second, she was able to apply the relation $a_{n+1} - a_n > 0$ for a monotone increasing sequence to prove that (a_n) of is bounded. From her formal rhetoric part , that is, behavioural knowledge related to proving, the following weakness was noticed. With respect to use of order axioms she used "or" instead of "and" in using the order property: $ab < 0 \Rightarrow a > 0$ and $b < 0$ or $a < 0$ and $b > 0$. Tanya rather wrote, $a_n - 2 < 0$ or $a_n + 1 < 0$ $a_n < 2$ or $a_n < -1$. Tanya's proof behaviour points to the eternity of the external conviction symbolic proof among student teacher informants (Harel & Sowder, 1998, 2007).

A sequence (a_n) of real numbers is defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$ Prove that (a_n) converges and find its limit.	• Follow up interview Researcher: You had $a_n - 2 < 0$, $a_n + 1 < 0$ [...] and if I take you back to the order properties, $ab < 0$ implies that, what? Tanya: $a > 0$ and $b < 0$ aaaa, sorry, $b > 0$ Researcher: […] you wrote here; discard the value a_n that is negative. […] Why did you write that statement? Tanya: [silent] Because we are mapping for sequences, we are mapping for real numbers to natural numbers. Researcher: We are mapping what? $\mathbb{R}$ to naturals. Are you saying so? Where is the domain of the set of sequences? Tanya: the vice versa, naturals to reals. Researcher: […]. So why are you discarding the negative? What led you to that conclusion? Tanya: […] Because I had found that the first term for the sequence was $\sqrt{2}$ which is greater than -1, there was no way that an can take any value that is below $\sqrt{2}$.	The focus of the follow up interview was to seek clarification from the student on some flaws noted in the written response data source. First, Tanya could not provide clarification on order properties of real numbers as she had $a < 0$ and $b < 0$ in her response to the question her $ab < 0$ implis what? Further, features of her response like "aaa, sorry $b > 0$" ponted to weak command of order properties by the student. Second, Tanya justified the decision to discard the solution $a_n < -1$ by explaining that the first term $a_1 = \sqrt{2}$. However, she had suggested earlier on from written response data source that the reason for disregarding -1 stems from the fact for sequnces we are mapping real numbers to natural numbers which is exactly the opposite.	From the follow up interview Tanya's command of order axioms was also revealed to be somewhat weak as could be seen from such remarks as "aaaa, sorry, $b > 0$." This remark reinforced the inference that her understanding of order axioms was fragile. During the follow up interview Tanya could explain why the solution $a_n < -1$ should be neglected by noting that $a_1 = \sqrt{2}$ and (a_n) is a monotone increasing sequence. Tanya could access relevant conceptual insights and technical handles (Hanna & Mason, 2014). Tanya managed also to clarify the formal definition of a sequence, a mapping from the set of natural numbers to real numbers that she had described as a mapping from real numbers to natural numbers. Overall, Tanya's responses during the follow up interview revealed that she had established contact with mathematical objects she used to tackle the task (Wilkerson & Wilensky, 2011).

Use the definition of appropriate limit to prove that $lim_{x\to\infty}\frac{\sqrt{3x^2+4}}{x}=\sqrt{3}$.	• Written response Let $\varepsilon > 0$ be given, we need to determine $X \in \mathbb{R}$ s.t if $x > X$ then $\left\|\frac{\sqrt{3x^2+4}}{x}-\sqrt{3}\right\| < \varepsilon$ $\left\|\frac{\sqrt{x^2}\sqrt{3+\frac{4}{x^2}}-\sqrt{3}}{x}\right\| < \varepsilon$ $\left\|\sqrt{3+\frac{4}{x^2}}-\sqrt{3}\right\| < \varepsilon$ …………………………………… $\left\|\frac{4}{x^2\sqrt{3+\frac{4}{x^2}}+\sqrt{3}}\right\| < \varepsilon$ $\left\|\frac{4}{\sqrt{3x+4}+\sqrt{3}}\right\| < \varepsilon$ $\frac{4}{\varepsilon} < \sqrt{3x+4}-\sqrt{3}$ $\frac{4}{\varepsilon}+\sqrt{3}\sqrt{3x+4}$ $\left(\frac{4}{\varepsilon}+\sqrt{3}\right)^2 < 3x+4$ $\frac{1}{3}\left[\left(\frac{4}{\varepsilon}+\sqrt{3}\right)^2-4\right] < x$	Tanya stated the formal definition of limit of f as $x \to \infty$ and the definition was correctly applied to the problem context as shown by the substitutions made. Algebraic manipulations were properly done up to : $\left\|\sqrt{3+\frac{4}{x^2}}-\sqrt{3}\right\| < \varepsilon$. She was at this stage supposed to apply the identity $\sqrt{x}-\sqrt{y}=\frac{x-y}{\sqrt{x}+\sqrt{y}}$. Failure to use this identity led to *chaos* or complete *mess* as student failed to express x in terms of ε and hence could not determine $X \in \mathbb{R}$ for which if $x > X$ then the condition $\|f(x)-L\| < \varepsilon$. Hence Tanya could not state the conclusion.	From the description of Tanya's proof effort it can be noted that she tried to use formal axiomatic reasoning as was seen from correct substitutions made into the formal definition. The goal the proof effort pursued was well articulated by Tanya so she had a sense of what the proving effort sought to accomplish (Selden & Selden, 2009). Progress in the proving effort was impended by the student's limitations in algebraic manipulations that required the use of the identity $\sqrt{x}-\sqrt{y}=\frac{x-y}{\sqrt{x}+\sqrt{y}}$. It can thus be noted that although Tanya was aware of the proof framework, because he had realized that the method of direct deduction and adhered to the logical structure by drawing from the formal structure of the definition algebraic limitations impeded progress as student could not deploy the right resources at the right time Tanya lacked access to relevant technical handles (Sandefur et al., 2013).
	• Chalkboard demonstration [student is silent after writing statement on chalkboard] {So we are going to first of all define the limit as $x \to \infty$ of a function } [student writes and verbalizes] *Let* $\varepsilon > 0$ *be given, we need to determine* $X \in \mathbb{R}$ *s.t. if* $x > X$ *then* $\|f(x)-L\| < \varepsilon$ (coughs) {And} [student verbalizes and writes the following] *if* $x > X$ *then* $\left\|\frac{\sqrt{3x^2+4}}{x}-\sqrt{3}.\right\| < \varepsilon${From the numerator, we are going to factor out x^2, the square root of x^2 so that inside the bracket we have}$\left\|\frac{\sqrt{x^2}\sqrt{3+\frac{4}{x^2}}}{x}-\sqrt{3}\right\| <$ ε{If we simplify the numerator $\sqrt{x^2}$ becomes x and x and the denominator [referring to x] , will cancel each other. Then we have }	Similar to the written response data source, Tanya could verbalize and write the formal definition of limit of a function as $x \to \infty$ and could apply it correctly to the task. Algebraic manipulations were much better than they were in the written response section. The identity $\sqrt{x}-\sqrt{y}=\frac{x-y}{\sqrt{x}+\sqrt{y}}$, was correctly stated and applied correctly to obtain $\left\|\frac{4}{x^2\sqrt{3+\frac{4}{x^2}}+\sqrt{3}}\right\| < \varepsilon$, which is flawed in the sense that term x^2 should have multiplied both terms in the denominator, that is, a correct representation should have been $\left\|\frac{4}{\left(\sqrt{\sqrt{3+\frac{4}{x^2}}+\sqrt{3}}\right)}\right\| < \varepsilon$. Once again an impasse was then reached. At this stage the student relooked at earlier working erased some parts	Although algebraic manipulations were better than was previously reported an impasse was also reached here. Tanya failed to realise that $\left\|\frac{1}{\sqrt{3+\frac{4}{x^2}}+\sqrt{3}}\right\| < \left\|\frac{1}{\sqrt{\frac{4}{x^2}}}\right\|$ This limitation in algebra manipulations impeded progress even though Tanya could articulate the formal rhetoric part she failed to act on it (Selden & Selden, 2009). It can be noted that Tanya had a strong command of the hierarchical structure of the proof as she could describe that "What we want to do here is that we want to find ,eee, x in terms of ε […] I am failing to, uuu, simplify but, but I want, what I want is that I must get x in terms of ε." Therefore, Tanya was aware of the goal the proving effort was meant to accomplish but failed to act on this knowledge. In other words Tanya lacked access to relevant technical facility (Raman, 2003).

	[Student writes] $\left\|\sqrt{3+\frac{4}{x^2}}-\sqrt{3}\right\|<\varepsilon$ {We have that identity which states that } [student verbalizes and writes] $\sqrt{x}-\sqrt{y}=\frac{x-y}{\sqrt{x}+\sqrt{y}}$ {So we are going to apply this identity here. Which means we are going to have} $\left\|\frac{3+\frac{4}{x^2}-3}{\sqrt{3+\frac{4}{x^2}}+\sqrt{3}}\right\|<\varepsilon$ {In the numerator, 3 and 3 goes so we are left with} $\left\|\frac{4}{x^2\sqrt{3+\frac{4}{x^2}}+\sqrt{3}}\right\|<\varepsilon$ {What we want to do here is that we want to find ,eee, x in terms of ε. This is going to be (coughs)} $\left\|\frac{4}{\sqrt{3x^2+4}+\sqrt{3}}\right\|<\varepsilon$ $\frac{4}{\sqrt{3x^2+4}+\sqrt{3}}$ [Student makes an effort to simplify, relooks at the working and says] {I don't why…[inaudible]. I am failing to, uuu, simplify but, but I want, what I want is that I must get x in terms of ε. So we multiply this, this bracket} [silent and staring at the working on the chalkboard. Student erases what has been written. Student is apparently stuck i.e. an impasse].	of the board and ultimately gave up the proof attempt.	
Define a sequence (x_n) inductively by $x_1=1$ and $x_{n+1}=\frac{2x_n+3}{4}$. Prove that (x_n) is a bounded monotone sequence is bounded and hence determine its limit.	• Written response $x_1=1$ $x_2=\frac{5}{4}$ $x_3=\frac{11}{8}$ $x_1<x_2 \Leftrightarrow x_2>x_1$ $x_2<x_3 \Leftrightarrow x_3>x_2$ Since $x_1<x_2$ and $x_2<x_3$ it holds for P_1 and P_2. Assume it holds for $n=k$, then it is true for P_k. We need to prove that it holds for $n=k+1$ $x_{k+1}-x_k=\frac{2x_k+3}{4}-\frac{2x_{k-1}+3)}{4}$ =……………………….. $=\frac{2}{4}(x_k-x_{k-1})$ ∴ (x_n) is a monotone increasing sequence. Because x_n is a monotone sequence $x_{k+1}-x_k>0$	Tanya determined the terms x_2, x_3. She then used these specific examples to observe that $x_1<x_2$ and $x_2<x_3$. These empirical evaluations constituted the base step of the Principle of Mathematical Induction. She was then able to conclude that P_1 and P_2 were valid. Tanya then just alluded to the induction hypothesis. The implication statement $P_k \Rightarrow P_{k+1}$ was well articulated and well executed. Student could establish through mathematical induction that the sequence is monotone increasing. Definition of a monotone	The chunk of reasoning displayed by Tanya was very much similar to the one noted above with the following improvements. The proof effort was not impeded by limitations in algebraic manipulations. After successful application of mathematical induction to prove that the sequence is monotone increasing and is bounded above Tanya capitalized on those processes successfully completed to determine the limit of the sequence. There was coherence in reasoning displayed by Tanya as mathematical induction process made it possible for the student to prove that the sequence is monotone increasing. The much desired interplay between technical

	$\frac{2x_n+3}{4}$ -$x_n > 0$ $\frac{2x_n-4x_n}{4} + \frac{3}{4} > 0$ -$\frac{2x_n}{4} > \frac{-3}{4}$ $x_n < \frac{3}{2}$ $\therefore 1 < x_n < {}^3/_2$ $\therefore$ x_n is an increasing monotone sequence that is bounded and it converges to its leas upper bound equal to ${}^3/_2$.	increasing sequence was then used to prove that (x_n) is bounded. This then led to the formulation $1 < x_n < \frac{3}{2}$. The conclusion that the sequence (x_n) converges to $\frac{3}{2}$. Tanya provided proper justification for the conclusion. No flaws were noted her argument.	handles and conceptual insight was illuminated here (Hanna & Mason, 2014; Raman, 2003). This in turn led to use of the relation $x_{k+1} - x_k > 0$ to prove that the sequence is bounded with least upper bound $\frac{3}{2}$ which was deduced to be the limit.

Table 5.19: Mid-instruction assessment data matrix for Getrude on Real Analysis proof tasks

Task	Student's response (written, oral, actions)	Profiles of students' proving	Proof scheme elements present
Determine whether the following statement is true or false. For all real numbers a and $b, a - b > 0 \Rightarrow a^2 - b^2 > 0$.	• Written responses If $a - b > 0$ then $a > b$ (by the trichotomy law) $a.a = a^2$ and $b.b = b^2$ $a.a - b.b > 0 \Rightarrow a.a > b.b$. Similarly $a^2 - b^2 > 0$ $a^2 - 0 > b^2$ $\Rightarrow a^2 > b^2$ $a^2 - 0 > b^2$ holds $\therefore a^2 - b^2 > 0$ is a true statement. If it were not true the $-(a - b) > 0$ would hold (by the trichotomy law) $-(a - b) > 0 \Rightarrow b - a > 0$ and $b > a$ which is a contradiction to the given $a - b > 0$ $\therefore$ the above proof holds and therefore it is true that $a - b > 0 \Rightarrow a^2 - b^2 > 0$	Getrude engaged in symbolic manipulations and claims meant to justify the symbolic manipulations did not resonate well with each other. For instance, "if $a - b > 0$ then $a > b$" The student claimed the statement holds because of the Trichotomy Law. However the application of the Trichotomy to link the two statements $a - b > 0$ then $a > b$ not clear. Student then stated that the statement is true. She then set out to prove this assertion by contradiction. The effort to use the method of proof by contradiction to the antecedent statement (IF part) did not show a link with the consequent statement $a^2 - b^2 > 0$	From the description of Getrude's proof effort it can be noted that efforts to justify mathematical claims were flawed because the student could not deploy the right resources at the right time (Wilkerson-Jerde & Wilensky, 2011). For instance the "resource" Trichotomy Law of $\mathbb{R}$ as an ordered field was irrelevant as it did not show why $a - b > 0 \Rightarrow a > b$. The same argument also applies to the situation where the student tried to apply method of proof by contradiction to establish that $a - b > 0 \Rightarrow a^2 - b^2 > 0$. The student's purported proof by contradiction did not connect the antecedent statement to the consequent statement. Overall, it can be inferred that Getrude applied mathematical objects indiscriminately as she tried to prove that $-b > 0 \Rightarrow a^2 - b^2 > 0$ (Duffin & Simpson, 2000).
Describe whether the following statement is true or false. For	• Follow up interview Researcher: […] for instance they say prove that $a - b > 0$ then $a^2 - b^2 > 0$, then you started by saying by our trichotomy law or by your order properties when it was easy to use	Getrude was asked to explain why she had decided to use axioms and definitions when it was much easier to use examples. She attributed her propensity to use axioms to lack of experience with tasks of this nature and to the way the proposition was	From the follow up interview, it can be noted that Getrude explained her proclivity for use of axioms in terms of lack of experience with proof

all real numbers a and $b, a - b > 0 \Rightarrow a^2 - b^2 > 0$.	a counter example Getrude: So it's inexperience the more I am more involved in these problems the more I know how to solve them. When asked I will definitely I will use examples and not axioms so maybe it's inexperience on the concept. Researcher: But when do you decide as a student to say here I think I want an example and this other situation I think it requires axioms? [...] How do you make a choice between the use of examples and the use of axioms in a given a task? Getrude: Maybe the way it is asked will really direct you on what to use [...].	formulated. She stated that the use of inequality symbols persuaded her to use order axioms. Limited knowledge about how to go about proving can force the "learner to try to come up with a solution without the correct knowledge of the method to use, just maybe beating about the bush…"	tasks that demand counter-argumentation. Getrude also suggested that the tendency to use axiomatic reasoning in proof tasks is caused by the manner in which the question is framed. For instance the inequality symbol ">" in the task persuaded her to use order properties of real numbers in place of specific examples. In other words, Getrude was persuaded to use axioms by the contextual clues in the form of the formulation; $a - b > 0 \Rightarrow a^2 - b^2 > 0$, in the proof task.
Determine whether the statement is true or false. If x *is an integer, then* $x^2 - x$ *is an integer. Justify your answer.*	• Written responses $x^2 - x \Rightarrow x(x+1)$. Let us set x to be 1. If $x = 1$ then $1(1+1) = 1 \times 2 = 2$ an even. If $x = 2$ then $2(2+1) = 2 \times 3 = 6$ an even Setting $x = k$ yields $k(k+1)$=$k^2 + k$ …………….(1) Setting $x = k + 1$ $k + 1(k + 1 + 1)$ $= k + 1(k+2)$ $=k^2 + 3k + 2$ ... (2) Letting $k \in \aleph$ and substituting in (1) and (2) Let $k = 1$ $k^2 + k \Rightarrow (1)^2 + 1$ =2 (even) $k^2 + 3k + 2 = (1)^2 + 3(1) + 2 = 6$ even. Let k=2 $k^2 + k \Rightarrow (2)^2 + 2 = 6$ even $k^2 + 3k + 2 = (2)^2 + 3(2) + 2$ =12 even ∴ if for any value of k. $(k^2 + k)$ and $(k^2 + 3k + 2)$ are both even it is true that $x^2 - x$ is an even number.	Getrude started by factorizing: $x^2 - x \Rightarrow x(x+1)$. A plus sign was written instead of a minus sign. Specific examples: $x = 1$ and $x = 2$ were substituted into the expression $x(x+1)$ to get 2 and 12 respectively. Student then set $x = k$ into the expression to get $k^2 + k = k(k+1)$. She then substituted the value $x = k + 1$ into the expression and expanded to get: $k^2 + 3k + 2$ for $k \in \mathbb{Z}$. Student then switched back to particular instantiations: $k = 1, 2,$ into expressions $k^2 + k$ and $k^2 + 3k + 2$ and the student obtained 6 and 12 respectively. Student then concluded that since $(k^2 + k)$ and $(k^2 + 3k + 2)$ both yielded even numbers it is true that if $x \in \mathbb{Z}$, then $x^2 - x$ is an even number.	It can observed from Getrude's proof effort initially used particular instantiations, presumably these were intended to constitute the base step of the principle of mathematical induction. This is because soon after the empirical verification, Getrude wrote: "Setting $x = k$ yields $k(k+1)$=$k^2 + k$". This step was presumably constituted the induction hypothesis. Next she wrote: "Setting $x = k + 1 = k + 1(k + 1 + 1) = k + 1(k+2)$ =$k^2 + 3k + 2$." Again, presumptively this represented the induction thesis. However, Getrude then started to use specific examples and concluded that: If x is an integer, then $x^2 - x$ is an integer. Although the student did not state precisely that he tried to apply the method of proof by induction it can be inferred from her working that there was a switch from formal deductive reasoning

			(axiomatic proof scheme) to the empirical proof scheme (CadawalladerOlsker, 2011).
Prove that the sequence defined by $(u_n) = \frac{n^2-1}{2n^2+3}$ *converges.*	• Written response Lets first find the limit L s.t. $\lvert u_n - L\rvert < \varepsilon$ ……….to find limit we divide by n^2 $\frac{1-\frac{1}{n^2}}{2-\frac{3}{n^2}}$ as $n \to \infty$ $\frac{1}{n^2} = 0$ $\frac{3}{n^2} = 0$ ∴ the sequence converges to $1/2$ ∴ Our $lim = 1/2$ Proof $\left\lvert\frac{n^2-1}{2n^2+3}\right\rvert < \varepsilon$ for $n \leq N(\varepsilon)$ $\left\lvert\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right\rvert < \varepsilon$ $-\varepsilon + \frac{1}{2} < \frac{n^2-1}{2n^2+3} < \varepsilon + \frac{1}{2}$ Considering only the positive by our triangle inequality $\frac{n^2-1}{2n^2+3} - \frac{1}{2} < \varepsilon \, \frac{2n^2-2}{2n^2+3} - 1 < 2\varepsilon$ $\frac{2n^2 - 2 - 1(2n^2+3)}{2n^2+3} < 2\varepsilon$ …………………………………… $\frac{-5}{2n^2+3} < 2\varepsilon$ $-5 < (2n^2+3)2\varepsilon$ …………………………………… $\frac{-5-6\varepsilon}{4\varepsilon} < n^2$ $-\left(\frac{5}{4\varepsilon} + \frac{6}{4}\right) < n^2$ $i\sqrt{\frac{5}{4\varepsilon} + \frac{6}{4}} < n$	Getrude started by expressing her goal: to find the limit L such that $\lvert u_n - L\rvert < \varepsilon$. She then proceeded to find the limit by dividing through out by the dominant term and evaluated the limit of the expression: $\frac{1-\frac{1}{n^2}}{2-\frac{3}{n^2}}$ as $n \to \infty$. For the proof part, the expression $\left\lvert\frac{n^2-1}{2n^2+3}\right\rvert < \varepsilon$ for $n \leq N(\varepsilon)$ has the following flaws. First, the inequality $n \leq N(\varepsilon)$ is wrong, exactly the opposite is true, that is it should have been stated as $n > N(\varepsilon)$ Student did not specify the symbol she engaged, e.g., Getrude was supposed to state that $N(\varepsilon) \in \aleph$ and the quantity $\varepsilon > 0$ described as a small radius. In her symbol manipulations, Getrude wrote $\left\lvert\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right\rvert < \varepsilon$ that was then transformed to $-\varepsilon + \frac{1}{2} < \frac{n^2-1}{2n^2+3} < \varepsilon + \frac{1}{2}$. Getrude then referred to the second part of the inequality as "the positive of the triangle inequality". Then algebraic manipulations led to $i\sqrt{\frac{5}{4\varepsilon} + \frac{6}{4}} < n$; a senseless answer because the number n concerned is a natural number but Getrude got a complex number. Getrude could not state the conclusion, that is, she could not use results of her symbol manipulations to determine whether the sequence (u_n) converges.	Getrude's proof attempt revealed some limitations in her understanding of the concept of convergence of a sequence. The flaws include failing to describe meaning of symbols central to the definition such as $\varepsilon > 0$, the natural number, $N(\varepsilon)$ and omission of modulus symbol. Her working revealed poor algebraic and technical manipulations that led to a complex solution which is senseless in real sequences. Consequently it can be inferred that Getrude had not built a coherent network of mathematical resources to deal with the proof task (Duffin & Simpson, 2000; Wilkerson-Jerde & Wilensky, 2011).

Table 5. 20: End of instruction assessment data matrix for Getrude on Real Analysis proof tasks

Task	Student's response (written, oral, actions)	Profiles of students' proving	Proof scheme elements present
Prove that $f(x) = x^2 + 2x - 5$ *is uniformly continuous on* $[0, 3]$.	• Written responses If f is uniformly continuous on [0,3] if $x, y \in [0,3]$ ∃ $\delta(\varepsilon) > 0$ st if $\lvert x-y\rvert < \delta(\varepsilon)$ then $\lvert f(x) - f(y)\rvert < \varepsilon$ $\lvert x^2 + 2x - 5 - (y^2 + 2y - 5)\rvert < \varepsilon$ ……………………………… …………………………… $\lvert (x+y)(x-y) + 2(x-y)\rvert < \varepsilon$ Let $x = y = 3$ $\lvert (3+3)(x-y) + 2(x-y)\rvert < \varepsilon$ ……………………………… ……………………………. $\lvert x-y\rvert < \frac{\varepsilon}{8}$ which is independent of [0,3]	The formal definition of uniform continuity written by Getrude had one important aspect missing. The quantity $\varepsilon > 0$, that is fundamental to definition was not specified explicitly but only appeared when the student wrote $\lvert f(x) - f(y)\rvert < \varepsilon$. However, other aspects of the definition were all in order, e.g., the student stated that $x, y \in [0,3]$. Algebraic manipulations were correctly done and Getrude obtained $\lvert x-y\rvert < \frac{\varepsilon}{8}$. The comment about $\delta(\varepsilon)$ obtained that "which is independent of [0,3]" is not a correct characterisation of the idea of uniform continuity, rather it should have been described as a quantity that is independent of arbitrarily elements $x \in [0,3]$.	From Getrude's solution attempt it can be seen that her conceptual knowledge of concepts pertinent to the task was somewhat limited. Lack of access to conceptual insights could be seen from such comments as "$\lvert x-y\rvert < \frac{\varepsilon}{8}$ which is independent of [0,3]" when she was supposed to explain that $\delta(\varepsilon) > 0$ determined was independent of an arbitrary element drawn from the interval [0,3]. Some steps were not justified, e.g., "Let $x = y = 3$." Getrude did not also explain how $\delta(\varepsilon) > 0$ determined showed that $f(x)$ is uniformly continuous. It can therefore be inferred that Getrude had limited conceptual knowledge of the mathematical ideas involved. Her working demonstrated she had instrumental knowledge of uniform continuity because she could engage in algebraic and determined the size of $\delta(\varepsilon) > 0$ but could not interpret this quantity in terms uniform continuity. It can be concluded that Getrude failed to provide an epistemological justification as to how answers generated were solutions to the task (Koichu, 2012; Pfeiffer, 2010).
Define a sequence (x_n) *inductively by* $x_1 = 1$ *and* $x_{n+1} = \frac{2x_n+3}{4}$. Prove that (x_n) is a bounded monotone sequence and hence determine its limit.	• Written response Inductively $x_1 = 1$ $x_2 = \frac{2(1)+3}{4} = \cdots = 1.25$ $x_3 = \frac{2(1.125)}{4} = \frac{5.5}{4} = 1.375$ $x_1 < x_2$, $x_2 < x_3$ ∴ it holds for x_1, x_2 , and x_3 ⇒ $x_n > x_{n-1}$ is true. Assume it is true for $n = k$ it implies that $x_k > x_{k-1}$ also holds $x_{k+1} - x_k = \frac{2x_k+3}{4} - \frac{2x_{k-1}+3}{4}$ …………………………… $x_k - x_{k-1} = {}^1\!/_2\,(x_k - x_{k-1})$ which implies $x_{k+1} > x_k$, but by induction $x_k > x_{k-1}$ and hence $x_{k+1} > x_k$. ∴ it also holds for $n = k+1$. Prove for x_n is bounded monotone sequence	Student first wrote "inductively" and then determined the terms x_2 and x_3. Getrude then noted $x_1 < x_2$ and $x_2 < x_3$. Student then used the specific instantiations to conclude that $x_n > x_{n-1}$ in other words Getrude claimed that the sequence is monotone increasing. She then tried to carry out steps of mathematical induction to establish the sequence (x_n) is monotone increasing. Method of proof by induction was not properly executed e.g.,	Getrude's command of the proof framework was weak. The proof framework refers to the conventions of doing things in mathematics. It can be inferred from her working that she did not adhere to the logical structure of doing proofs because Getrude concluded on the basis of particular instantiations that " ∴ it holds for x_1, x_2 , and x_3 ⇒ $x_n > x_{n-1}$ is true." She then set to establish that $x_n > x_{n-1}$ by mathematical induction. Other aspects of her proof attempt relate to the hierarchical order of the proof, that is, the goals the proof effort pursued (Selden & Selden, 2009). Getrude did not specify that the sequence is monotone increasing. Although

	$x_{n+1} - x_n > 0$ $\frac{2x_n+3}{4} - x_n \geq 0$ ……………………………………… $\frac{3 - 2x_n}{4} \geq 0$ …………………………………………………. $3 - 2_{x_n} > 0$ ……………………………………. $x_n \leq {}^3\!/_2$. Since $x_1 = 1$ and $x_n < {}^3\!/_2$ it implies $1 \leq x_n \leq {}^3\!/_2$ The sequence is bounded and its limit is ${}^3\!/_2$.	$x_k - x_{k-1} = {}^1\!/_2\,(x_k - x_{k-1})$ is wrong statement used to deduce that $x_{k+1} > x_k$. She did not also mention whether the sequence (x_n) is monotone decreasing or increasing. However, when Getrude set out to prove that the sequence is bounded the inequality written: $x_{n+1} - x_n > 0$ indicated that the sequence is monotone increasing. From the inequality the student substituted and got $x_n \leq {}^3\!/_2$. Getrude then deduced that $1 \leq x_n \leq {}^3\!/_2$.	Getrude had challenges noted about matters to do with the proof framework and the hierarchical structure, her working revealed a good command of conceptual knowledge (du Toit, 2009). She determined that the sequence is bounded by capitalizing on the fact the sequence (x_n) is monotone increasing sequence. After proving that the sequence is bounded Getrude then used the idea a bounded sequence to deduce that it has limit ${}^3\!/_2$. The connection in concepts described supports the the inference that Getrude had a strong conceptual knowledge of the ideas involved in the task.
Define a sequence (x_n) *inductively by* $x_1 = 1$ *and* $x_{n+1} = \frac{2x_n+3}{4}$.	• Follow up interview Researcher: […] you have: {You have $1 \leq x_n \leq \frac{3}{2}$}]. How did this lead now to your conclusion that the limit is $3/2$. Getrude: If I remember I checked on the behaviour of the function, whether it was convergent. So in that [inaudible] used this inductive method. I realised that it was monotone increasing sequence. So if a sequence is a monotone increasing, it means that it is bounded above and least upper bound becomes the limit. Researcher: In other words, a bounded monotone increasing sequence converges that's what [interjection from Getrude] Getrude: Yes it converges to its least lub [least upper bound]. […………] So this one was like if it lies between this range $[1 \leq x_n \leq \frac{3}{2}]$ it means the terms are starting from up to up to $3/2$ which 1 ½ . So this one, $3/2$ becomes the limit because of the what, the monotone increasing pattern.	The focus of the interview was on how Getrude had drawn the conclusion that ${}^3\!/_2$ is the limit of sequence (x_n). She was able to explain that a sequence that is bounded above converges to its least upper bound. However, some statements produced by the student were not true. For instance, Getrude stated that, "If a sequence is monotone increasing it means it is bounded above." This statement by Getrude is not necessarily true because a monotone increasing sequence does not always converge. A monotone sequence converges only if it is bounded.	The follow up interview on the task revealed that Getrude had grasped the convergence criterion for a monotone bounded sequence in ℝ. When probed about how she drew the conclusion that the sequence (x_n) converges she explained that "Yes it converges to its least lub [least upper bound]. So when I worked out to see the pattern, and the convergence, I realised that uhu, my sequence was lying between this range. So this one was like if it lies between this range $[1 \leq x_n \leq \frac{3}{2}$] it means the terms are starting from 1 […] up to $3/2$ which 1 ½ . So this one, $3/2$ becomes the limit…" Her explanation showed good command of ideas pertinent to the task. However, some utterances by Getrude during the interview showed that she lacked understanding of aspects on sequences. For instance claims such as "So if a sequence is a monotone increasing, it means that it is bounded above." This is not necessarily true. So while Getrude exhibited good command of convergence criterion for bounded monotone sequence, some limitations were also revealed, e.g., she claimed that every monotone sequence converges.

Use the definition of appropriate limit to prove that $lim_{x\to\infty}\frac{\sqrt{3x^2+4}}{x} = \sqrt{3}$.	• Written response Let $\varepsilon > 0$ be given. $\exists X \in \mathbb{R}$ such that if $x > X$ then $\|f(x) - L\| < \varepsilon$ $\left\|\frac{\sqrt{3x^2+4}}{x} - \sqrt{3}\right\| < \varepsilon$	The formal definition of the limit of a function f as $n \to \infty$ correctly stated and the function $f(x)$ and the limit L also correctly substituted into the expression. However, the student made no further progress toward determining $X \in \mathbb{R}$ after making the substitutions described above.	Getrude had a grasp of the formal rhetoric aspect of the proof task but she could not act on this behavioural knowledge (Selden & Selden, 2009) . She substituted correctly into the formal definition but could not progress to find $X \in \mathbb{R}$ such that if $x > X$ then the function will approach. It is important that a prover acts on the behavioural knowledge rather than just articulating it (Fukawa-Conelly, 2012).
Use the definition of appropriate limit to prove that $lim_{x\to\infty}\frac{\sqrt{3x^2+4}}{x} = \sqrt{3}$.	• Follow up interview Researcher: […] can you suggest reasons why you could not make progress in question (a) after writing the definition. Getrude: [Laughing] Ok, aaaa, the fact that there are two square roots was a challenge at first. But if I had enough time I could have solved it. However, I didn't want to use the identity that say [$\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$] why because this was one [ ] is a fraction and this one seems to be not this one is a fraction because of the x here and this one is not. So the process was for one to put it under one denominator then simplify then use the identity later on. But I knew from my notes when I was revising that x^2 is supposed to be factored out but I couldn't remember how, so because of time I told myself to leave this for a moment, do the other simpler ones and then I was going to come since I left this space. Researcher: You seem to be very much aware of processes. Can you suggest reasons why people do not make progress despite having a lot of knowledge about....[interruption from Getrude] Getrude: Lack of computations in algebra. Even if I know the formula if I can't operate the algebraic performances I can't proceed.	Upon being asked to account for her lack of progress with the task, Getrude explained that the two square roots put her off. She even could describe how she could have simplified the expression $\left\|\frac{\sqrt{3x^2+4}}{x} - \sqrt{3}\right\| < \varepsilon$ in order to determine $X \in \mathbb{R}$ for which the function converges to the limit L as $x \to \infty$. She could also describe an extremely useful identity : $\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$, that could have made it possible for her to determine the real number X in terms of the radius $\varepsilon > 0$, chosen. She attributed lack of success with the solution process to computational challenges in algebra.	During the follow up interview, Getrude explained the cause of her impasse during her efforts to solve the task as follows. She pointed out that she was discouraged in her efforts by two square roots: $\frac{\sqrt{3x^2+4}}{x}$ and $\sqrt{3}$. However, she showed a grasp of the appropriate resources that could have been deployed in order to tackle the task such as the identity $\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$ (TH) but she could not execute this behaviour because she wanted to focus her attention on what she called "simpler ones." I emphasize that behavioural knowledge pertinent to mathematical problem no matter how well articulated but not executed is not important (Selden & Selden, 2011). So although Getrude could describe how she thought she could have tackled the task, executing her plan could have given insights into her thinking as she engaged with the task.

5.2 Results: Research question two

How do the undergraduate student teachers develop their proof schemes?

The focus of the study was on how the mathematical object emerges? Data from the reflective interview guide on students' proof experiences were used to address research question two. Summative content analysis (Berg, 2009) was applied to the verbatim transcriptions of reflective interview textual data. Summative content analysis of verbatim transcriptions started with actual phrases and words in the textual data. First, students' conceptions of mathematical proof were explored. The intent was to establish the nature of "being" of the object, that is, its ontology among the student teachers. The actual words and phrases were then used to inductively build the categories of students' definitions of mathematical proof. Second, the student teachers' proof experiences at pre 'A'-level, 'A'-level and undergraduate level were reported in this section. Furthermore, this section also reports on student teachers' inconsistencies in student teachers' formal rhetoric aspects.

5.2.1 Students' conceptions of mathematical proof

Table 5.21: Inductively developed categories of student teachers' conceptions of mathematical proof ($n = 10$).

Category	Examples of responses	frequency
Facts	Bea: […] a mathematical proof refers to some ideas, eee, supporting , that is, in favour of or against a statement	2
Procedures	Taku: Mathematical steps required to reach a conclusion.	3
Logical argument	Tino: Logical argument needed to convince that something is true	4
Formula	Tina: Finding out if formula is correct	1
	Total	10

Researcher's comments

Table 5.21 reveals that mathematical proof was thought of mainly in terms of logic, that is, in terms of activities with a focus on establishing the truth or falsity of a mathematical proposition as indicated by (4 out of 10) responses in this category. The table also illustrates that mathematical proof was thought of as steps or procedures needed to establish mathematical theorems, (3 out of 10) were in this category. Definitions are complete descriptions of the behaviour or structure of the focal mathematical idea that accounts for all instances of that idea (Wilkersen-Jerde & Wilensky, 2011, p. 31). One's understanding of a concept determines the manner in which one engages with problems for which the concept is focal. I had therefore, anticipated that the students' descriptions of a mathematical proof could in turn illuminate how their proof schemes emerged. The following section focused on the students' experiences with mathematical proof at various scholastic levels. The specific item in the interview guide that captured their pre 'A'-level experiences was: *Describe your pre 'A'- level experiences with the concept of a proof*

5.2.2. Undergraduate student teachers' pre 'A'-level experiences with proof

Table 5.22: Inductive codes developed through content analysis of the data for $(n = 10)$

Category	Example of student utterances	relative frequency
Calculations	Taku: measuring and calculations , no axioms	2
Formula	Tendai:[…] we were just substituting into the formula	9
Procedures/steps	Bea: […] could easily follow steps	2
Not challenging	Bea: Proofs not challenging could easily follow steps	1
No proofs	Cortney: No proofs […] we were exposed to the stuff	10
Empirical/experimental	Taku: drawing and measuring	3
Applying memorized facts	Taku: We were just given (Cosine rule, Pythagoras) and told to memorize and apply them	4
	Total	31

Researcher comments

Of the inductive categories derived from content analysis it can be noted at the pre 'A'-level period there was "no proof" exploring experiences as shown by (10) responses. Students reported that they did not engage with the concept of mathematical proof during the pre 'A'-level period. Emphasis was on applying formulas to obtain answers as represented by category labelled "Formula." An interview extract of Tanya revealed the point that the major mathematical activity of the pre 'A'-level involved applying given formulas.

Tanya: Aaaa, I would like to believe I did not deal with them so much, maybe at the end I would just be given the result.

Researcher: Ok.

Tanya: Then just use the result.

Researcher: To do what?

Tanya: Maybe, like we have the quadratic formula where we want to solve for the roots of the equation. Then you will just be given how to go about and come out with the proof. I didn't know that much.

Researcher: In other words, before you came to this level [undergraduate studies], you did not know how to prove the quadratic formula, how to prove the cosine rule, the sine rule, but you were using those?

Tanya: Yaaa. I was using them but we didn't really concern ourselves with the part of proving that.

This extract reinforces the idea that mathematical concepts were transmitted to student with little regard for their underpinnings and verification (Varghese, 2009). These inferences were implied by the phrases such as "Then you will just be given…" and "we didn't really concern ourselves with the part of proving" from the extract. Utterances by Tino reinforce the point that there were no proofs in the primary and secondary school mathematics curricula. He mentioned that the concepts: Pythagoras' theorem, Cosine rule were used without proving as their proofs were not included in the syllabus.

Another student teacher Bea who had defined mathematical proof in terms of steps needed to reach a conclusion pointed that out proving at pre 'A'- level was not challenging as she could follow steps easily. Yet another inductive code derived from summative content analysis of data was labelled "Calculations." This category had a strong link with the category involving use of given formula as illustrated by Taku's utterances during the reflective interviews.

> **Taku:** Aaaa, actually I didn't experience much about proofs but, aaa, the mathematical concept which were actually used to, like the formula, was already provided. You could be provided with the sort of formula then you could just substitute the required numbers then you come up with an answer. At that stage I had no much experience as far as proofs were concerned.

Taku's utterances revealed that focus was on doing calculations using given formulas and there were no proofs done. Therefore, it can be, concluded that from the students' definitions of a mathematical proof that proving was not part of their early learning experiences. The students' accounts are aligned to their reported conceptions of mathematical proof in Table 5.21. Students conceived mathematical proof in terms of use of facts (axioms and definitions) to validate mathematical statements that are often provided in a logical manner. This view of proof can be used to explain why "calculations" and formulas given were not seen as part of proof activities by student teachers. The next section focused on students' 'A'-level experiences with proof.

5.2.3 'A'-level experiences

Inductive categories derived from summative content analysis of the data are presented in the table below

Table 5. 23: Student teachers' A-level experiences with mathematical proof ($n = 10$)

Category	Example of student responses	Relative frequency
Applying facts	Getrude: There were some facts given. […] applying given facts	5
Solving equations and identities	Taku: Proof of trig ratios not part of syllabus but solution of trig equations and identites	4
Few proofs	Bea: There were not too many proofs	3
Deriving formula	Taku: […] developing $\frac{1}{2}bh$ into $\frac{1}{2}absin\alpha$	2
Not challenging	Bea: […] proofs not challenging	1
Low intensity	Cortney: The only proof I remember is induction	2
	Total	17

Researcher comments

Applying given facts dominated learning of mathematics at A-level (5 out of 17 responses). Emphasis at 'A'-level was also on solving equations and identities as shown by (4 out 17) responses in this category. Thus, not much in terms of proving featured in 'A'-level school mathematics learning. There were few proofs at 'A'-level which were within what (Ball, Thames & Phelps, 2008) call students' conceptual reach as illustrated in Bea's extract of the interview.

> **Bea**: There are not many proofs considering the paper which [silent] I sat for exams paper,

> pure maths. It didn't have many proofs. So the proofs are [pause] at that level not challenging again, there are just a few proofs at 'A'- level.

The level of difficulty was also low as pointed by Bea. It can therefore be inferred from the responses, that proving was within students conceptual reach because it consisted of "applying given facts" as mentioned by Getrude. Another response that revealed that the activity of proving had a low intensify at 'A'- level came also from Cortney who said; "The only proof I remember is induction." Cortney even stated that she did not understand the underlying ideas of proof by induction.

At 'A'-level, students concentrated on memorising formulas from sources as was revealed by Taku's utterances: "We were told we were going to be given some formulae e.g. $cos^2\alpha + sin^2\alpha = 1$ was not proved … we were given." The idea that theorems, formulas and facts were provided to students without providing their basis, could also be seen in responses such as:

> **Tafa:** […] I do not remember any real proof. […] may be you were asked to differentiate.
>
> **Researcher**: So how were you learning the double angle identities?
>
> **Tafa:** Some of them were just given.

This interview excerpt with Tafa summarises, how the object of proof scheme existed at A- level, that is, low in intensity and mainly thought of in terms of applying given facts and techniques through processes such as differentiation. However, from the interview it can be noted that though low in intensity proving had begun to emerge. The development was followed through to students' current [at the time of the study] scholastic level, so next I asked the students to describe their undergraduate experiences with proof.

5.2.4 Undergraduate proof experiences

Summative content analysis of verbatim transcriptions gave rise to the following categories of the students' experiences with the notion of a proof at undergraduate level. I re-cap that ontology is the study of the existence of the object "being," the categories of being and entities within those categories. The intent was to develop an understanding about how the world fits together (Porta & Keating, 2008). Put in the context of the study, the focus was to develop an understanding of the various categories in students' differential experiences with proof and describe the mechanisms and processes that operate within a given category.

Table 5. 24: Student teachers' undergraduate experiences with proof ($n = 10$).

Category	Examples of student utterances	Relative Frequency
Axioms and definitions	Tafa: We need to know axioms, aaa, all definitions, the lemmas and so forth	2
Difficulty/challenging	Bea: […] challenging. Maybe I think its [pause] because something I am not familiar to,	4
Use of examples	Tendai: I can fix, I write $2 > 1$. I can fix numbers	1
Time	Taku: Proofs take more time to comprehend	2
Justification Skills	Debra: […] we were just using without knowing where they were coming from […] we can now prove why we are using the concepts or some of the formula	2
High intensity	Tanya: I have used them so much [referring to proofs], for example in linear Algebra […] lots of proofs	2
	Total	13

From Table 5.24, it can be seen that proving at undergraduate level was challenging for the students. Challenges and difficulties students faced with proofs at this level were revealed in the following student utterances. For example, Bea mentioned that proof was difficult and proving was not an extension of her early learning experiences as illustrated in the utterances below:

Bea: […] challenging because it's something I am not familiar with. Taking the whole course with proofs, failing to understand.

Bea mentioned also that in spite of repeating reading notes many times she had failed to comprehend proof of the Cut Property in $\mathbb{R}$. The difficulties Bea experienced with proofs were revealed in the following excerpt.

Researcher: Can one generate a proof?
Bea: I do not think, its possible. At my level. No, without reference to some written source.

The difficulties she had in comprehending her notes influenced her into thinking that producing a proof to a statement autonomously was impossible, and that one could only do so with the aid of a written source. This excerpt reveals the tenacity of external conviction authoritative proof scheme (Harel & Sowder, 1998, 2007) as Bea thought that she could only produce a proof with the help on a written resource. Yet, another excerpt that uncovered the difficulty undergraduate students experienced with proofs is now produced.

Tino: Proofs were challenging
Researcher: What was challenging?
Tino: Trying to argue logically to convince."

Therefore, the process of putting together axioms and definitions to build arguments that were convincing posed a challenge to student teachers. Further, students mentioned that proving mathematical statement posed serious difficult to the extent that it was difficult to figure out the start and end of the proving process —a result similar to that reported by Doruk and Kaplan (2015). Cortney's utterances revealed the point made above when she said:

Cortney: First it was hard; I couldn't tell where we were coming from and where we were going […]

Another characteristic of the category denoted as challenging is that students' experience some discomfort with proofs that forced them to react by resorting to what Tendai called "own understanding" which is explained after presenting the following excerpt.

> **Tendai** […] now maybe the theorem, If you did not understand the theorem and it's difficult, maybe you can use your own understanding.
> **Researcher:** What do you mean by your own understanding?
> **Tendai:** If you say you want to give may be an ordered field theorem then, you use if you want to say $a > b$ then I can fix, write $2 > 1$, I can use numbers.

Tendai's utterances revealed the fact that when she faced challenges in arguing in a formal deductive manner, in her case by using order axioms, she could then turn to particular instantiations of the order properties. Another characteristic of proof at undergraduate revealed during the reflective interviews concerns the need for justification skills. The following students' utterances reflected the need to justify claims made when proving.

> **Tanya**: Things no longer taken for granted.
> **Debra:** I can say that I have been learning that things which we were just using without knowing where they were coming from, we can prove why we are using the concepts or some of the formula.

Content analysis of textual data also led to the emergence of an inductive category denoted as "high intensity" that encompassed matters related to increased proving activity at undergraduate level. Unlike at the previous scholastic levels at undergraduate level, Taku mentioned that, "There are proofs after proofs." The increased proving activities were also mentioned by Bea as shown in the following interview excerpt.

> **Bea**: Taking the whole course with proofs.

Bea's utterance points to the fact that the Fundamental of Analysis course is a proof laden course. Another feature of undergraduate proof experiences relates to time taken to comprehend proof. For, instance, Tina mentioned, "You take your time on an aspect. Proofs take time to comprehend." Finally, it can be noted that the use of examples in proving had diminished as students preferred to use axioms and definitions to carry out proofs. In pursuance of how undergraduate students develop their proof schemes, it was necessary to trace differences they had noted as they dealt with proofs at various scholastic levels. So presented next are inductive categories from content analysis of student's verbatim transcriptions from perspective of students' differences in proof encounters.

Table 5. 25: Distinctive features of undergraduate proof experiences during mid-instruction interview ($n = 10$).

Category	**Example of student utterances**	**Relative frequency**
Level of detail	Debra: At lower lower […] proofs were simple but now we are doing proofs in detail.	5
Logical presentation of ideas	Tafa: […] but now […] we appreciate the concept of presenting arguments logically	1
Justification	Cortney: What is unique here [undergraduate level] we are looking at the basis where he theorems are coming from	5
Challenging	Getrude: Now I can support myself though it's still challenging.	2
Volume of work	Cortney: Yaa since here we are taking this as a course and there it was topic	1
	Total	14

From Table 5.25, it can be noted that students' proof experiences differed principally in level of detail and in the terms of fostering justification skills among students. With respect to level of detail, students pointed out that while the focus of pre 'A'-level and 'A'-level Mathematics was on applying rules and procedures of steps to obtain answers, at undergraduate level particular attention needed to be paid to level of detail. Getrude's utterances illustrate the point that has been just made.

> **Getrude**: At A level we were somehow spoon-fed […] given a fact and told how to use it.

Commenting on her undergraduate experiences with proofs, Tanya stated that:

> **Tanya**: Maybe it just depends on the content. Proofs at the lower level do not demand that much. At this level [undergraduate level] we actually prove things, we don't just take things for granted, that is obvious to say $1 > 0$ but now we have to prove that $1 > 0$. We need to give a lot of explanation.

Another excerpt that involved Debra reinforces the point that proof differed in terms of the level of detail.

> **Debra:** Yaa, I think, eee, at a lower level proofs were simple and not detailed but now we are doing proofs in detail.
> **Researcher**: What do you mean by detail?
> **Debra**: Ok, I think we are now being given enough information to show why we say a given concept has to be used.

It can be noted also that although the level of detail had increased the student was given that information. From Debra's responses cited above we see that Debra played a passive role in generating proofs because she was given facts to apply. Debra's response points to a robust external conviction authoritative proof scheme (CadawalladerOlsker, 2011; Harel & Sowder, 2011).

Second, students' proof experiences also differ primarily in the way statements are justified at the various scholastic levels. Students reported that they rarely justified claims made. For instance Getrude mentioned that she used to take it to be obvious that $-1 < 1$ but at a undergraduate she had to convince others that $-1 < 1$. She mentioned that "Now I can support myself though it's still challenging." Tino's utterance reinforced differences highlighted by Getrude in student proof experiences in terms of justifying claims and processes engaged in during proving.

> **Tino**: If you look at 'O'-level and 'A'-level Mathematics you have to know rules and procedures, the steps involved in solving. But now at this level [undergraduate level] I think you have to understand the reasons behind what you are doing […] depth of conceptual understanding is much deeper.

Therefore, unlike at 'O'-level and 'A'-level where emphasis is mainly on obtaining the answer (product) focus at undergraduate level as mentioned by Cortney was on "looking at the basis where the theorems are coming from." Debra's utterance reinforces the point made here that students' verbatim transcriptions illuminated differences in the extent to which justification skills are fostered at different scholastic levels in the following interview transcript.

> **Debra:** I can say that I have been learning that we were just using them [formulas] without knowing where they were coming from, we can prove why we are using the concepts or some of the formulas.

Another category that emerged from content analysis of verbatim transcriptions denoted as Logical presentation was about differences noted in presenting ideas when proving. At undergraduate level students described that there was logical presentation of arguments by means of axioms while at 'O'- and 'A'-levels focus was on what Bea described as "plugging numbers" in formulas, referring here to calculations. Tendai's response reinforces the point made here about use of particular instantiations at lower levels of mathematics learning while at undergraduate level use of axioms and definitions in building arguments became prominent (Alcock, 2010).

> **Tendai**: Maybe at 'O'-level its only substituting maybe we were only calculating.

One of the categories that emerged during summative content analysis encompassed the challenges faced by students when they dealt with proofs at the various scholastic levels. Thus, while students explained that they could justify mathematical claims made and provided detail at undergraduate level, they expressed the fact that such efforts were challenging. For instance, Cortney commenting on her experiences when she was dealing with axioms of a field mentioned that:

> **Cortney**: But I can recall it was a bit challenging because we did not have the basis (pause) somewhat.

By "basis" Cortney was referring to relevant prior knowledge that was pertinent to the learning of axioms of a field. I conclude this section on students' undergraduate proof experiences by noting that volume of work on proof was another category that emerged from summative content analysis of the data. Student highlighted that at 'O' and 'A'-levels proof was treated as a topic while at undergraduate proof was dealt with as a course. The inductive category 'volume of work' is similar to Selden and Selden's (2003) description of Real Analysis as a proof laden course in which students spend substantial amount of time reading and writing proofs. Another dimension that was considered to illuminate the nature of existence, that is, ontology of proof scheme at various scholastic levels are, ways used by student teachers to reach conviction about the truth of a mathematical statement. So in the next section I present summative content analysis results of verbatim transcriptions from the perspective of means of reaching conviction.

5.2.5 Ways to gain conviction

Table 5.26: Mid-instruction reflective interview on ways used by students to attain conviction ($n = 10$).

Category	Example of student utterance	Relative frequency
Use of theories	Tanya: […] should be able to use given theories which support the statement	3
	Getrude: use facts to come up with theorem	4
	Cortney: Provided facts that support the statement	
Use of examples	Taku: trying out integers	3
Methods/stages calculation	Tina: Through the channels or steps reaching a certain answer. […] between the problem and the answer there are steps required […]	3
Logical presentation	Tafa: […] logical presentation of statements that are linked […] arranging argument	4
Contradicting	Tafa: Contradicting original statement	2
	Total	19

Researcher Comments

The dominant means of attaining conviction regarding the falsity/truth of a mathematical proposition was through logical presentation of an argument (4 out 19 utterances) and by proving using facts rules and laws pertinent to the proof task at hand. Student teachers expressed that attaining conviction through use of particular instantiations emerged as a significant category.

From the categories; Methods/stages with (3 out 19 utterances) and Use of theories which had (3 out 19 responses), it was noted that student teachers shared the view that the methods facts and rules should be given to the learner. In other words, the student teachers felt they should be supplied with tools to use. This category reveals the dominance of the authoritative warrant type (Alcock, 2010; Weber & Mejia-Ramos, 2011). Describing methods as means of reading conviction, Taku said:

> **Taku**: Applying methods you have been given.

Tanya expressed the view that to reach conviction theories should be given to the prover to aid the proving activity as was noted in the following utterance.

> **Tanya**: […] should be able to use given theories which support the statement.

What is common in the two responses by Taku and Tanya is the idea that methods and theories employed in proving came from external sources such as the textbook or the teacher. This shows the dominance of the external conviction authoritative proof scheme (Harel & Sowder, 2007).

Next, I present results on inconsistencies observed among undergraduate student teachers as they worked on given proof tasks. Prior to the presentation of results on the inconsistencies, it is crucial that I describe briefly what motivated me to data analysis from this perspective. It was noted during the pilot study and even during main study data collection that student teachers displayed inconsistent behavioural tendencies in the following manner. Student teachers had a tendency to use axioms and definitions in proof tasks that demanded use of counter argumentation. Conversely, student teachers used particular instantiations to tackle tasks that needed the formal deductive approach. These contradictory behavioural tendencies motivated me to explore students' thinking in this regard in order to develop an explanation on students' mental constructs around the notion of proof. So Table 5.27 presents the results of content analysis of students' accounts of the inconsistencies in their formal rhetoric aspects with respect to proof construction.

5.2.6 Mid-instruction interview on student teachers' inconsistent formal rhetoric aspects

Summative content analysis gave rise to the following descriptors that were used to form categories used to account for inconsistent student behaviour when solving proof tasks.

Table 5.27: Mid-instruction interview results on inconsistent student formal rhetoric aspect $(n = 10)$

Category	**Example of student utterances**	**Relative frequency**
Lack of practice	Tendai: I think its lack of practice	1
Culture (use of examples)	Tafa: using numbers […] more closer home	2
Question/statement formulation	Taku: […] its because someone would not yet have grasped […] even the question, what the question requires you to do.	2
Lack of understanding	Tendai: Maybe someone did not understand the axioms.	5
	Total	10

Table 5.27 reveals that the inconsistent behavioural tendencies displayed by students during proof attempts can be explained mainly by lack of knowledge of axioms, definitions, lemmas and theorems pertinent to the proof. The learner could lean towards examples because he/she would have failed to master axioms. For instance the following extracts illustrate the point made here.

> **Getrude**: […] when the learner has mastered the axioms there won't be a problem or a challenge straight away the learner will solve the task accordingly.
> **Taku**: […] if they ask you for a certain aspect application of aspect which seem to be difficult for my own understanding I can simplify it by using examples. For example, they can ask me to apply a certain axiom yet I don't know the axiom, I use examples.

The two extracts reinforce the idea that students used examples in situations that required use of formal deductive reasoning because of their limited knowledge about definitions and axioms pertinent to the proof tasks. The student teachers also explained their use of examples instead of formal deduction in terms of the influence of early learning experiences, that is, 'A'-level and pre

‘A’-level mathematics experiences where calculations were dominant. Tanya’s interview extract illustrates the point made above.

> **Tanya**: I think it’s a result of maybe people not being exposed to deductive reasoning from an early stage because you would find that from the lower levels they are just taught to use calculations in whatever you want to show. Then it is at higher stages that you, start, eee, introduced to axioms.

Summative content analysis revealed that question formulation had led students to use axioms in proof tasks where counter arguments could have been strategic. The following interview extract supports the point made here.

> **Tafa**: Maybe first it’s the statement that leads people to decide whether to use examples. Aaa, at first I thought about axioms and […] I was stuck. I started using actual numbers; they were closer home than axioms. The bottom line is the statement.

From Tafa’s extract we see the effect of a combination two factors: limited knowledge implied by the word “stuck” and the natural tendency to use examples to validate statements because Tafa referred to examples as being closer home. Another category that emerged from summative content analysis and with a lowest frequency (1 out 13) was lack of practice with proof tasks. Tendai explained the effect of lack of practice as failure by students to distinguish between tasks that require counter argumentation from those that could be solved by formal deductive means. I pursued the inconsistencies in students’ formal rhetoric aspects in an effort to develop an explanation about why student approached proof tasks in the manner they did. Table 5.28 presents summative content analysis results of interview audits of students’ accounts of contradictory behavioural tendencies.

Table 5.28: Reflective interview audits on contradictory behavioural tendencies by student teachers ($n = 10$)

Category	Example of student response	Relative frequency
Calculations	Cortney: […] we are used to doing calculations […] you always think of calculations	3
Knowledge of axioms	Tanya: […] so one is just using them but one does not fully understand the axioms so I think maybe its failure to understand the axioms	7
Lack of practice	Debra: So if there is lot of practice there will less problems …	5
Question interpretation	Getrude […] In fact it’s the problem of understanding the question	7
Lack of confidence	Taku: They have no confidence in […] if they use simple examples	2
	Total	24

Researcher comments

It can be noted from the table that the use of examples to tackle tasks that demanded use of axioms and definitions and use of formal deductive argumentation in situations that demanded proof by refutation can be explained mainly through question interpretation and lack of knowledge about axioms and definitions. With respect to lack of knowledge of axioms and definitions student teachers mentioned that in such circumstances students would tend to apply axioms and definitions indiscriminately as illuminated in the following extract.

> **Researcher**: […] You said we have been used to calculations. But how do you explain this scenario where one is supposed to use examples, one now uses axioms?
> **Tanya**: I think it's maybe because of not really understanding them and then one just maybe feels let me use them. But you would find that in most cases that person uses those axioms that won't be the correct approach. So one is just using them but one does not fully understand the axioms so I think maybe it's failure to understand the axioms.

The extract reveals that because of limited knowledge of the axioms students would use them to resolve tasks that demanded use of counter examples. Regarding the inductive code of question interpretation, the student teachers expressed the sentiments that the way the task that has to be solved or the proposition that has to be proved was formulated had consequences on how a prover could approach it. The following extract has students' accounts of how they decided to use axioms when a counter example could have been easier.

> **Researcher**: How do you make a choice between use of examples and use of axioms when given a task?
> **Tanya:** Maybe the way it is asked will direct you on what to use […]. The sign, yaa, because of the sign like $a - b > 0$ you think of order properties of the relationships of the greater than sign.

Tanya was referring to the Mid-instruction assessment task: *For all* $a, b \in \mathbb{R}$, $a - b > 0 \Rightarrow a^2 - b^2 > 0$. The task demanded students to determine its truth/falsity and students were required to justify their answer. As indicated Tanya used what (Alcock, 2010) calls structural reasoning when the task could be proved by refutation. Another reason that can be used to account for use of examples instead of axioms during proving or vice versa relates to lack of practice with proof tasks. Tendai's sentiments were that lack of enough exposure to proving situations interwoven with questions interpretation discussed could be attributed to inconsistencies in student teachers' approaches to proof tasks. The following extract illustrates the impact of lack of exposure to proving situations on students' formal rhetoric behaviour

> **Researcher**: How do you account for the use of examples and axioms interchangeably during proving?
> **Tendai**: Maybe someone didn't understand the question so she may choose to solve either to using examples than axioms.
> **Researcher**: You may choose [interjected]
> **Tendai**: Easier, easier way.
> **Researcher**: Ok, I understand what you are saying Tendai but the question says if $a - b > 0$ then $a^2 - b^2 > 0$. […] Tendai wasn't it easier and neat to use examples? I am responding to your explanation that it's much easier to use axioms?
> **Tendai:** It's much easier to use examples.
>
> **Researcher:** But why then did some of you use, say, order properties?
> **Tendai:** Maybe someone didn't understand where to use axioms about the given question.

Tendai's response revealed that lack of practice twinned with question interpretation would lead to students failing to distinguish tasks that needed proof by refutation from those that required formal deductive argumentation to resolve them. Another inductive category that emerged from summative content analysis was denoted as "calculations." Students' utterance revealed that use of specific numbers had become a "culture" to them and had also become a comfort zone they could turn to when stuck as illustrated by Tanya's utterance.

> **Tanya:** I think it's a result of maybe people not being exposed to deductive reasoning from an early stage because you would find that from the lower levels they are just taught to use calculations in whatever you want to show. Then it is at the higher stages that you start (eee) introduced axioms.

Finally, an inductive category that emerged with minimum frequency encompassed issues related to lack of confidence among students. For instance, Taku stated that students had low confidence in use of simple examples during proof constructions. The inconsistencies in students' approaches to proof tasks were further explored during End-of-instruction reflective interviewing process to try and develop a full picture about why students behaved in the manner they did when they tackled the tasks assigned in the task-based interviews. Table 5.29 illustrates summative content analysis results of students' utterances on reasons for their inconsistent formal rhetoric aspects.

5.2.7 End-of-instruction reflective interview on contradictory proof behaviour

The intent was to account for the use of examples in situations requiring formal deductive reasoning. The other goal was to build an explanation for the use of axioms and definitions to solve proof tasks that can be solved through counter-argumentation. Summative content analysis of textual data led to the following categories.

Table 5.29: End-of-instruction reflective interview on contradictory proof behaviour ($n = 10$).

Category	Example of students' utterances	Relative frequency
Limited knowledge of axioms and definitions.	Tafa: […] you won't be having enough information to put into an argument so you kind of learn towards what is easier. Tanya: […] the fact that people really do not understand […]The axioms one might use even when they do not apply	6
Lack of practice	Debra: […] lack of practice	1
Culture	Tendai: Maybe we are used in examples a lot.	2
Over emphasis of form deductive reasoning	Taku: You need axioms particularly when dealing with analysis.	4
	Total	13

With respect to use of examples in situations requiring use of axioms and definition it, can be noted from Table 5.29 that students' limited knowledge of axioms and definitions contribute significantly as shown by (6 out 16) responses. Students mentioned that they are forced to use examples instead of axioms and definitions because of lack of in-depth knowledge about axioms and definitions that are pertinent to the statement or proposition whose proof a prover would be trying to establish. The point made above can be illustrated using an excerpt from Tafa.

> **Tafa**: […] eee, its just a case of lack of knowledge […] That means you won't be having enough information to put into an argument so you kind of lean towards what is easier.

Tanya reinforced the idea that students have difficult with formal deductive reasoning when she stated that:

> **Tanya**: […] examples easier to use than theorems, easier than axioms.

Turning to the idea, of "culture" that had emerged earlier, it was inferred from students' utterances that examples are sometimes used by student teachers instead axioms and definitions because according to Tanya students are used to the empirical-numeric proof scheme as revealed by the following extract.

> **Tanya**: We are used to solve mathematical problems using numbers. It's a culture, yes.

It can be noted here that students' earlier experiences with proof which are prominently about calculations and procedures of obtaining answers has a bearing on how they approached proof tasks. The tendency by students to use examples to tackle problems that require use of axioms was also attributed to lack of practice that made it difficult for them to differentiate and classify problems according to whether they can be solved using formal deductive means or solved by instantiations (Alcock, 2010; Morselli, 2006).

The other goal was to elicit data that would help to account for use of axioms and definitions when counter argumentation was needed. From Table 5.29, overemphasis of formal deductive reasoning in proof laden undergraduate mathematics courses such as Fundamentals of Analysis emerged as the main constituting factor. For instance, Taku uttered "mathematically when you use an example to testify a proof then its wrong […] You need axioms practically when you are dealing with analysis you do not have to use an example."

Taku's response revealed that any proof constructions that did not involve formal deductive reasoning were not valid and therefore did not count as proof to a proposition. An excerpt from Tino reinforced the over-emphasis of axioms and definitions at the detriment of counter argumentation. In other words Taku had relative conviction in arguments that did not involve axiomatic reasoning (Weber & Mejia-Ramos, 2015).

Researcher: How do you explain the use of axioms and definitions in proving when one is supposed to use an example.

Tino: Maybe because one may fail to recognize that this thing requires an example or woo you are trying to show that you have done some proofs. Uhuu, so may be the idea of trying to pick an example to counter may not cross his mind …

Tino's utterance revealed that formal deductive reasoning was overemphasized to the extent that students could shy away from use of examples as being inferior to someone who had studied mathematical analysis. Limited knowledge about the axioms could have resulted in their indiscriminate use by the students. That could mean axioms and definitions might be used instead of examples because student had a weak command of the axioms as illuminated in Tanya's utterances.

Tanya: I think even the fact people really do not understand or do not have crossed well the axioms one might tend to use them even when they do not apply.

Therefore lack of a profound grasp of the axioms and definitions could force student teachers to use them in situations where counter argumentation was needed more so if that had been coupled by the desire to demonstrate that one had done Analysis. Finally, the use of axiomatic reasoning instead of instantiations could be explained in terms of lack of appreciation of counter examples as illustrated by Cortney's utterances.

Cortney: […] you will always think that what if I just come up with a counter example […] I have to make use of axioms so that the answer [proof] gets balanced.

Cortney's utterance revealed that one would still have the urge to use axioms and definitions even after an appropriate counter example had been found. The lack of appreciation of proof method by refutation expressed by student teachers was also reported by Harel and Sowder (1998, 2007).

In this chapter directed content analysis results of students' responses to Mid-instruction and End-of-instruction proof tasks has been presented and analysed. Analysis of data from the three sources: written responses, chalkboard demonstrations and follow up interviews on students' proof construction efforts were used to address research question: *What kinds of proof schemes characterise undergraduate student teachers' conceptualisations of mathematical proof?* Students were prompted through guiding questions on the basis of their written responses and chalkboard demonstrations. These reflective interview questions aimed to clarify some responses thereby helping students to interrogate what they had written which in turn helped them to construct and re-construct proofs of statements posed in the tasks. Presented in this chapter also are summative content analysis results of verbatim transcriptions of audio-recorded reflective interview data about students' proof experiences as they went through the various scholastic levels. The focus of the study in this regard was on research question two: *How do the undergraduate student teachers* *develop their proof schemes?* The intent of this research question was to develop a hypothesis about how students' proof schemes emerge. Addressing these two research questions were part of the efforts to establish the terms in which student teachers think about proving at undergraduate level.

Chapter Six

Discussion and Conclusion

6.0 General Approach

The study aimed at developing a story from the data that is a coherent representation of students' experiences with mathematical proof. This representation of student teachers' thoughts reflects the grit and complexity in student teachers' encounter with the concept of a mathematical proof. One of the goals of the study was to develop an explanatory theory about the kinds of proof schemes held by undergraduate student teachers. The intent was to establish a set of causal links in student teachers' categories of proof schemes. This goal was pursued by addressing research question one: *what kinds of proof schemes characterise undergraduate student teachers' conceptualisations of mathematical proof?* The idea was to account for students' proof behaviours. So the focus with respect to research question one was on identifying how students construct and explain why they construct proofs in the manner they do. Another goal was to formulate a hypothesis grounded in data about how student teachers develop their proof schemes. This goal was pursued by addressing research question two; *how do undergraduate student teachers develop their proof schemes?*

In this chapter, I discuss findings from the two research questions. The main aim was to explore and explain the kinds of proof schemes held by undergraduate students as well as explaining how students' thinking around mathematical proof evolves. Consequently the discussion is organised as follows. First, individual student's conceptions of mathematical proof are discussed using information on students' proving profiles drawn from column 4 entries of student data matrices for both Mid-instruction and End-of-instruction assessments. Column 4 entries comprised inferences I made about proof scheme elements from column 3 entries of student teachers' data matrices. Individual student conceptions are compared for similarities and differences in order to discern a revealing picture about the kinds of proof schemes held by students (Corbin & Strauss, 2008; Miles, Huberman, & Saldana, 2014). During the discussion the inferences I made about student teachers' proof scheme elements are mapped to theoretical constructs from the analytic framework that include the ideas of a proof event, technical handles and conceptual insights and the notion of intellectual need (Bostic, 2016; Koichu, 2012; Raman, 2003; Sandefur, Mason, Stylianides & Watson, 2013). These constructs were considered within the realist process framework. This part of the discussion leads to a formulation of the overall conclusion to research question one.

Second, with respect to research question two, a summary table was constructed comprising 3 columns. In column 1, the main aspects explored were recorded that include pre-'A'-level and 'A'-level and undergraduate student experiences with mathematical proof. Column 2 consisted of main observations made about students' proof experiences. Finally, column 3 entries consisted of comments I made on students' proof experiences. Findings are then mapped to existing literature in order to identify similarities and distinctive features in the manner in which students' proof schemes evolve (Corbin & Strauss, 2008; Yin, 2009). In-vivo codes are used to support conclusions drawn about the way undergraduate students' proof schemes emerge (Berg, 2009; Corbin & Strauss, 2008). The discussion of results on students' proof experiences leads to formulation of the overall conclusion to research question two.

Efforts to account for the overall conclusions to the two research questions then led to the overall conclusion about the nature of students' formal praxis with respect to proof and proving in mathematics. In other words, interpretation of conclusions to research questions one and two led to the overall conclusion about the terms in which undergraduate student teachers think around the notion of a mathematical proof.

6.1 Discussion of Research Question One Results

Research question one: *what kinds of proof schemes characterise undergraduate student teachers' conceptualisations of mathematical proof?* A realist process approach was employed to observe events and processes in students' proof construction efforts. The notion of a learning event, particularly the concept of a proof event (Bostic, 2016; Moore, 1994) was central in identifying causal mechanisms from students' proof attempts (Maxwell & Mittapalli, 2007, 2010). A proof event is said to have occurred when a conjecture and a justification has been produced (Bostic, 2016). In the context of this study a proof event referred to the production of a justification because the study did not involve conjecturing but rather involved justifying given statements.

6.1.1Tino's proof scheme elements

Tino's Mid-instruction assessment data matrix for the proof tasks is discussed. From Table 5.1, in Chapter Five, Tino's approach to the task: *Describe whether the following statement is true or false. For all real numbers* a *and* $b, a - b > 0 \Rightarrow a^2 - b^2 > 0$; was primarily syntactic because Tino used the structural mode of thinking (Alcock, 2010 in Fukawa–Conelly, 2012). Tino drew from the formal structure of the definitions involved and applied formal deductive reasoning based on order axioms of the real field $\mathbb{R}$. However, structural thinking by Tino lacked criticalness because he did not question, for instance, the implications of multiplying both sides of the inequality by $a \in \mathbb{R}$ to get $a^2 > ab$. Therefore lack of critical thinking (Alcock, 2010) led to the violation of order properties because Tino did not question implications of multiplying by $a < 0$.

It can be noted that Tino could not operate within and did not consider all possible instances from reference theory $\mathbb{R}$. For instance multiplying by $a < 0$ could change the inequality $a > b$ to

$a^2 < ab$. This was a manifestation of the external conviction symbolic proof scheme. Although the student later acknowledged that he had made a mistake during the follow up interview, the working by the student revealed that he did not deploy the right resources at the right time (Wilkerson-Jerde & Wilensky, 2011). From Tino's proof attempt it was also inferred that the premises and the conclusion were disjointed. He had deduced through use of order axioms that $a^2 > b^2$ but he wrote "therefore the statement is false." Furthermore, Tino's proof behaviour revealed some inconsistency in the formal rhetoric aspect (Selden & Selden, 2009). The task could have been resolved through use of counter arguments but Tino opted for axiomatic or structural reasoning (Alcock, 2010).

From Table 5.1, Tino's proof behaviour exhibited during his attempts to the task: *If x is an integer then $x^2 - x$ is even;* revealed that he was not convinced that a single deductive statement can constitute a proof. After deducing that: "Multiply two consecutive integers you get an even number," he then engaged in symbol manipulations to prove the same task an indication he was not convinced by the statement he had written as a proof to the task.

From the same Table 5.1 it can be seen that Tino engaged in non-goal directed dynamic explorations of proof task: *Determine whether the following is true or false. For all real values of x, $f(x) \equiv 2x^2 + 7x - 4$, implies that $f(x) > 0$.* Tino factorised the expression but did not use the factor form to conclude that was false. During these non-goal directed dynamic explorations Tino could have been testing the ground without knowing exactly what to find (Garuti et al.,1998). This might explain why the counterexample picked was not substituted into the factor form of the expression. Further, Tino violated the proof framework by first stating the conclusion: "This is false" prior to proving the statement (Selden & Selden, 2011).

Regarding the student's attempt to the task on sequences: *Prove that the sequence defined by $(u_n) = \frac{n^2-1}{2n^2+3}$ converges*; Tino had an awareness of what the proof construction exercise sought to accomplish, that is, to determine a natural number $N(\varepsilon)$ for which $n > N(\varepsilon) \Rightarrow |u_n - L| < \varepsilon$. While the student engaged in accurate algebraic manipulations he failed to get a sense of how the expressions for the natural number $N(\varepsilon)$ determined showed that the sequence converges. Consequently, Tino could not articulate the conclusion. He admitted during the follow up interview that "Aaa at that stage I had problems in producing that conclusion." Tino also failed to articulate the sort of problems he had in drawing the conclusion and rather concentrated on procedural aspects. Therefore according to the manipulating-getting a sense of-articulating (MGA) construct by Sanderfur, Mason, Stylianides and Watson (2013), Tino could handle the manipulations (M), but failed to get a sense of (G) and therefore could not articulate (A) the conclusion. In other words, according to Koichu (2012) Tino's engagement in the problematic situation (S) represented by the proof task led him to construct a piece of knowledge, $n > \sqrt{\frac{1}{2}\left(\frac{5}{2\varepsilon} - 3\right)}$. However Tino failed to see how the piece of knowledge generated resolves the problem situation S. In other words, Tino lacked an appreciation of the epistemological justification as to why the answer he had constructed resolved the proof task.

I now discuss Tino's End-of-instruction assessment data matrix for the proof tasks. From Table 5.2, Tino's proof attempt to the task: *Define a sequence* (x_n) *inductively by* $x_1 = 1$ *and* $x_{n+1} = \frac{2x_n+3}{4}$. *Prove that* (x_n) *is a bounded monotone sequence and hence determine its limit;* it can be noted that the type of argument used by Tino is empirical. While empirical justifications are known to serve important purposes such as illuminating the underlying mathematical property— in this case empirical evaluations should have revealed that the sequence is monotone increasing but they failed to serve such a purpose because the numerical tests were not accurately performed (Morselli, 2006, p. 6). For example, inaccurate numerical evaluations such as " $x_3 = x_{2+1} = \frac{2x_2+3}{4} = \frac{2\left(\frac{5}{4}\right)+3}{4} = 22$, $x_4 = x_{3+1} = \frac{2x_3+3}{4} = \frac{2(22)+3}{4} = \frac{47}{3}$," led to the wrong conclusion "(x_n) is bounded below and not bounded above. " Hence, Tino's attempt to use structural-intuitive reasoning to explore properties of the sequence was not successful because of wrong empirical evaluations (Weber & Mejia-Ramos, 2011). In other words, Tino failed to benefit from the crucial interplay between use of particular instantiation and formal deductive reasoning because of inaccurate inductive explorations (Goethe & Friend, 2010).

From the same Table 5.2, Tino's proof attempt to the task: *Use the definition of an appropriate limit to prove that* $lim_{x\to 1}\left(\frac{x^3}{x-1} - \frac{1}{x-1}\right) = 3$, is now considered. An ontological oscillation was noted when student switched from use of particular instantiations to the syntactic approach. Structural reasoning (Alcock, 2010) was used to find $\delta(\varepsilon) > 0$ for which the limit of the function exists. Once again in terms of the MGA construct by Sandefur, Mason, Stylianides and Watson (2013), Tino failed to get a sense of how the value of $\delta(\varepsilon) > 0$ he had determined proved that the limit of the function exists. Thus while Tino could access technical handles he had no grasp of the sense of the structural relationship and hence could not articulate the conclusion. Similarly, Tino's engagement in the proof task led to the construction of the piece of knowledge K represented by $\delta(\varepsilon) = \min\left\{1, \frac{\varepsilon}{4}\right\}$ but the student could not see how the piece of knowledge resolved S.

According to scientific realism emotions are part of reality (Maxwell & Mittapali, 2007). Moments of silence, mumbling in an effort to produce the definition of a limit of a function during the chalkboard demonstrations pointed to weak grasp of the notion of a limit by Tino and therefore he could not articulate the conclusion, despite exhibiting deep procedural knowledge. Thus the two forms of knowledge did not complement each other as suggested by Weber and Alcock (2005). In other words there should be interplay between formal and informal mathematics. From both the chalkboard demonstrations and the written attempt Tino failed to provide an epistemological justification as to why $\delta(\varepsilon)$ he had found proved that the function has a limit as $x \to 1$.

From the foregoing discussion of Tino's Mid-instruction and End-of-instruction assessment matrices the following emerged as persistent characteristics of his proving profiles. First, Tino displayed the tendency to use the structural mode of thought in situations that called for use of particular instantiations. Second, the dominance of the external conviction symbolic proof scheme was shown by failure to articulate conclusions. Third, Tino failed to see how answers generated resolved the proof tasks he engaged with (Koichu, 2012).

6.1.2 Tafa's proof scheme elements

From Table 5.3, Tafa's proof efforts to the task*: Determine whether the following statement is true or false. For all real numbers a and $b, a - b > 0 \Rightarrow a^2 - b^2 > 0$;* revealed that the student switched from a higher level to a lower level proof scheme. According to Duval (2002)'s cognitive analysis of argumentation in proving, Tafa's argument that, $a > b \Rightarrow a^2 > b^2$ revealed serious limitations, in students' micro reasoning. Micro reasoning includes the ability to check conditions in which the theorem applies. Process of squaring could have been avoided if the student had checked with other elements from reference theory. Use of deductive argumentation was then abandoned and student switched to semantic approach (Sandefur, Mason, Stylianides & Watson, 2013). Particular instantiations were not also accurately done. For instance, Tafa wrote: $-\frac{1}{2} - \frac{1}{4} > 0$ instead of $-\frac{1}{2} - \frac{1}{4} < 0$. The follow up interview revealed that the ontological oscillation noted was caused by the student's difficulties with formal deductive argumentation. Another violation of the conventions of proving statements in mathematics was observed when the student started arguing from the consequent statement $a^2 - b^2 > 0$, yet the consequent statement conclusion should logically follow the premises (Selden & Selden, 2011).

From the same Table 5.3, Tafa's attempt of the task: *Determine whether the statement is true or false. If x is an integer, then $x^2 - x$ is an integer. Justify your answer:* revealed another violation of the proof framework (Selden & Selden, 2009), because the conclusion was stated prior to adducing evidence. Empirical evaluations done were not consistent with definition of an integer stated by Tafa. The student defended the dominance of empirical proof scheme as a "culture in us" and pointed out he was not used to the other side," referring to deductive reasoning. Fundamental limitation of empirical scheme had not been grasped.

Tafa's written response to the task: *For all real values of* $x. f(x) \equiv 2x^2 + 7x - 4$, *implies that* $f(x) > 0$*;* revealed that the student had access to technical handles pertinent to the proof task (Sandefur et al., 2013). The student had a grasp of the sense of the structural relationship involved and hence could convert the conceptual insight into a proof through the use of strategic technical handles (order axioms) (Birky et al., 2009; Raman, 2003; Sandefur et al., 2013). Hence structural-intuitive reasoning shown through the use of graphical instantiation on the interval $[-4, ½]$ led to refutation of the proposition. In other words Tafa deployed the right resources at the right time (Wilkerson-Jerde & Wilensky, 2011). According to Duffin and Simpson (2000)'s components of mathematical understanding Tafa had developed a repertoire of mathematical resources (counter argumentation skills, order axioms) which were enacted at the appropriate time in a given problem context. Both the chalkboard demonstration and the written response revealed that the student had a profound grasp of the problem centred aspect of the proof task (Selden & Selden, 2009) and that his procedural knowledge and conceptual knowledge complemented each other (Weber & Alcock, 2005).

From Tafa's proof effort to the task: *Prove that the sequence defined by* $(u_n) = \frac{n^2-1}{2n^2+3}$ *converges*; it can be seen that while the student had a grasp of the hierarchical structure of the proof, he articulated correctly what the proving effort was supposed to accomplish but he could not reach the stated goal because he had not built the right resources which one should have enacted to determine the natural number $N(\varepsilon)$ for which the sequence converges. Hence, limitations in his formal rhetoric aspect of the proof attempt shown by inaccurate algebraic manipulations that led to a complex solutions and Tafa could not reach his intended goal (Duffin & Simpson, 2000; Selden & Selden, 2009; Wilkerson-Jerde & Wilensky, 2011). In terms of the MGA construct by Sandefur et al (2013), the student failed to manipulate (M) and hence could not get a sense (G) that the sequence (u_n) converges.

From Table 5.4, it can be observed from Tafa's End-of-instruction assessment data matrix, that he confused the concept of uniform continuity with the process of differentiation. Instead of applying the definition of uniform continuity, Tafa differentiated the function $f(x) = x^2 + 2x - 5$ in order to show that $f(x)$ is uniformly continuous on $[0, 3]$. This "mess" or "chaos" from Tafa's proof attempt illustrates that he had no means of actually creating the proof. In other words, he had a weak command of the construction path of the proof task (Selden & Selden, 2009). Hence, Tafa could not draw from the formal definition of uniform continuity within the reference theory of real-valued functions. In other words, he had not built a repertoire of mathematical resources which he was supposed to enact to the problem context. The non-availability of right resources, Tafa was supposed to deploy points to absence of relevant technical handles and conceptual insights pertinent to the concept of uniform continuity the student had to apply. Because of lack of accessibility to appropriate technical facilities, Tafa differentiated the function. In other words, the student's engagement in the problematic situation did not lead to the construction of a piece of knowledge that could resolve the problem situation (Koichu, 2012). In Koichu's terms, Tafa's did not realize the intellectual need and hence he engaged in irrelevant processes such as differentiation.

From the same Table 5.4, it can also be observed that Tafa's efforts to produce the formal definition of limit of f as $x \to \infty$ contained errors such as "$\delta(\varepsilon) \in \aleph$ such that $x, X \in \aleph$." The articulated goal: "to determine $\delta(\varepsilon) > 0$," is not connected to the solution, $X = \frac{2\sqrt{\varepsilon}}{\varepsilon}$. Hence, while the algebraic manipulations were accurately done, the student did not have a sense of the structural relationships (conceptual insight) of mathematical ideas relevant to the proof task as suggested by Sandefur, Mason, Stylianides and Watson (2013). Thus, while the student deployed appropriate technical handles to determine X in terms of ε, the resources deployed showed no connection with the goal the student had set out to pursue. In terms of MGA construct by Sandefur et al. (2013), the student could perform the algebraic manipulation M, but failed to get the sense of the problem and hence could not articulate the conclusion. This explains why he just wrote "therefore set $X = \frac{2\sqrt{\varepsilon}}{\varepsilon}$," without further elaboration on how X found proved that $f(x)$ has a limit $\sqrt{3}$ as $x \to 1$. It can therefore be argued that although the student could find X in terms of ε he did not understand the meaning or properties of those resources. Michner (1978) cited in Wilkerson-Jerde & Wilensky (2011) notes that mathematical understanding is not only about processing and connecting between different mathematical resources but requires an awareness of the different purposes served by those resources or mathematical items.

The overall characteristics observed from the discussion of Tafa's two data matrices are as follows. First, Tafa's proving profiles revealed "chaos" or "mess" in the student's proving behaviour shown by claims such as $\delta(\varepsilon) \in \aleph$, $x, X \in \aleph$ which are senseless claims that demonstrate that student had weak grasp of the scope of the statements pertinent to the proof task. Second, Tafa's proof attempts showed a serious violation of the proof framework, e.g., consequent statements were provided before providing the premises yet the premises should logically imply the conclusion. Third, Tafa showed that he had no means of creating proofs illustrated by differentiating when he was supposed to draw from the definition of uniform continuity. Therefore Tafa thought of mathematical proof in terms of syntactic derivations using mathematical objects such as $\delta(\varepsilon),\ \varepsilon$. However the manner in which these mathematical items are deployed revealed that Tafa did not have a sense of their meaning.

6.1.3 Tendai's proof scheme elements

From Table 5.5, it can be seen that from Tendai's effort to resolve the proof task: *Determine whether the following statement is true or false. For all real numbers a and $b, a - b > 0 \Rightarrow a^2 - b^2 > 0$*, that Tendai started to argue from the consequent statement $a^2 - b^2 > 0$ represented as a difference of two squares $(a + b)\,(a - b) > 0$. Tendai then removed brackets, expanded the expression and finally wrote: $a^2 - b^2 > 0$. Selden and Selden (2009) assert that one of the five aspects a prover needs to handle mentally and technically is the proof framework which encompasses the conventions of proving statements in mathematics. The proof framework includes the logical structure of different methods of proof. Tendai's proof attempt reveals serious violations of the proof method of direct deduction because as indicated above she started to argue from the consequent statement instead of deducing it from the premises.

From the same Table 5.5, it can be noted that Tendai's written response to the task: *Prove that the sequence defined by $(u_n) = \frac{n^2-1}{2n^2+3}$ converges*; revealed that she had not built a network of resources, in other words, she had not developed an understanding of the concept of convergence beyond instrumental techniques for finding the limit L of the sequence (Duffin & Simpson, 2000). In terms of Duval's (2002) levels of geometric competency in proving she had not even attained the first level of competency in proving as she could not set the premises and conclusion into a sequence of deductive steps. Her proof attempt just comprised two disjointed flawed statements, "$\lfloor u_n - L \rfloor > \varepsilon$", instead of $\lfloor u_n - L \rfloor < \varepsilon$ and "$\frac{1}{2} > \varepsilon$" whose purpose Tendai could not articulate and yet according to Michemer (1978) mathematical understating can be demonstrated by explaining the purpose different mathematical resources serve in resolving a given problem.

Tendai's written response and follow up interview to the task: *For all real values of x, $f(x) \equiv 2x^2 + 7x - 4$, implies that $f(x) > 0$*; illustrates that Tendai had not established contact with underlying ideas pertinent to the proof task. Selden and Selden's (2011) theory of actions in proof construction posits that proving comprises a sequence of mental and physical actions such as reflecting on earlier proofs and taking appropriate physical and mental action. Selden and Selden describe that as the prover acquires experience in proving the sequence of physical and mental actions mould into small situation–action pairs called behavioural schemas. These consist of recognising a situation and then taking an appropriate physical or mental action. Tendai's proof effort was in stark contrast to the notion of behavioural schemas proposed by Selden and Selden in their theory of actions because her proof effort consists of disjointed statements. For instance the supposition: Suppose $f(x) = 2x^2 + 7x - 4 \Rightarrow f(x) > 0$ is not connected to the conclusion ½ $>$ 0. Further the link between statement "$x = ½\ or - 4$ critical" and the conclusion not illuminated.

Scientific realism asserts that emotions and utterances are part of reality and that the emotions and utterances are causally relevant to the explanation of student proof behaviour. During the follow up interview meant to uncover reasons for the impasse experienced by Tendai, the following exchange took place;

> **Researcher**: How did it then lead to this answer [referring to $\therefore \frac{1}{2} \geq 0$]?
>
> **Tendai**: I prefer to use ½ because it is a positive.
>
> **Researcher**: Oh, since it was written greater than 0. So you prefer to take ½?
>
> **Tendai**: Because it is positive than -4.
>
> **Researcher**: But when you look at -4, you will realise that the function will be positive also below -4. So why did you opt for the half?
>
> **Tendai**: [silent looks stuck]

The extract of the follow up interview reveals the *chaos* that characterised Tendai's proof effort when she said: " I prefer to use $\frac{1}{2}$ because it is positive". Tendai had a wrong interpretation of the proof task. She thought that $f\ (x) > 0$ meant that only positive values of x were required, yet the assertion $f\ (x) > 0$ can also hold for negative values of x. Further, emotions expressed; looking stuck and being silent were confirmatory evidence of chaotic proof behaviour and impasse experienced by Tendai. Next, we consider Tendai's proof attempt to the proof task: *Determine whether the statement is true or false. If x is an integer, then $x^2 - x$ is an even integer. Justify your answer.*

Tendai used a single example to assert that: *If x is an integer, then $x^2 - x$ is even*. Her solution showed that she had not developed an understanding that empirical verifications should not be elevated to the status of a mathematical proof (Stylianides, 2011). During the chalkboard demonstrations only two instantiations were done, also further reinforcing the idea.

Presented next is a discussion of Tendai's End-of-instruction assessment data matrix from Table 5.6. First, we consider Tendai's proof attempt to the task: *A sequence (a_n) of real numbers is defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. Prove that (a_n) converges and find its limit*. It can be noted from the proof effort that semantic reasoning or the referential proof scheme was used to explore behaviour of the sequence (a_n). However, the instantiating reasoning process lacked accuracy as shown by $a_2 = 1.84 \; and \; a_3 = 1.832$. This is contrary to Tendai's assertion that (a_n) is a monotone increasing sequence. This shows that Tendai did not reflect on particular instantiations made in making the conclusion that (a_n) is monotone increasing. Hence, there was lack of critical thinking because Tendai did not relate the conclusion made to the particular instantiations she had done (Alcock, 2010).

Furthermore, Tendai violated the proof framework, as the conclusion that the limit is 2 was drawn prior to adducing evidence. A flawed formulation was then used to try and determine the limit of (a_n). Structural mode of thinking (Alcock, 2010) was then applied through use of order axioms which were wrongly applied. For instance, Tendai wrote "$a_n + 1 < 0, \quad 2 - a_n < 0$." The intersection is a null set but Tendai wrote "$a_n = 2 \; \therefore$ this implies that $a_1 < a_2 \quad a_n = 2$ which is the limit because a_n converges." Tendai did not reflect on the meaning of mathematical process she engaged the task but had to drive the ball towards the goal post, "a_n converges" regardless of the legitimacy of means used to reach the goal.

Tendai's proof attempt on the task: *Use the definition of appropriate limit to prove that* $lim_{x\to\infty} \frac{\sqrt{3x^2 + 4}}{x} = \sqrt{3}$; shows that she had not developed a network of resources focal to the notion of limit involving infinity (Duffin & Simpson, 2000). This was shown by the fact that while the student should have set out to find $X \in \mathbb{R}$, she wrongly stated that she needed to determine $\delta(\varepsilon)$. Similar comments apply to Tendai's proof efforts to the task: *Use the definition of appropriate limit to prove that* $lim_{x\to 1}\left(\frac{x^3}{x-1} - \frac{1}{x-1}\right) = 3$. Once again the attempt to produce the formal definition revealed limitations in her command of the notion of a limit. For instance the expression "$0 < |fx - L| < \delta(\varepsilon)$" is flawed in the sense that it contains the quantity $\delta(\varepsilon)$ associated with the domain of the function. Such flaws reinforce the inference that Tendai had not built a repertoire of mathematical resources that could be enacted in resolving proof task (Duffin & Simpson, 2000).

6.1.4 Cortney's proof scheme elements

From Table 5. 7, it can be seen that a semantic approach was employed by Cortney to tackle the task: *Determine whether the statement is true or false. If x is an integer, then $x^2 - x$ is even.* Cortney's grasp of the proof method by counter-argumentation was weak because she continued with empirical evaluations after an appropriate counter had been found: "$0^2 - 0 = 0$ which is neither even nor odd." A vague conclusion "$\therefore x^2 - x$ is an even holds for $x \in$ integers with 0 excluded," was then drawn. The conclusion shows that Tendai had a weak command of the use of counter examples because once a statement fails to hold for just one single case it is refuted.

Cortney's proof attempt to the task: *Determine whether the statement is true or false. For all real numbers a and b, a – b > 0 = > $a^2 - b^2 > 0$;* shows that she had a fragile grasp of the proof method by counter–argumentation. She picked an appropriate counter example: $a = -2$ and $b = -3$ and wrote "we have $(-2)^2 - (-3)^2 > 0$, which is false." This is a wrong statement since $(-2)^2 - (-3)^2 = -5 < 0$. Hence according to Koichu (2012) while Cortney had generated a counter example which refutes the proposition, she did not see how the generated piece of knowledge resolves the proof task as illuminated by her somewhat vague formulations of the conclusion: "∴ $a - b > 0$, implies $a^2 - b^2 > 0$ for for all positive real numbers." Cortney's utterances which are part of reality during the follow up interview confirmed her shaky grasp of the proof method by counter argumentation (Maxwell & Mittapali, 2007). When probed about her apparent confusion Cortney mentioned that she used particular instantiations because order axioms were not within her conceptual reach.

Cortney's written response to the task: *Prove that the sequence defined by* $(u_n) = \frac{n^2-1}{2n^2+3}$ *converges*; revealed the eternity of external conviction symbolic proof scheme (Harel & Sowder, 1998, 2007). Cortney could handle the instrumental aspect comfortably as she could determine the limit of the sequence to be $\frac{1}{2}$. The symbolic expression "∴ $|u_n - L| < \varepsilon$" then sprang from nowhere and no explanation provided. In other words the purpose of the quantity $\varepsilon > 0$ was not specified. Flawed algebraic manipulations then ensued. According to MGA construct by Sanderfur et al (2013), Cortney engaged in algebraic manipulations (M) without getting a sense of the underlying ideas related to the convergence of a sequence and hence could not articulate (A) the conclusion. In terms of Duffin and Simpson's (2000) categorisation of mathematical understanding, Cortney had not developed a network of appropriate mathematical resources she could enact on the given proof problem. Further, Cortney's lack of awareness of the purpose of mathematical objects she manipulated led her to drift out of the reference theory seen by the complex solution $n^2 < \frac{-5.}{4\varepsilon}$. Cortney did not question such a result (Alcock, 2010, Michner, 1978 in Wilkerson-Jerde & Wilensky, 2011; Weber & Alcock, 2005).

From Table 5.8 Cortney's proof effort to the task*: Use the definition of appropriate limit to prove that* $lim_{x\to\infty} \frac{\sqrt{3x^2 +4}}{x} = \sqrt{3}$ reinforces the point made earlier that Cortney engaged in symbolic manipulations without establishing contact with their meaning (Sandefur, Mason, Stylianides & Watson, 2013). The symbolic manipulations were non-goal oriented because while Cortney had stated that "to determine $x \in \mathbb{R}$" she wrote her final answer as "Set $X = \frac{2}{\varepsilon}$" revealing a lack of connection between the articulated goal and the final answer (Garuti, Boero & Lemut, 1998).

Cortney did not show how "$X = \frac{2}{\varepsilon}$" proves that the limit of $f(x)$ as $x \to \infty$ is indeed $\sqrt{3}$. In other words, she did not provide an epistemological justification (Koichu, 2012). The chalkboard demonstration had similar features to the written response. For instance, there was no alignment between articulated goal: "to determine "$x \in \mathbb{R}$" and the final answer "$X = \sqrt{\frac{4}{\varepsilon}}$. " Further, algebraic manipulations done were flawed. For instance, Cortney squared each term inside the modulus sign to get $\left|\frac{3x^2 +4}{x^2} - 3\right| < \varepsilon^2$. Thus Cortney did not get a sense (G) of the underlying ideas and hence could not articulate (A) the conclusion (Sandefur et al., 2013). In other words, the student could not see how the generated piece of knowledge $X = \sqrt{\frac{4}{\varepsilon}}$ resolved the proof task (Koichu, 2012).

Next we consider Cortney's written response to the task: *Prove that f(x) =* x^2 *+ 2x -5 is uniformly continuous on [0, 3].* The external conviction symbolic proof scheme was exhibited (Harel & Sowder, 1998, 2007). Severe flaws were noted in her attempt to produce the formal definition. For instance "$\exists\ x, y \in \aleph$" instead of $x, y \in [0, 3]$. A false condition $0 < |x - y| < \delta(\varepsilon)$ was also stated and $\delta(\varepsilon)$ was not specified. Further, the goal of proving activity not stated. This evidence affirms the inference that Cortney had an external conviction symbolic proof scheme. She could not use the piece of knowledge constructed: "Set $\delta(\varepsilon) = \frac{\varepsilon}{6}$" to justify that the function is uniformly continuous. In other words, she did not discern sense in her manipulations and hence, could not see how the piece of knowledge resolves the proof task (Koichu, 2012; Sandefur et al., 2013).

The mode of thinking exhibited by Cortney's in her proof attempt to the task: *A sequence* (a_n) *of real numbers is defined by* $a_1 = \sqrt{2}$ *and* $a_{n+1} = \sqrt{2 + a_n}$ *. Prove that* (a_n) *converges and find its limit;* was formal deductive reasoning (Weber & Mejia-Ramos, 2011). Cortney demonstrated a good command of the hierarchical structure by articulating what the proving process sought but revealed some limitation with respect to the proof framework (Selden & Selden, 2009). While the student managed to go through the process of mathematical induction, the argumentation process was flawed at the base step when Cortney concluded on the basis of particular instantiations $(a_1 = \sqrt{2},\ a_2 = 1.84775, a_3 = 1.99036)$ that the sequence holds for $a_1\ , a_2, a_3, a_4, \ldots. a_n$. Student then formulated the induction hypothesis and proved the implication. That is, student completed the process of mathematical induction when she had concluded at the base step the sequence was monotone increasing. This was a violation of the conventions of proving things in mathematics whereby the conclusion should logically follow from the premises (Selden & Selden, 2011; Stylianides & Stylianides, 2009). However, Cortney seemed to have developed beneficial behavioural schemas of the convergence criterion of bounded monotone sequence (Selden & Selden, 2011). Beneficial behavioural schemas could be seen in her ability to produce the statement $\sqrt{2} < a_n < 2$, which she used to infer that (a_n) is bounded above and has least upper bound 2. Cortney correctly deduced that (a_n) had limit 2. The interpretation is that Cortney had a sense (G) of manipulations done and hence could articulate the conclusion. In other words, her engagement in the proof task led her to enact appropriate mathematical resources that enabled her to construct a piece of knowledge $\sqrt{2} < a_n < 2$. Further, her awareness of the purpose of the resources employed allowed her to see how the piece of knowledge constructed resolved the proof task (Duffin & Simpson, 2000; Koichu, 2012; Wilkerson-Jerde & Wilensky, 2011;).

6.1.5 Bea's proof scheme elements

Next the discussion of results focus on Bea's proof attempts from her Mid-instruction assessment and End-of-instruction assessment data matrices for the proof tasks. From Table 6.9, Bea's proof attempt to proof task: *Determine whether the following statement is true or false. For all real numbers a and $b, a - b > 0 \Rightarrow a^2 - b^2 > 0$* indicates that an external conviction symbolic proof scheme was exhibited as affirmed by flawed algebraic manipulation done. For instance, for $a^2 > b^2$, an appropriate counter argumentation $\sqrt{3^2} > \sqrt{2^2}$ can be used to refute Bea's claim. Bea also violated the logical structure of proof framework by building her argument from the consequent statement. It can, therefore, be concluded that Bea did not develop a sense of her manipulations (Sanderfur, et al., 2013). Further, Bea did not enact the right resources on the proof problem (Duffin & Simpson, 2000) and that led to a wrong conclusion: "the statement is true."

Second, we focus on Bea's proof effort to the task: *Determine whether the statement is true or false. If x is an integer then $x^2 - x$ is even.* The proof method of counter argumentation was correctly applied. Initially, the student teacher did not justify the conclusion that the statement was false but managed to do so during the follow up interview when Bea explained that: "$x^2 - x$ is an even number is false because 0 is not an even number." From the written effort and utterances during the follow up interview it can be inferred that Bea managed to enact appropriate resources (counter examples) and through her manipulations got a sense (G) of ideas pertinent to the proof task and hence could articulate the correct conclusion (Duffin & Simpson, 2000; Sandefur et al., 2013).

Third, regarding the task: *Prove that the sequence defined by $(u_n) = \frac{n^2-1}{2n^2+3}$ converges;* Bea could only engage in instrumental techniques to get the limit $L = ½$. Beyond this Bea could not even access appropriate technical handles to prove that (u_n) converges (Hanna & Mason, 2014). Hence

Bea had not built the necessary mathematical resources to enact on the proof task (Duffin & Simpson, 2000). Probing during the follow up interview revealed that Bea had no contact with the concept of convergence of a sequence as the following excerpt affirms.

> **Researcher:** How do you define convergence of a sequence?
>
> **Bea**: [silent] I am not well versed in that.

Bea could not describe the underpinnings of convergence of a sequence. For instance, for $\varepsilon > 0$ there is need to find a natural number, $N(\varepsilon)$ for which $n > N(\varepsilon)$ implies that $|u_n - L| < \varepsilon$. Bea admitted that she had not grasped these crucial ideas. Her utterances tended to be centred on procedural aspects rather than on conceptual ideas driving the procedural efforts. The follow up interview thus affirmed the inference that Bea had not developed the concept of convergence of a sequence.

From Table 5.10, we now discuss Bea's End-of-instruction assessment proof attempts. First, we focus on the task*: A sequence* (a_n) *of real numbers is defined by* $a_1 = \sqrt{2}$ *and* $a_{n+1} = \sqrt{2 + a_n}$ *. Prove that* (a_n) *converges and find its limit.* One of the theorems covered in the Real Analysis course was reproduced but the student did not bring it to bear on the task. It can thus be inferred that Bea had not developed an understanding of the theorem recalled (Hanna & Mason, 2014). The theorem reproduced was not accompanied by relevant and accessible technical handles which could have allowed Bea to tackle the proof task. In other words, Bea had not built a network of relevant resources to enact on the proof task (Duffin & Simpson, 2000). According to scientific realism emotions are part of reality (Maxwell, 2004). Bea's failure to access relevant technical handles to deal with the proof task was also seen during the chalkboard demonstration. She read the question slowly reproduced the same theorem which she did not connect to the proof task. She erased the chalkboard many times and uttered senseless statements like: "$a_3 = \sqrt{2 + \sqrt{2} + \sqrt{2}} < a_3$" and *"$a_2 < a_2$"* which pointed out Bea's severe limitation in her knowledge of the convergence criterion for monotone sequences. Statements such as those indicated here illustrate that Bea had not developed the ability to do micro reasoning during her proving efforts (Duval, 2002). She failed to identify crucial elements in her reasoning by not following through the statement "$a_3 = \sqrt{2 + \sqrt{2} + \sqrt{2}} < a_3$" in order to realize that it would lead to $a_3 < a_3$ which is a senseless formulation. This lack of micro reasoning and failure to access relevant technical handles were affirmed during reflective interview when Bea mentioned that she could not remember the concepts. She admitted that she had serious challenges to a point where she could not figure out how to begin proving process.

The last row of Bea's End-of-instruction assessment data matrix contained entries of proof efforts to the task: *Use the definition of appropriate limit to prove that* $lim_{x \to 1}\left(\frac{x^3}{x-1} - \frac{1}{x-1}\right) = 3$. Bea confused the limit of f as $x \to x_o$ with the limit of a function f as $x \to \infty$ as indicated by the expression: "Required to find $x \in \mathbb{R}$ such that if $x > X, |f(x) - L| < \varepsilon$." Consequently, the symbolic manipulations that followed were done without a clear goal and hence the student reached an impasse. It can therefore be concluded that Bea had a weak command of the construction path, that is, the actual means of producing the proof (Selden & Selden, 2011), because of her limited knowledge of the concept of limit of f as $x \to x_o$. Manipulations were done without a grasp of the hierarchical structure of the proof task (Selden & Selden, 2011).

6.1.6 Taku's proof scheme elements

Next, the discussion section focuses on Taku's Mid-instruction assessment data matrix for the proof tasks. From Table 5.11, the first row of this matrix contained entries on Taku's proof efforts to the task: *Determine whether the statement is true or false. For all real numbers* a *and* $b, a - b > 0 \Rightarrow a^2 - b^2 > 0$. Taku's written response revealed severe challenges in Taku's micro reasoning abilities (Duval, 2002). Taku was not mindful of conditions in which the proposition applies squared both sides to get $a^2 > b^2$ in an instrumental fashion without paying attention to the representation system of the proposition (Hoyles & Kuchemann, 2002).

The second row entries contained information on Taku's proof efforts to the task: *Determine whether the statement is true or false. If x is an integer then* $x^2 - x$ *is even*. Taku concluded on the basis of two integers plugged into the expression. The empirical evaluation " (1 – 1) = 0" was an appropriate counter example to warrant refuting the proposition. Taku could not capitalise on this opportunity because the numeric tests were done without micro reasoning (Duval, 2002; Hoyles & Kuchemann, 2002). Taku did not reflect on the scope of the proposition and hence drew a false conclusion. Further, the student had a fragile grasp of the fundamental limitation of the empirical proof scheme that numeric tests cannot be used to represent the general case (CadawalladerOlseker, 2011; Stylianides, 2011, p. 2).

The last row entries of Taku's Mid-instruction assessment data matrix is for the proof task: *Prove that the sequence defined by* $(u_n) = \frac{n^2-1}{2n^2+3}$ *converges.* Taku succeeded in applying the instrumental technique to find the limit of (u_n). However efforts to reproduce the formal definition of convergence of a sequence revealed severe limitation in Taku's knowledge of this concept as can be seen by weird formulations such as: "there is a natural number $\varepsilon > N(\varepsilon)$. The expression "there is "$\varepsilon > 0$" for a natural number "$n > 0$", points to a weak command of elementary ideas such as if

$n \in \aleph$ then by implication $n > 0$. Further, the claim that there is a natural number $\varepsilon > N(\varepsilon)$ also points to Taku's shacky grasp of the radius $\varepsilon > 0$. The student then engaged in manipulations of these symbols in a procedural manner without demonstrating, an awareness of the conceptual insights, that is without a grasp of the structural relationship (Birky et al, 2009; Hanna and Mason, 2014). This affirms Sandefur et al. (2013)'s warning in their description of MGA construct that a learner can disguise and carry out symbolic manipulations without acquiring a true sense of the underlying relationships, and can still articulate the conclusions without developing an understanding of the ideas involved (Ndemo & Mtetwa, 2015). Here Taku could conclude that (u_n) converges on the basis of $N(\varepsilon)$ found but had no contact with the concept of convergence of a sequence.

From Table 5.12, we discuss Taku's end of instruction data matrix. First we consider results for the proof task: *Prove that* $f(x) = x^2 + 2x - 5$ *is uniformly continuous on* $[0, 3]$. The student's conception of the notion of uniform continuity can be described as chaotic because he did not specify the set from which arbitrary elements x and y were selected. The conditions $|x - y| < \varepsilon$ and $|f(x) - f(y)| < \varepsilon$, show that the radius $\varepsilon > 0$, worked for both the domain and range of the function f. Further, the expression for $n>N$ suggests that the student might have confused the convergence of a sequence with the concept of uniform continuity. The interpretation is that Taku was seriously wanting in micro reasoning he failed to identify crucial elements in reasoning during proof efforts as illustrated by $|x - y| < \varepsilon$ and $|f(x) - f(y)| < \varepsilon$ (Duval, 2002). In other words, in terms of the MGA construct by (Sanderfur et al., 2013), Taku accessed relevant technical handles without developing a profound understating of the structural relationships of ideas pertinent to the proof task but could still manage to get $\delta(\varepsilon) = \frac{\varepsilon}{8}$. Because Taku had no sense, G, of the symbols he had handled technically he could not articulate, A, the conclusion. In Koichu's (2012) terms Taku could not see how the constructed piece of knowledge "$\delta(\varepsilon) = \frac{\varepsilon}{8}$" resolves the proof task. Taku's fragile grasp of uniform continuity was revealed when he confused uniform continuity with the idea of a limit. His utterances also indicated that he thought of $\delta(\varepsilon)$ as a natural number, a severe limitation given that this concept is fundamental to the course.

The third row of the End-of-instruction assessment matrix for Taku contains information about his proof attempt to the task: *A sequence* (a_n) *of real numbers is defined by* $a_1 = \sqrt{2}$ *and* $a_{n+1} = \sqrt{2 + a_n}$. *Prove that* (a_n) *converges and find its limit.* The empirical proof scheme manifested. Taku determined the terms a_2, a_3, a_4. Inaccurate computations led to another senseless formulation "$a_1 < a_2 > a_3 > a_4$." Hence, while one of the purposes served by using specific examples is to reveal the mathematical properties or patterns of the focal mathematical idea involved, the particular instantiations failed to serve that purpose because of inaccuracy. The claim "Upper bound is 1.848" is inconsistent with the conclusion that (a_n) is monotone decreasing. It can therefore be concluded Taku failed to identify critical elements of his reasoning (micro-reasoning). For instance, he should have questioned the existence of an upper bound for what he called a monotone decreasing sequence." This is so because a bounded monotone decreasing sequence converges to the greatest lower bound. The proof effort also revealed that Taku had not developed a connected network of mathematical resources to apply to the problem (Duffin & Simpson, 2000; Hoyles & Kuchemann, 2002; Wilkerson-Jerde & Wilensky, 2011).

Fourth, Taku's response to the task: *Define a sequence* $(x_n\ )$ *inductively by* $x_1 = 1$ *and* $x_{n+1} = \frac{2x_n+3}{4}$. *Prove that* (x_n) *is a bounded monotone sequence and hence determine its limit*, is now examined. Taku used specific examples to conclude that "$x_1 < x_2 < x_3 = a_n < a_{n+1}$." This expression shows lack of consistency in notation. Sequence was given in terms of (x_i) but Taku ended up having a_i. It is also vague in the sense that he wrote $x_3 = a_n$ which is somewhat a weird statement. The student concluded on the basis of numeric tests that the sequence "is a monotone increasing sequence." Taku then started to prove by induction that the sequence is monotone increasing. The base step was flawed since he wrote $P(1) < P(2) \Rightarrow a_n < a_{n+1}$. The induction hypothesis was not well formulated and hence the student faced difficulty in proving the implication statement $P_{(k)} \Rightarrow P_{(k+1)}$. The boundedness property was not explored. Efforts to find the limit revealed more chaos as student wrote "$\lim_{x\to\infty} \frac{2x+3}{4} = \infty$," without regarding the subscript. It can be concluded that Taku conceived proving in terms of handling symbols without drawing meaning from the symbols —which is a typical characteristic of the symbolic proof scheme (CadawalladerOlsker, 2011). Duval's (2002) idea of micro-reasoning can be used to explain the weird answer $\lim_{x\to\infty} \frac{2x+3}{4} = \infty$. The student failed to interpret the notation $x_{n+1} = \frac{2x_n+3}{4}$ which he decided to treat the sequence as a linear function, $f(x) = \frac{1}{4}(2x+3)$.

The chalkboard demonstration by Taku on the task affirmed the tenacity of the empirical proof scheme as the student used two instantiations only to conclude that the sequence is monotone increasing. Taku wrote "$x_1 < x_2 < x_3$" and mentioned that "in essence we can just say "$x_n < x_{n+1}$." As was the case with his written effort, he failed to carry out the process of mathematical induction. Consequently he could not prove that (x_n) is bounded and hence he could not find the limit. It can, therefore, be concluded that despite a well articulated hierarchical structure shown by the student's description of what the proof process sought to accomplish, Taku could not prove that the sequence was bounded and also could not determine the limit because he could not access relevant technical handles. The failure to access relevant procedural techniques might have been caused by weak command of convergence criterion of bounded monotone sequences and also weak command of proof by mathematical induction. In Duffin and Simpson's (2000) categorisation of mathematical understanding we describe this as having no connections of mathematical resources that can be used in solving problem. I can argue that the mathematical resources: student's knowledge of induction and the concept of bounded monotone sequence did not exist as a coherent structure and hence could not be enacted on the proof task posed (Duffin & Simpson, 2000).

Follow up interview on the task revealed that the student had indeed confused the concept of limit of a function f as $x \to x_0$ with the concept of convergence of a sequence. It can thus be noted that Taku failed to draw on the representation system (reference theory) of the proof task, that is, sequences and shifted to functions (Selden & Selden, 2009).

6.1.7 Debra's proof scheme elements

Presented now is a discussion of Debra's Mid-instruction assessment data matrix for her proof efforts. From Table 5.13, we first examine her response to the task: *Determine whether the following statement is true or false. For all real numbers a and b,* $a - b > 0 \Rightarrow a^2 - b^2 > 0$. Although Debra violated the proof framework or logical structure of an argument by first concluding that the statement is false prior to adducing evidence for the conclusion, the argument developed to support the conclusion was valid and in terms of arbitrary elements $a, b \in \mathbb{R}$ (Stylianides & Stylianides, 2009). Debra capitalised on the order property: If $a > b$ and $c < 0$ then $ac < bc$ to deduce that $a^2 - b^2 < 0$ which contradicts the consequent statement. The interpretation of Debra's efforts is that she had developed a coherent network of mathematical resources with respect to the order axioms which she mobilised and deployed at the right time to resolve the proof task (Duffin & Simpson, 2000); Wilkerson-Jerde Wilensky, 2011). In other words Debra's engagement with the problematic situation represented by the proof task led her to construct a piece of knowledge and Debra could see how the piece of knowledge constructed resolves the proof task as shown by a correctly articulated conclusion (Koichu, 2012).

Next is Debra's proof attempt to the task: *Prove that the sequence defined by* $(u_n) = \frac{n^2-1}{2n^2+3}$ *converges.* A meaningful aspect of Debra proof attempt was the instrumental technique used to find the limit of the sequence. What followed this procedural technique was complete mess characterised by wrong definition of convergence of a sequence. Debra wrote: "Let $n \in \mathbb{R}$ then there exist $N(\varepsilon) \in \mathbb{R}$ if it converges." According to Wilkerson-Jerde and Wilensky (2011) definitions are complete descriptions of the behaviour or structure of a focal mathematical idea that accounts for all instances of the idea. The definition of convergence of a sequence stated by Debra revealed that she had a fragile grasp of the concept of a sequence, which is a mapping from natural numbers to real numbers and yet Debra thought of the domain of sequences as $\mathbb{R}$ as can be seen by "Let $n \in \mathbb{R}$." She had also a fragile grasp of the concept of convergence of a sequence, also seen by "there exist $N(\varepsilon) \in \mathbb{R}$ " instead of $N(\varepsilon) \in \aleph$. Debra's shacky grasp of these fundamental ideas of sequences might explain the coming into the picture of awkward and disjointed expressions "$\left|\frac{1}{2} + L\frac{1}{2}\right| < \varepsilon$" and "$-N(\varepsilon) \leq L \leq N(\varepsilon)$" which were not explained by Debra. The interpretation of Debra's effort is that her fragile understanding of the crucial ideas made it difficult to access relevant technical facilities to enact on the problem. This was confounded by weak contact with the structural relationships of ideas as shown by wrong definitions and disjointed statements.

Third, we consider Debra's proof attempt to the task: *Determine whether the statement is true or false. For all real values of* $x, f(x) \equiv 2x^2 + 7x - 4$ *implies that* $f(x) > 0$. Debra first applied procedural techniques (technical handles) to factorize the quadratic expression. Debra then tried to apply order axioms to the factor form: $(2x-1)(x+4) \geq 0$. A weak command of order properties was shown as can be seen from the omission of the case, $a < 0, b < 0$ for the property $ab > 0$, for $a, b \in \mathbb{R}$. A semantic approach in which structural–intuitive mode of thinking was employed could be seen from the illustration of inequalities on the number line. Debra did not interpret the question correctly as can be seen from "$x \geq \frac{1}{2}$." This conclusion was not consistent with the consequent part of the proposition $f(x) > 0$, for all $x \in \mathbb{R}$. Hence, an irrelevant conclusion " $\therefore f(x)$ has a solution $x \geq \frac{1}{2}$," was drawn. In other words, Debra did not determine whether the proposition that: *For all real values of* $x. f(x) \equiv 2x^2 + 7x - 4$, *implies that* $f(x) > 0$ was true but rather stated one of the solutions of the inequality $f(x) > 0$.

Finally from Debra's Mid-instruction assessment data matrix we consider her proof attempt to the task: *Determine whether the statement is true or false. If* x *is an integer, then* $x^2 - x$ *is even.* Semantic approach was employed by the student where structural-intuitive reasoning was used to deduce that the statement is true (Weber & Mejia-Ramos, 2011). However, Debra's proof effort revealed that she had not grasped the limitation of particular instantiations because she concluded on the basis of this empirical evaluation that the statement is true. Proof method by refutation was not within her conceptual reach at that instant.

Table 5.14 shows Debra's End-of-instruction assessment matrix for the proof tasks. The first row of the matrix contains information about Debra's attempt on the task: *A sequence* (a_n) *of real numbers is defined by* $a_1 = \sqrt{2}$ *and* $a_{n+1} = \sqrt{2 + a_n}$. *Prove that* (a_n) *converges and find its limit.*

Debra stated the definition which she did not use to tackle proof task but rather she used numeric tests. She determined the terms a_1 and a_2 of the sequence (a_n). The numerical evaluations lacked accuracy and this led to the wrong conclusion that (a_n) is decreasing. Order axioms were applied to the inequality, $(a_n + 1)(a_n - 2) > 0$, formed after applying the definition of a monotone decreasing sequence. Debra then got $a_n + 1 < 0$ and $a_n - 2 < 0$ which was supposed to yield $a_n < -1$ and $a_n < 2$. As was also noted with her earlier attempt to apply order axioms in the Mid-instruction assessment data matrix, Debra used one case of the order properties, precisely the case: $ab > 0 \Rightarrow a > 0$ and $b > 0$. The other case of the order properties ($ab > 0 \Rightarrow a < 0$, and $b < 0$ was not considered. Structural-intuitive reasoning that is, the number line was used. The student wrote: " $-1 < a_n < 2$" which is not a logical consequence of the working presented. It can be concluded that Debra engaged in technical symbolic manipulations without reflecting on their meaning.

From Table 5.14, the third row of Debra's End-of-instruction assessment data matrix for the tasks contains information about her response to the proof task: *Prove that* $f(x) = x^2 + 2x - 5$ *is uniformly continuous on* $[0, 3]$. Debra's proof attempt revealed that she had not developed a coherent network of mathematical resources around the concept of uniform continuity (Duffin & Simpson, 2000;Wilkerson-Jerde & Wilensky, 2011). Consequently, she could not access relevant technical handles because of reported limitations in her grasp of the structural relationships of the focal mathematical idea (uniform continuity). She ended up confusing the concept of uniform continuity with the notion of Cauchy sequences (Hanna & Mason, 2014). For instance, Debra wrote: "to find $\delta(\varepsilon) > 0$ such that $n, m \in [0,3]$ and $n, m > M \in \aleph$, $f(n) < \frac{\varepsilon}{2}$, $f(m) < \frac{\varepsilon}{2}$ $|f(n) - f(m)| < \varepsilon$". The expressions "$n, m > M \in \aleph$, $f(n) < \frac{\varepsilon}{2}$, and $f(m) < \frac{\varepsilon}{2}$," suggest the evocation of the concept of a Cauchy sequence which was then confused with uniform continuity. According Duval's (2002) cognitive analysis of argumentation, Debra failed to do micro reasoning, that is, to check conditions in which concept applies. This might explain why there was a mix up of ideas on uniform continuity and Cauchy sequences. Once again according scientific realism mental concepts are real entities that are causally relevant to behaviour or emotions (Maxwell, 1999, 2004). During the follow up interview the chaos or mess discussed that characterised Debra's proof attempts was affirmed by moments of being silent and stuck and she retorted "So I was just trying to write down something." She attributed this sort of behaviour to lack of practice.

The fourth row entries have information about Debra's proof efforts to the task*: Use the definition of appropriate limit to prove that* $lim_{x \to 1}\left(\frac{x^3}{x-1} - \frac{1}{x-1}\right) = 3$. Despite a few noted flaws in the definition such as $0 < x - x_0 < \delta(\varepsilon)$, which is true for a right hand limit, and inappropriate condition "Set $\delta(\varepsilon) = 1$" instead of $\delta(\varepsilon) \leq 1$, Debra's proof effort reveals that she was aware of the hierarchical structure of the proof as the goal of the proving exercise was well articulated (Selden & Selden, 2009). Further, Debra had a strong command of the formal-rhetoric as can be seen by her ability to mobilise relevant technical facilities, successful factorisation and expressing the expression for $|f(x) - L| < \varepsilon$ as a function of $(x - 1)$. However her knowledge of the structural relationship (conceptual insight) was weak as she could not provide an epistemological justification as to how the constructed piece of knowledge $\delta(\varepsilon) = \frac{\varepsilon}{4}$ showed that the limit of f as $x \to 1 = 3$ (Hanna & Mason, 2014; Koichu, 2012).

6.1.8 Tina's proof scheme elements

Presented next is a discussion of the Mid-instruction assessment data matrix for Tina for the proof tasks. From Table 5.15, we consider Tina's proof efforts to the task: *Determine whether the following statement is true or false. For all real numbers a and $b, a - b > 0 \Rightarrow a^2 - b^2 > 0$.* While Tina's opening statement indicates that he intended to accomplish the proof by contradiction, the mode argumentation that followed illustrates that proof by direct deduction was employed in terms of arbitrary elements a and b. This lack of consistency might account for the impasses reached. Attempts to use structural mode of thought or deductive justification (Alcock, 2010; Weber & Mejia-Ramos, 2011) were then shelved and Tina resorted to instantiating (Alcock, 2010). He was then able to refute the proposition. Thus an ontological oscillation was noted as Tina had to slide down from a higher level proof scheme (axiomatic proof scheme) to a lower level proof scheme —empirical-numeric proof scheme. The chalkboard demonstration and the written response data sources had many similar features. One point that can be made about the chalkboard demonstration is that it revealed the fact that Tina's confidence and appreciation of the method of proof by refutation was doubtful because, when he had found an appropriate counter example, he failed to capitalise on earlier proof attempt and repeated deductive argument that he had used in the written response section.

Second, the discussion focuses on Tina's proof efforts to the task: *Determine whether the statement is true or false. For all real values of x, $f(x) \equiv 2x^2 + 7x - 4$, implies that* $f(x) > 0$. Although the instantiations used revealed that the statement is false, Tina drew a somewhat vague conclusion: "This implies that $f(x) < 0$ for values of $x < 0$ so ≥ 0 for $x > 0$ values." It is therefore doubtful whether Tina had a correct interpretation of the proof task. His conclusion indicates he did not interpret the question properly. Responses pointing to this fact from the follow up interview include utterances such as: "The overall conclusion there might be from $-\infty$ to $+\infty$." Hence, it became clear during follow up interview that Tina had not interpreted the question correctly. Therefore it can be concluded that Tina had a weak command of the hierarchical structure that is, he was clueless of what the proof task demanded (Selden & Selden, 2009).

Third, we focus on Tina's proof attempt to the task: *Prove that the sequence defined by* $(u_n) = \frac{n^2-1}{2n^2+3}$ *converges.* Tina could employ the instrumental technique for determining the limit. What followed after determining the limit $L = \frac{1}{2}$ was a complete mess. Symbolic expressions just sprang from nowhere. For instance Tina wrote "$\varepsilon < \frac{3}{n^2}$" and "$\varepsilon < \frac{1}{n^2}$." He did not describe what the symbols represented, that is, the purpose served by the symbols was not specified as well. The interpretation of Tina's proof behaviour is that he had not developed a network of mathematical resources around the notion of convergence of a sequence and hence, had no grasp of the hierarchical structure, that is, what the proving exercise intended to accomplish (Duffin & Simpson, 2000; Selden & Selden, 2009). This interpretation of Tina's behaviour was affirmed during the follow up interview through responses such as "Actually I was writing for the sake of writing the question." Presumably, Tina referred to the idea that he was putting down something, as some sort of ritual undertaking without establishing contact with underlying ideas. This sort of behaviour is a typical characteristic of the external conviction symbolic proof scheme. The inference that his efforts pointed to the external conviction symbolic proof scheme was supported by Tina's failure to explain meaning and purpose of the quantity, $\varepsilon > 0$.

Fourth, Tina's proof attempts to the task: *Determine whether the statement is true or false. If x is an integer, then $x^2 - x$ is even. Justify your answer*; are now discussed. It can be noted that Tina tried to use structural thinking (syntactic reasoning) to build the argument that an odd number a has the representation: "$odd = \frac{even}{even}$." This can be deduced from his claim that "$a = \frac{\alpha}{a-1}$" and "#." "Since we introduced α as divisible by 2 it means it is even. # Since we know that if 1 is subtracted from an odd number we get an even number, it means $a - 1$ is even." This is a flawed argument that can be refuted using the counter example 7 (odd) = $\frac{21}{3} = \frac{(odd)}{(odd)}$. It can also be noted that the premises were not connected with the consequent statement. In other words, Tina could not conclude. Proof behaviour exhibited by Tina revealed that his repertoire of mathematical resources (skills, objects and knowledge) was wanting in terms of representational forms of odd and even numbers.

According to Wilkerson-Jerde and Wilensky (2011) mathematical definitions are supposed to be complete descriptions of the structure or behaviour of the focal mathematical idea that cater for instances of that focal idea. Tina's definition of an odd number did not meet this criterion as shown by the counter example given. For instance, an odd number n could have been represented by $n = 2k + 1, k \in \mathbb{Z}$ and an even number m as $m = 2j$ where $j \in \mathbb{Z}$. Hence, Tina failed to draw from the formal structure of the definition in his attempts to build a deductive argument to determine the truth/falsity of the given proposition. In other words, Tina's command of the construction path was weak (Selden & Selden, 2009; Wilkerson-Jerde & Wilensky, 2011). Further, Tina's ability to engage in micro reasoning was weak. He should have been able to identify crucial elements in his reasoning especially in cases involving elementary ideas such as "odd $= \frac{\alpha}{a-1}$ " which is $\frac{even}{even}$ (Duval, 2002; Hoyles & Kuchemann, 2002). It can further be argued that such proof efforts lacked critical thinking (Alcock, 2010) as proof attempts reveal that Tina did not question implications of technical symbolic manipulations done.

When probed during the follow up interview about how he could show that the squares of odd numbers are also odd, Tina shelved structural thinking (Alcock, 2010) and used specific examples. Thus an ontological oscillation was noted here as Tina had to slide down on the proof scheme ladder from higher deductive proof scheme to the lower level empirical proof scheme.

From Table 5.16, Tina's End-of-instruction assessment data matrix for the proof tasks is now examined. The first row entries captured information on Tina's proof attempt to the task: *A sequence (a_n) of real numbers is defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. Prove that (a_n) converges and find its limit.* Tina's written effort revealed he had a weak command of the formal rhetoric aspect of the proof. He claimed that *(a_n)* is monotone increasing before adducing evidence. Empirical evaluations done did not back the claim Tina had made. For instance, Tina wrote "$a_2 \ldots = 2, a_3 \ldots = 2 \ a_n \ldots = 2,$" a sequence mapping to 2. Tina then provided an alternative proof that consisted of thick symbolic technical symbolic manipulations that were even overwhelming to the student to the extent that Tina failed to draw a conclusion. For instance, Tina introduced expressions such as "$L + \varepsilon = (a_{n-1} + 2 - \varepsilon^2)^{\frac{1}{2}}$." He then tried to find the binomial expansions resulting in complicated and lengthy expressions which he failed to handle technically.

In terms of Sandefur, Mason, Stylianides and Watson's (2013) MGA construct, Tina engaged in symbolic manipulations without developing a sense G of the symbolism. In other words, his engagement with the problematic proof task did not lead to the construction of a piece of knowledge that could resolve the proof task (Koichu, 2012). This might explain why no conclusion could be drawn. Tina's weak command of convergence criterion for bounded monotone sequence was affirmed by his utterances during the chalkboard demonstration such as “we know that whenever a sequence converges it must be a monotone sequence but we don’t know whether it is an increasing monotone or decreasing.” This is not a necessarily true statement because not all convergent sequences are monotone sequences. Such flaws point to severe limitations in the student’s critical reasoning and micro reasoning because he did not question implications of such statements. Further, Tina did not state the goal he intended to reach by engaging in the technical manipulations. It can also be noted that Tina had no grasp of limitation of empirical verification as he concluded on the basis of instantiations done that (a_n) converges. Similar to the written responses case, Tina shelved particular instantiations and then resorted to symbolic technical manipulations that were so overwhelming to him that he ended mixing up notation. For instance an awkward formulation, $\frac{2x_{n-1}+3}{4} - L < \varepsilon$, was written for this proof task involving sequence (a_n). Tina faced extreme difficulties with the proof task to the extent that he even felt embarrassed by this level of confusion and uttered that: “Aaa actually I am mixing up.” The student looked confused and embarrassed by the mix up. According to scientific realism, mental events, and processes are real entities which are causally relevant to the explanation of individual behaviour (Maxwell, 2004) therefore, severe limitations in Tina’s knowledge of bounded monotone sequences caused serious discomfort to the student to the point of feeling embarrassed.

The third row of Tina’s End-of-instruction assessment data matrix contains information on his written response to the task *Define a sequence* (x_n) *inductively by* $x_1 = 1$ *and* $x_{n+1} = \frac{2x_n+3}{4}$. *Prove that* (x_n) *is a bounded monotone sequence is bounded and hence determine its limit.* Tina’s proof attempt revealed that he had not developed a grasp of method of proof by induction and also that he had not developed an awareness of the fact that empirical verifications cannot be elevated to the status of a proof. Tina concluded on the basis of a single instantiation "$x_2 = \frac{2x_1+3}{4} = \frac{2(1)+3}{4} = \frac{5}{4} = 1\frac{1}{2}$," that was not even accurate, that (x_n) is monotone increasing. The argumentation was loaded with false assertions. For instance, Tina wrote; “As $n \to \infty$ $\frac{3-x_n}{4} < 0$, which means there is a contradiction, so x_n is bounded.” The conclusion drawn is false because Tina had disregarded the subscript and treated sequence (x_n) as a linear function $\mathrm{f}(x) = \frac{3-x}{4}$. This chaotic mix up of concepts made his proof attempt so clumsy that the claim that the sequence (x_n) is bounded cannot be deduced from his working. Attempts to reason structurally were abandoned and an ontological oscillation was observed when Tina tried to use the formal definition of a sequence that was also flawed. For instance Tina wrote “Given $\varepsilon > 0$, there exist a number which is a natural number n such that $n > N$, therefore $x_n - L < \varepsilon$.” This is a flawed definition because n is implied by the sequence (x_n) given so focus should be on finding $N \in \aleph$. The definition given is flawed in the sense that the condition $x_n - L < \varepsilon$ does not accommodate terms of the sequence that are less than L. These terms will render $\varepsilon < 0$ which is a senseless formulation. Rather, Tina should have stated that $| x_n - L| < \varepsilon$.

The flaws in argumentation and false assertions discussed point to a multitude of challenges (chaotic proof behaviour) that Tina faced with the proof task. The proof behaviour shown by Tina can be explained in terms of Duffin and Simpson's categorisation of mathematical understanding. It can be inferred that Tina had not developed a coherent connection of mathematical resources which he could enact in the given proof task. In other words, his lack of contact with the structural relationships of the focal mathematical ideas meant that he could not access relevant technical handles to map the conceptual insight (relationship) into a mathematical proof (Hanna & Mason, 2014; Wilkerson-Jerde & Wilensky, 2011). The mess displayed in the written response and chalkboard demonstration became the central focus of the follow up interview. Probing during the follow up interview affirmed the challenges and flaws in Tina's argument discussed. The following interview excerpt reveals the mix up of the concepts of linear function and sequences by Tina.

> **Researcher**: […] how then did you reverse the sign [referring to how $\frac{3-x_n}{4} > 0$ ended up being $\frac{3-x_n}{4} < 0$].
>
> **Tina**: Aaa actually I was considering a sequence (x_n). Aaa, if you subtract it from 3 then actually as $x_n \to \infty$, it gets larger so you will be subtracting a bigger number.

The excerpt reveals severe limitations in Tina's micro reasoning abilities (Duval, 2002; Hoyles & Kuchemann, 2002). Tina failed to identify crucial elements in his reasoning in the following manner. While he talked about (x_n) as a sequence the fact that "it gets larger" as $n \to \infty$ implies that Tina was in fact thinking of (x_n) as the linear variable, x. Once gain this argument supports the inference that Tina did not get a sense (G) of the underlying ideas of bounded monotone sequences as he engaged with symbolic manipulations. The proof behaviour manifested is typical of the external conviction symbolic proof scheme.

Finally, from Tina's end of instruction assessment data matrix his proof attempt to the task *Prove that* $f(x) = x^2 + 2x - 5$ *is uniformly continuous on* $[0, 3]$ is now discussed. Another mix of concepts was observed. To prove uniform continuity, Tina differentiated the quadratic function (Alcock, 2010). An irrelevant structural-intuitive warrant type, that is, a graphical instantiation of the derivative was then used to illustrate that $f(x) = x^2 + 2x - 5$ is uniformly continuous. Severe limitations were unravelled during the follow up interview such as wrong conception of continuity that can be seen from the following interview excerpt:

Researcher: When do we say a function f is continuous on a set?
Tina: Aaa it is continuous if aaa, if a certain let's say if a given range of numbers aaa, tend to increase within a certain range.

These utterances reveal lack of knowledge of the underlying ideas of uniform continuity Tina engaged with. So Tina had not developed mathematical resources with respect to uniform continuity (Duffin & Simpson, 2000). According to Selden and Selden's (2011) theory of actions in proof construction Tina had not established any behavioural schemas around the notion of uniform continuity and thus could not recognise the problem situation so that he could take appropriate technical handles to map the conceptual insight (situation) into a proof. Tina just acted out of his imagination as can be seen by the utterance "Aaa, I just imagined that when we are saying uniform continuity, something that is uniform is just as gradient at any point is the same as we proceed." These ideas have no connection whatsoever with the concept of uniform continuity.

6.1.9 Tanya's proof scheme elements

Tanya's Mid-instruction assessment data matrix for the proof tasks is now discussed. From Table 5.17, the first row entries captured Tanya's efforts to the task: *Determine whether the following statement is true or false. For all real numbers a and $b, a - b > 0 \Rightarrow a^2 - b^2 > 0$.* Tanya used a single particular instantiation involving $a = 5$ and $b = 3$ to decide that the statement is true. Tanya's proof behaviour points to a weak grasp of the limitation that empirical explorations cannot be used to represent the general case. Similar to what has been noted with this task, Tanya also used a single instantiation $x = 1$ to refute the proposition that if x is an integer then $x^2 - x$ is even. In the written response section, no justification was provided for the conclusion "∴ the statement is false." Tanya's chalkboard demonstration gave two important insights. First, while Tanya declared that "I am going to try to prove by induction" she as a matter of fact did not use the principle of mathematical induction. Rather, she used just a single empirical–numeric test. Second, Tanya justified the conclusion that: If $x \in \mathbb{Z}$, then $x^2 - x$ is not even by saying that "Because $x^2 - x = 0$ when $x = 1$, then the statement is false." Tanya's proof behaviour during her attempts to the task, *If $x \in \mathbb{Z}$, then $x^2 - x$ is even*, reveals that she had a strong command of method of proof by counter-argumentation as she just used a single instantiation to refute the proposition. However, her proof behaviour shows she had not developed the schema for proof method by induction because she confused it with use of examples.

The fourth and fifth rows of the Mid-instruction assessment data matrix for Tanya contain information on Tanya's proof attempt to the task: *Determine whether the statement is true or false For all real values of $x. f(x) \equiv 2x^2 + 7x - 4$, implies that $f(x) > 0$.* From the fourth row it can be seen that Tanya had access to relevant technical handles as shown by successful factorisation of the quadratic expression and use of order axioms to solve the inequality; $(2x - 1)(x + 4) > 0$. An appropriate structural intuitive warrant type, graphical instantiation of the solution led to the conclusion that "$f(x) > 0$ given $(x) = 2x^2 + 7x - 4$." This conclusion lacked clarity and therefore it became the focus on the follow up interview that gave rise to fifth row entries. Tanya explained by means of the graphical instantiation (Alcock, 2010) that it was possible to pick appropriate counter examples from the interval $(-4, ½)$ in order to refute the proposition, which is a convincing explanation.

Tanya's End-of-instruction assessment data matrix for the proof tasks is now discussed. From Table 5.18, the first row entries that capture Tanya's proof efforts relating to the task, *A sequence* (a_n) *of real numbers is defined by* $a_1 = \sqrt{2}$ *and* $a_{n+1} = \sqrt{2 + a_n}$. *Prove that* (a_n) *converges and find its limit.* Tanya's effort reveals that she had strong command of concepts pertinent to the proof task shown by being able to go through the process of proof method of induction to show that (a_n) is monotone increasing. Tanya then capitalised on what she had just established, that is, (a_n) is monotone increasing to prove that (a_n) is bounded. The fact that (a_n) is bounded was then utilised in determining the limit. It can thus be inferred that she had a coherent network of mathematical resources around the concept of bounded monotone sequences and she was able to mobilise and deploy these resources at the right time (Duffin & Simpson, 2000; Wilkerson-Jerde & Wilensky, 2011). In other words, Tanya had strong contact with the structural relationship (conceptual insight) that enabled her to access relevant technical handles (order axioms, factorisation process) needed to transform the conceptual insights into a mathematical proof. In terms of the MGA construct Tanya had a sense (G) of the symbolic technical manipulations done and hence was able to articulate (A) the conclusion that the monotone increasing sequence (a_n) is "bounded and converges to its least upper bound which is 2" (Sanderfur et al., 2013). However, Tanya's efforts revealed a few algebraic glitches, such as the induction hypothesis being alluded to and some slips in use of order axioms such as $a_n - 2 < 0$, or $a_n + 1 < 0$, instead of $a_n - 2 < 0$, and $a_n + 1 < 0$. That is, there was wrong use of the propositional connective "or" by Tanya.

Overall, Tanya's written response illustrates that she mobilised and deployed the right mathematical resources at the right time (Balacheff, 2008; Wilkerson-Jerde & Wilensky, 2011). Her mode of thought in building the arguments can be described as structural thinking (Alcock, 2010, p. 78). She drew from the structure of the formal definitions and used order axioms to tackle the proof task using formal deductive reasoning (Alcock, 2010 in Fukawa-Conelly, 2012). According to Duval's (2002) level of competency in proving Tanya's effort illustrates she had attained the second level of competency as can be seen by being able to turn the premises and conclusion into a proof.

The follow up interview affirmed the inference that Tanya had good command of the problem-oriented part as reflected in the discussion of written attempt. During the follow up interview Tanya demonstrated high micro reasoning abilities as illustrated in the excerpt now presented.

> **Researcher**: [...] so why discarding the negative solution? [Referring to $-1 < a_n < 2$]. What led you to this conclusion? [Referring to $\sqrt{2} < a_n < 2$].
> **Tanya**: [...] Because I had found that the first term for the sequence was $\sqrt{2}$ which is greater than -1, there was no way that (a_n) can take any value that is below $\sqrt{2}$.

High level of micro reasoning was demonstrated as Tanya checked conditions that apply to sequence (a_n) and she was to identify crucial elements in her reasoning by realising that "the first term for the sequence was $\sqrt{2}$ which is greater than -1"

Third row entries contain information on Tanya's proof attempt to the task: *Use the definition of appropriate limit to prove that* $lim_{x\to\infty} \frac{\sqrt{3x^2+4}}{x} = \sqrt{3}$. Tanya's formulation of the definition of f as $x \to \infty$ shows that she had a good command of the hierarchical structure, that is, the goal of the proving attempt being to "determine $X \in \mathbb{R}$ s.t. if $x > X$ then $\left|\frac{\sqrt{3x^2+4}}{x} - \sqrt{3}\right| < \varepsilon$." However, limitations related to the behavioural knowledge of proving (formal rhetoric part) impeded progress. For instance, failure to use the identity: $\sqrt{x} - \sqrt{y} = \frac{x-y}{\sqrt{x}+\sqrt{y}}$, hindered progress despite Tanya's awareness of the proof framework. Tanya's working revealed she had realized the need to use the proof method by direct deduction. So despite a profound grasp of the structural relationships she could not access relevant technical facility that could map the conceptual insight to a mathematical proof (Hanna & Mason, 2014; Raman, 2003). Tanya's proof behaviour affirms that lack of conceptual insight can as much be a hindrance to proving as lack of technical facility. Tanya's chalkboard demonstration kicked off in a similar fashion to the written response section with correct articulation of the intended goal. Symbolic algebraic manipulations improved as student applied the identity: $\sqrt{x} - \sqrt{y} = \frac{x-y}{\sqrt{x}+\sqrt{y}}$. So Tanya cruised comfortably up the point where she got $\left|\frac{4}{\left(\sqrt{3+\frac{4}{x^2}}+\sqrt{3}\right)}\right| < \varepsilon$. However failure to access the "surprise calculation" or conceptual insight $\frac{4}{\left(\sqrt{3+\frac{4}{x^2}}+\sqrt{3}\right)} < \frac{4}{\sqrt{\frac{4}{3x^2}}}$ impeded progress with the proving exercise. So although Tanya had a strong command of the hierarchical structure of the proof reflected in utterances such as "What we want to do here is that we want to find eee, x in terms of a [...] I am failing to uuu, simplify but but I want what I want is that I must get x in terms of ε." So despite a well-articulated formal rhetoric Tanya failed to act on this knowledge. Selden and Selden (2009) say it is not important that behavioural knowledge is articulated but it is essential that the prover acts on this knowledge.

Finally, we consider Tanya's proof attempt to the task: *Define a sequence* (x_n) *inductively by* $x_1 = 1$ *and* $x_{n+1} = \frac{2x_n+3}{4}$. *Prove that* (x_n) *is a bounded monotone sequence and hence determine its limit.* It can be noted that the chunk of reasoning displayed by Tanya was very much similar to that depicted in the third row with the following improvements. First, her proof construction efforts were no longer hindered by glitches in technical symbolic manipulation as illustrated by well executed proof method by induction that enabled Tanya to prove that (x_n) is a monotone increasing sequence. She then capitalised on the definition of monotone increasing sequence $x_{n+1} - x_n > 0$ to deduce that (x_n) is bounded. After showing that (x_n) is bounded, that is, $1 < x_n < 3/2$. Tanya inferred that the limit of the sequence is $\frac{3}{2}$, which is the least upper bound of (x_n). So Tanya had a profound grasp of the convergence criterion for bounded monotone sequences. From Tanya's proof behaviour it can be inferred she had built beneficial behavioural schemas (Selden & Selden, 2011).

According to the theory of actions in proof constructions by Selden and Selden (2011), Tanya had established behavioural schemas in the form of persistent mental structures around the concept of monotone sequence. Her proof behaviour illustrates that her behavioural schemas had moulded into small situation-action pairs as she could apprehend situation, e.g., recognising to the need to show that the sequence is monotone increasing and then took an appropriate mental or physical action such as implementing proof by induction (Selden & Selden, 2011). So Tanya's engagement with the problematic situation (proof task) led her to construct a piece of knowledge; $1 < x_n < 3/2$, that resolved the problem situation. Further, Tanya could see how the piece of knowledge generated resolved the proof task, as seen by a well-articulated conclusion that resulted from a coherent reasoning chunk. In other words, Tanya provided an epistemological justification for the proof task (Koichu, 2012). Tanya's proof behaviour also indicates that not only was she able to connect between different mathematical resources relevant to the convergence criterion of bounded monotone sequences, but she also showed an awareness of the purpose of those different resources (Michner, 1978 in Wilkerson-Jerde & Wilensky, 2011).

6.1.10 Getrude's proof scheme elements

The discussion now turns to Gertrude's two data matrices for the proof tasks. In Table 5.19, the first row entries of Getrude's Mid-instruction assessment data matrix contains data on Getrude's written response to the task: *Determine whether the following statement is true or false. For all real numbers* a *and* $b, a - b > 0 \Rightarrow a^2 - b^2 > 0$. From Getrude's proof effort it can be noted that flawed argumentation processes were employed e.g., "If $a - b > 0$ then $a > b$ (by the law of trichotomy)." It is not clear how the mathematical tool, Trichotomy law, led to the deduction that if $a - b > 0$ then $a > b$. The same argument can be used for the student's claim that application of Trichotomy law leads to a contradiction. This sort proof behaviour illustrates that Getrude manipulated symbols without establishing a sense of the symbols. False claims and flawed argumentation process show that Getrude had a weak command of the construction path of the proof task and hence, could not mobilise and deploy relevant resources in the form of technical handles and conceptual insights to tackle the proof task (Balacheff, 2008; Hanna & Mason, 2014; Selden & Selden, 2011). In other words, Getrude had not developed means for creating the proof and hence engaged in flawed processes because she had not built a connection of resources pertinent to the proof task (Duffin & Simpson, 2000). Getrude explained her indiscriminate use of mathematical resources as being a result of lack of experience with proof tasks that require counter argumentation. She was also persuaded to use structural reasoning because of the symbol " $>$ " that evoked order axioms from her proof and theorem images (Hanna & Mason, 2014).

Fourth row entries capture Getrude's written proof efforts to the task: *Determine whether the statement is true or false. If* x *is an integer, then* $x^2 - x$ *is even. Justify your answer.* Getrude's propensity for formal deductive argumentation was also seen in this task as she tried to apply mathematical induction to prove the task. The statements "setting x to 1," and "If $x = 2$ then…" presumptively constituted the base step of empirical evaluations and the expression "setting $x = k$" referred to the induction hypothesis. Getrude then used the substitution $x = k + 1$ in order to establish the implication statement $P_k \Rightarrow P_{k+1}$. Numeric tests were then used to evaluate whether $x^2 + 3k + 2$ is even when the implication statement $P_k \Rightarrow P_{k+1}$ had been stated. The argument developed by Getrude was flawed in the following manner. The use of mathematical induction when x is said to be integer was flawed because the principle of mathematical induction is used to prove propositions of the form $P_{(n)}$, for all $n \in D$ where D is the set of natural numbers and $P_{(n)}$ is an open statement asserting the relationship between elements of natural numbers (Stylianides, Stylianides, & Philippou, 2007). Getrude did not realise the fact that integers also include negative whole numbers and 0 which are not natural numbers. This shows lack of micro reasoning because the student failed to identify conditions in which the proposition applies (Duval, 2002; Hoyles & Kuchemann, 2002). The conclusion drawn that "∴ if for any value of k, $(k^2 + k)$ and $(k^2 + 3k + 2)$ are both even, it is true that $x^2 - x$ is an even number," is vague because it is not the correct articulation of the conclusion for proof the method of mathematical induction. The switch from structural reasoning to empirical proof scheme by Getrude illuminated another instance of an ontological oscillation where she had to move from an external conviction symbolic proof scheme to an empiric-numeric proof scheme (Alcock, 2010). Getrude's proof scheme cannot be labelled analytic proof scheme because, as noted, although she used arbitrary elements to build her arguments, she failed to recognise properties of the representation system of integers. Therefore she engaged in technical symbolic manipulations while disregarding the reference theory, which is a typical characteristic of the external conviction symbolic proof scheme.

Getrude's proof attempt to the task: *Prove that the sequence defined by* $(u_n) = \frac{n^2-1}{2n^2+3}$ *converges* revealed some limitations in her knowledge of convergence of a sequence. Flaws in argumentation include the condition $n \le N(\varepsilon)$ instead of $n > N(\varepsilon)$) and awkward formulation "considering only the positive by our triangle inequality for the expression; "$\varepsilon + \frac{1}{2} < \frac{n^2-1}{2n^2+3} < \varepsilon + \frac{1}{2}$," and the radius $\varepsilon > 0$, which is fundamental to the concept of convergence was not described. Further, severe flaws in her technical symbolic manipulations led to a senseless result "$i\sqrt{\frac{5}{4\varepsilon} + \frac{6}{4}} < n.$" Proof behaviour by Getrude reveals that she had no contact with ideas connected to the concept of convergence of a sequence. False claims about the triangle inequality and the inequality $n \le N(\varepsilon)$ point to the fact that Getrude had not built a network of resources to enact in the problem context (Duffin & Simpson, 2000). In other words, she manipulated the mathematical objects without getting a sense, G of the underlying ideas (Sanderfur et al., 2013). The senseless statement "$i\sqrt{\frac{5}{4\varepsilon} + \frac{6}{4}} < n.$" illustrates that she did not engage in micro reasoning in her attempts to build the argument. She should have identified crucial elements in her reasoning if she had checked conditions of the reference theory that is, a real sequence. She should have questioned the meaning of the complex solution given that (u_n) is a real sequence. Therefore, there was no critical reasoning by Getrude when she engaged with the proof task (Alcock, 2010; Duval, 2002; Hoyles & Kuchemann, 2002).

Finally, we consider Getrude's End-of-instruction assessment data matrix for the proof tasks. From Table 5.20, first, we consider Getrude's proof behaviour when she engaged with the task *Prove that* $f(x) = x^2 + 2x - 5$ *is uniformly continuous on* $[0, 3]$. Getrude's proof attempts revealed limitations in conceptual knowledge seen in such statements as "$|x - y| < \frac{\varepsilon}{8}$ which is independent of [0,3]," and the radius $\varepsilon > 0$ a fundamental idea not mentioned. The expression “which is independent of [0,3]" is not a correct a characterisation of property $\delta(\varepsilon) > 0$ with respect to uniform continuity. Getrude should have articulated that $\delta(\varepsilon) > 0$ she had determined should work for every $x \in [0\ 3]$. The fact that Getrude did not explain how the specified $\delta(\varepsilon) > 0$ demonstrates that $f(x)$ is uniformly continuous points to the idea that in terms of the MGA construct by Sanderfur et al. (2013), she engaged in symbolic manipulations without establishing contact with underlying ideas and hence, she could not articulate how $\delta(\varepsilon) > 0$ illustrates that the the quadratic function $f(x) = x^2 + 2x - 5$ is uniformly continuous on [0 3]. In Koichu’s (2012) terms Getrude failed to provide an epistemological justification because she did not see how the constructed piece of knowledge $\delta(\varepsilon) = \frac{\varepsilon}{8}$ resolves the problematic situation (proof task) she engaged with. The chalkboard demonstration affirmed limitations discussed under the written response section, such as, failure to specify $\varepsilon > 0$ and the conclusion that “set $\delta(\varepsilon) = \frac{\varepsilon}{8}$ which is independent of [0 3]." The essential condition $\delta(\varepsilon) \leq 1$ was not also specified. However, one improvement in her proof attempt was that of being able to justify the step: “Let $x = y = 3$.” This step was supported by the utterance: “Then from the interval I am going to choose a value that is going to give me a minimum of the $\delta(\varepsilon)$.”

The omission of the crucial condition $\delta(\varepsilon) \leq 1$ resulted in the value of $\delta(\varepsilon)$ being given as $\delta(\varepsilon) = \frac{\varepsilon}{8}$ instead of $\delta(\varepsilon) = \min\{1, \frac{\varepsilon}{8}\}$. Such omissions point to low level in micro reasoning abilities about uniform continuity which are consequences of weak conceptual knowledge. Hence, Getrude had strong procedural knowledge as shown by being able to access relevant technical handles but her weak grasp of heuristic ideas may account for the flaws in her argument (Duval, 2002; Hanna & Mason, 2014; Raman, 2003).

Getrude’s proof attempt to the task: *Define a sequence* (x_n) *inductively by* $x_1 = 1$ *and* $x_{n+1} = \frac{2x_n+3}{4}$. *Prove that* (x_n) *is a bounded monotone sequence and hence determine its limit* is discussed next. Getrude had a weak command of the proof framework, that is, the conventions of doing proofs (Selden & Selden, 2011). She violated the logical structure by concluding on the basis of a particular instantiation that the sequence (x_n) is monotone increasing. After concluding she then tried to prove by induction that (x_n) is monotone increasing. Proof by induction was not well executed as seen by statements such as "$x_k - x_{k-1} = 1/2\,(x_k - x_{k-1})$ “a wrong statement used to deduce that "$x_{k+1} > x_k$,” at the induction thesis stage.

Despite the challenges noted about the proof framework her proof behaviour showed a good command of conceptual ideas pertinent to the proof task. She demonstrated a coherent chunk of reasoning by capitalising on the connection: monotone increasing → bounded → determination of limit. That is after proving that (x_n) is monotone increasing, she used the definition of a monotone increasing sequence, $x_n > x_{n-1}$, to prove that (x_n) is bounded. She then used the fact that (x_n) is bounded above by $3/2$ to deduce that its limit is $3/2$.

The coherent chunk of reasoning illustrates that Getrude had developed a coherent (status of resources) network of mathematical resources that she could access in the form of relevant technical handles and conceptual insights and brought to bear (enacting) on the proof task (Duffin & Simpson, 2000; Hanna & Mason, 2014). In terms of Theory of Actions in proving Getrude had developed beneficial behavioural schemas seen by her ability to recognize situations (e.g. the need to explore boundedness) and then taking an appropriate mental or physical situation (e.g., using the definition of monotone increasing sequence to establish that $1 \leq x_n \leq {}^3/_2$). Therefore Getrude had a sense (G) of the technical symbols she manipulated (M) and hence could articulate (A) the ideas involved. For instance, she remarked "I realised that uhuu, my sequence was lying between this range [referring to $1 \leq x_n \leq {}^3/_2$] […] so this one, 3/2 becomes the limit because of what the monotone increasing sequence." However some statements by Getrude were not true. For instance, "If a sequence is monotone increasing it means it means it is bounded above." This is not a necessarily true statement because there are monotone sequences that are not bounded.

Finally, we discuss Getrude's proof behaviour when she tackled the task: *Use the definition of appropriate limit to prove that* $lim_{x\to\infty} \frac{\sqrt{3x^2+4}}{x} = \sqrt{3}$. Her written attempt reveals that she was aware of the goal the proof construction exercise intended to achieve, that is, she had a good command of the hierarchical structure because she stated the need to determine "$X \in \mathbb{R}$ such that if $x > X$ then $|f(x) - L| < \varepsilon$. $\left|\frac{\sqrt{3x^2+4}}{x} - \sqrt{3}\right| < \varepsilon$." Not only was she able to state the formal definition, she also managed to make correct substitution to get, $\left|\frac{\sqrt{3x^2+4}}{x} - \sqrt{3}\right| < \varepsilon$. She could not make progress when she reached this stage, so the following remarks are from the follow up interview meant to probe reasons behind the impasse reached. Getrude explained that she was discouraged by the two square roots referring here to $\sqrt{3x^2+4}$ in the expression for $f(x)$ and $\sqrt{3}$, the limit. She gave her main challenge as failure to handle the technical symbolic manipulation. The following interview excerpt illuminates these challenges:

Researcher: You seem to be very much aware of processes. Can you suggest reasons why people do not make progress despite having a lot of knowledge about [interruption from Getrude].

Getrude: Lack of computations in Algebra even if know the formula, I can't operate the algebraic performances, I can't proceed.

In conclusion, it can be noted that although she could articulate the proving exercise and describe useful identities needed to prove that the limit exists, it is crucial that the prover should have executed the behaviour articulated so that insights into her thinking as she engaged the task would be gained. From the discussion of students' proving profiles the following table of students' composite profiles was constructed.

Table 6.1: Composite profiles of students' proof behaviours from written responses

Student teacher	Summary of proof behaviours observed (proof scheme elements observed)
Tino	• Syntactic thinking (Weber & Alcock, 2004) therefore inconsistent formal rhetoric • violation of proof construction conventions • lack of critical thinking • did not adhere to properties of reference theory • symbolic technical manipulations done without getting a sense of object • inaccurate empirical-numeric tests • external conviction symbolic proof scheme
Tafa	• No sense of structural relationship (Hanna & Mason, 2014) • Ontological oscillation observed, e.g., student had to slide from higher level axiomatic proof scheme to a lower level symbolic proof scheme • Inconsistent formal rhetoric • Limitations in micro reasoning (Duval, 2002) • Violations of proof conventions e.g., conclusion drawn before adducing evidence • Could access both technical handles and conceptual insights for some tasks • Articulated goals not reached with some proof tasks • No awareness mathematical resources employed, that is, no micro reasoning done • Mix up of confusion e.g., differentiated as a way of establishing uniform continuity
Tendai	• No network of resources around pertinent mathematical ideas to the proof task • No proving beyond instrumental techniques for finding to the limit L • Disjointed and flawed statement formulations e.g., disconnection between answer obtained and goals articulated • Wrong interpretation of proof task • Single example used to draw conclusion • Students did not reflect on meaning of instrumental techniques used • Serious flaws in definitions, e.g., $0 < \|fx - L\| < \delta(\varepsilon)$ –indiscriminate use of symbols • Particular instantiations used for the general case
Cortney	• Referential proof scheme with structural-intuitive mode of thinking (Weber & Mejia-Ramos, 2011) • Weak command of proof by refutation, e.g., continued with empirical-numeric tests even after finding an appropriate counter • Flawed technical symbolic manipulations • Lack of micro reasoning abilities e.g., did not question $n^2 > \frac{-5}{4\varepsilon}$ (complex solution outside reference theory of real sequences)
Bea	• Flaws in algebraic manipulations (external conviction proof scheme) • Inconsistent formal rhetoric e.g., use structural or analytic argumentation when refutation is needed • Could not access relevant technical handles

	• Weak construction path (Selden & Selden, 2009) ended up mixing ideas e.g., limit of f as $\to \infty$ confused with limit of f as $x \to x_o$
Taku	• Micro reasoning challenges e.g., did not check conditions of reference theory in squaring $a > b$ to get $a^2 > b^2$ (Hoyles & Kuchemann, 2002) • Inconsistent formal rhetoric aspect • Empirical proofs used for the general case • Weird formulations no contact with meaning of symbols e.g., "$\varepsilon < N(\varepsilon)$" • "Chaotic" mix up of ideas e.g., $\lvert x - y \rvert < \varepsilon$ and $n > N$ in uniform continuity • Ambiguity in formulation of conclusion e.g., a monotone increasing which started and then decreased (micro reasoning challenges)
Debra	• Violation of logical structure • Coherent network of mathematical resources developed • Chaotic proof behaviour, e.g., uniform continuity confused with Cauchy sequences • Weak contact with structural relationships " I was just try into write down something • Structural-intuitive reasoning shown • Use of particular instantiations
Tina	• Ontological oscillation; structural mode of thought shelved and instantiations used • Grasp of proof method by refutation weak continued instantiating when counter had been found, chalkboard demonstration student repeated formal deductive proof attempt • Proof is some form of symbol manipulation without contact with underlying ideas-external conviction ritual proof scheme Inconsistent formal rhetoric part • Micro reasoning challenges • No coherent connection of mathematical resources e.g., weak command of convergence criterion of bounded monotone sequences • Empirical evaluations used to represent general case • "chaotic" proof behaviour e.g., mixing concepts-uniform continuity confused with differentiation. "Aaa, actually Im mixing up" [looks confused and embarrassed] • Complicated technical symbolic manipulations
Tanya	• Single instantiation elevated to status of proof • Schema for mathematical induction not developed, it was confused with use of examples • Relevant technical and conceptual insights accessed e.g., graphical instantiation of $f(x) = 2x^2 + 7x - 4$ well linked to factorised form and use of order properties i.e. fruitful interplay, "pairing" between semantic and syntactic forms of argumentation (Alcock & Weber, 2005) • Strong command of method of proof by refutation • A coherent connection of mathematical resources around the concept of bounded monotone sequence had been built (Duffin & Simpson, 2000) • Had a sense of technical symbolic manipulations but algebraic glitches with use of order axioms • Drew from formal structure of definition • High level micro reasoning shown by considering the representation system (reference theory) when proving e.g., epistemological justification provided for neglecting solution $-1 < a_n \leq 2$ and accepting $\sqrt{2} \leq a_n \leq 2$ • Failure to access "surprise calculations" e.g., student could not recognize that • $\left\lvert \frac{4}{\sqrt{3x^2+4} \ \ +\sqrt{3}} \right\rvert < \left\lvert \frac{4}{\sqrt{3x^2}} \right\rvert$
Getrude	• Flawed argumentation processes "chaotic" proof behaviour e.g., induction confused with use of example, "$n \leq N(\varepsilon)$" instead of $n > N(\varepsilon)$ for convergence of a sequence • Indiscriminate use of mathematical symbols e.g., use of order axioms of a field when a counterexample was appropriate for the task: $a - b > 0 \Rightarrow a^2 - b^2 > 0$ • Severe flaws in technical symbolic manipulations e.g., $i\sqrt{\frac{5}{4\varepsilon} + \frac{6}{4}} < n$ senseless result given that reference theory is real sequences and n should be a natural number • No contact with structural relationships between mathematics, student's focus was

	instrumental techniques • Micro reasoning challenges

6.1.11 Discussion of inconsistent student proof behavioural tendencies

Next, I present a summary of inductive codes that emerged from the summative content analysis of data on student's inconsistent proof behavioural tendencies displayed during their proof attempts. Briefly, the contradictory behavioural tendencies were first noted during piloting and they were also a prominent feature of the main study hence the motivation to do further exploration of this aspect.

Table 6.2: Reflective interview results on inconsistencies in student proof behaviour

Aspect	Main observations
Mid-instruction reflective interview	• Limited knowledge of axioms, definitions and theorems. Students face difficulties in using structural reasoning with these objects (Alcock, 2010) • Culture (use of examples). Students indicated that they lean towards what is familiar (Morselli, 2006) • Question formulation • Lack of practice
Reflective interview audits for mid-instruction reflective interview	• Limited knowledge of axioms • Question interpretation • Lack of practice • Calculations (use of examples) • Lack of confidence
End of instruction reflective interview	• Limited knowledge of axioms, definitions, and lemmas • Use of examples (culture) • Over emphasis of formal deductive reasoning • Lack of practice

Regarding the use of particular instantiations in proof tasks that required axiomatic argumentation, students' descriptions revealed that this inconsistent formal rhetoric behavioural tendency was caused by limited knowledge about axioms, definitions and lemmas pertinent to the proposition to be proved. These mathematical objects are essential ingredients for structural reasoning (Alcock, 2010; Weber & Mejia-Ramos, 2011). Limited knowledge about axioms emerged as a very robust causal factor during End-of-instruction assessment data collection phase thereby revealing the students' discomfort with proof laden Real Analysis course. The students' discomfort with the analytic proof scheme forced them to resort to particular instantiations. So failure to access these objects required for formal deductive reasoning would then force student teachers to react by leaning towards use of examples. Typical exemplifications of students' descriptions that revealed students' difficulty with axiomatic reasoning include:

Taku: […] if they ask you for a certain aspect application of aspect which seem to be difficult for my own understanding I can simplify it by using examples. For example, they can ask me to apply a certain axiom yet I don't know the axiom, I use examples.

Tafa: Maybe first it's the statement that leads people to decide whether to use examples. Aaa, at first I thought about axioms and […] I was stuck. I started using actual numbers; they were more closer home than axioms. The bottom line is the statement.

The two extracts indicate that Tafa and Taku resorted to instantiating when they had faced challenges in accessing axioms and definitions, but generally they would have shown a preference for deductive argumentation. So when students were 'stuck', they then switched to empirical explorations. During the three phases of interviewing limited knowledge about axioms emerged as a dominant factor in the search for an explanation for student teachers' tendencies to use examples in proof tasks that demanded formal deductive reasoning.

Question interpretation was another category that emerged during summative content analysis of textual data. Question interpretation featured during the Mid-instruction interviewing and the reflective interview auditing sessions and vanished during the End-of-instruction interviewing sessions. A possible reason why this strong causal factor (question interpretation) did not appear in the final reflective interviewing phase is that student teachers might have developed behavioural schemas (Selden & Selden, 2011), as they acquired experience in proving during their learning of the Real Analysis course

Regarding student teachers' tendency to employ the axiomatic proof scheme to tackle proof tasks that demanded proof by refutation an inductive code that can be used to account for such student proof behaviour was denoted as "over-emphasis." Students' sentiments during the reflective interviews showed that conceptual justifications or axiomatic proof schemes (Balacheff, 1998; Harel & Sowder, 1998, 2007; Stylianides & Stylianides, 2011) are highly appreciated in terms of mathematical sophistication so that other means of eliminating one's doubts about the truth of a mathematical statement that do not embrace axiomatic argumentation is not regarded as credible. Typical exemplifications of this main observation are as follows.

> **Taku**: You need axioms particularly when dealing with analysis
> **Cortney**: […] you will always think that what if I just end up with a counter example […]

Cortney and Taku's utterances point to the robustness of the category, "over-emphasis." Formal deductive reasoning had gained prominence in proving to a point where any proof attempt devoid of analytic reasoning was deemed futile by the undergraduate students as supported by Taku's utterance that "mathematically when you use an example to testify a proof then its wrong […] You need axioms particularly when you are dealing with analysis you do not have to use an example. I have to make use of axioms so that the answer [proof] gets balanced."

These responses indicate that the proving process is conceived in terms of use of axioms and definitions. Hence, the belief that proof construction is about validating propositions using structural thinking had become a leading belief among undergraduate student teachers. Briefly, the construct of a leading belief proposed by Furighetti and Morselli (2011) focuses on detecting reasons behind one's decisions. In the context of this study, a leading belief is a belief that influences the way a student teacher engages with the concept of a proof. Decisions driven by a leading belief may include decisions about proof method to use in a task. The belief that proof construction is primarily accomplished through formal deductive reasoning was firmly established in students to the extent that they no longer trusted proof by counter examples as can be noted from Taku's comment that "I have to make use of axioms so that the answer [referring to proof] gets balanced." Taku was referring his proof efforts after finding an appropriate counter example to a proposition.

Finally, a fragile grasp of axioms has emerged a strong causal factor to the inconsistent proof behaviour demonstrated. Hence, a fragile command of axioms that is typical of external conviction symbolic proof scheme emerged as another inductive category that can be used to explain why the students employed the analytic proof scheme (axiomatic argumentation) to proof tasks that require proof by counter-argumentation. The underlying idea revealed by the inductive code is that student teachers' knowledge of axioms, definitions and lemmas is not a coherent network of mathematical resources (Duffin & Simpson, 2000; Wilkerson-Jerde & Wilensky, 2011). The lack of coherence in mathematical resources is then manifested through indiscriminate use of the axioms as revealed by the following students' utterances;

> **Tanya**: I think even the fact people really do not understand or do not have crossed well the axioms one might tend to use them even when they do not apply.
> **Tendai**: Maybe someone didn't understand where to use axioms about the given question.

Hence, a weak command of the axioms and definitions can influence student teachers to employ structural reasoning to tackle proof tasks that demand use of counter argumentation. This causal link is coupled with the desire to demonstrate that one had done Analysis. I now discuss main observations made on students' thoughts about mathematical proof based on the discussion of the student teachers' proof attempts.

6.1.12 Main observations for research question one

From the discussion of results the following main observations were made about students' thoughts of mathematical proof from their proof attempts designated here as proof schemes.

(i) First, mathematical proof was thought of in terms of handling symbols which sprang from nowhere without relevant explanations for the purposes served by symbols manipulated. The symbols were handled in a mechanical way. For example, from Table 5.13 Debra wrote $-N(\varepsilon) \leq L \leq N(\varepsilon)$ instead of $|a_n - L| < \varepsilon$ in connection with the definition of convergence of a sequence. Thus the study has revealed the tenacity of external conviction symbolic proof scheme in which proof was thought in terms of symbolic manipulations without drawing meaning from the symbols manipulated. For instance, from Table 5.16 Tina wrote: $+\varepsilon = (a_{n-1} + 2 - \varepsilon^2)^{\frac{1}{2}}$. He then claimed to apply the binomial theorem to the right hand side and obtained the wrong expression $L = -\varepsilon + \left[\frac{1}{2}a_{n-1} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)(2-\varepsilon^2)}{2!} + \cdots\right]$, which has no apparent link to the binomial expression. Further, supporting evidence to the inference that the external conviction symbolic proof scheme was dominant in student teachers' responses can be deduced from the fact the student teachers did not reflect on meaning of symbols handled.

(ii). Second, the discussion has revealed that proving was conceived in terms of particular instantiations. The use of specific examples was done in such a manner that revealed that the notion of a counter example was not grasped as in some cases a single instantiation was considered to be a valid proof. For example from Table 5.16 Tina wrote:

$a_2 - a_1 = \sqrt{2+\sqrt{2}} - \sqrt{2}$

{Since this [$\sqrt{2+\sqrt{2}} - \sqrt{2}$], gives a positive number, it means that it converges.

Tina's claim is not necessarily true since $a_2 - a_1 > 0$ does not imply that the sequence converges. Further, while quantitative evaluations are good at unwrapping the underlying property that forms the crux of a proof they failed to serve this purpose because in the majority of the cases student teachers' particular instantiations were not accurate. For example, Taku's proof profile in Table 5.12 indicates that lack of accuracy made it difficult to show that the sequence is monotone increasing and bounded. Taku wrote:

$a_1 = \sqrt{2} = 1.414.$

$a_2 = \sqrt{2+\sqrt{2}}$ =1.848

$a_3 = \sqrt{2+\sqrt{2}} = 1.832$

$a_4 = 1.801,$

where the inaccuracy is in Taku's quantitative evaluations of a_3 and a_4. From Taku's working it can be seen that $a_2 > a_3$ and $a_3 > a_4$, which is not true for a monotone increasing sequence. Furthermore, the terms of the sequence, a_2 and a_3 have the same representation "$\sqrt{2+\sqrt{2}}$", that led to two different answers 1.832 and 1.848. Taku's proof behaviour indicates that instantiations were done without reflecting on their essence.

(iii) Third, the discussion of results has revealed that proving was thought of in terms of logic, that is, in terms of steps and procedures with a focus on establishing the truth or falsity of a mathematical proposition. In this regard, proof construction was conceived in terms of processes and mechanisms through which mathematical objects in the form of axioms, definitions and previously proven theorems were handled in order to validate or refute mathematical statements. In this respect, proving was understood in terms of the provision of tools and theories required to establish the truth of a mathematical assertion. A prominent feature of this category of proof schemes held by undergraduate student teachers is that the facts in the form of axioms, definitions and methods of proving should be given or endorsed by an external authority such as a teacher and text book. This revealed the dominance of the external conviction authoritative proof scheme. In other words, proving was seen as truth seeking exercise in which details pertinent to proof task at hand are provided.

6.1.13 Overall conclusion to research question one

The kinds of proof schemes that characterised undergraduate student teachers' conceptualisations mathematical proof were the axiomatic, external conviction symbolic and authoritative proof schemes. Although the study has uncovered the axiomatic proof scheme as a strong leading belief, students' weak grasp of the axiomatic proof scheme forced them to resort to the empirical-numeric and external conviction symbolic proof schemes. Furthermore, the external conviction authoritative proof scheme was dominant as student teachers expressed that axioms and definitions must be given or endorsed by instructors or written sources.

These kinds of proof schemes that characterised undergraduate student teachers' conceptualisations of mathematical proof were manifested in students' proof behaviours that include the following.

First, the student teachers' proof behaviour was characterised by indiscriminate use of mathematical symbols. Technical and symbolic manipulations were done without establishing contact with the structural relationships of mathematical ideas pertinent to the proof task. In other words, the crucial interplay between conceptual insights and technical handles was not evident. For instance, lack of profound grasp of conceptual insights resulted in flawed argumentation process marked by a mix up of mathematical concepts. For example, the case of Taku: for the task involving uniform continuity of a function f he wrote $|x - y| < \varepsilon$. This is a flawed statement because the quantity $\varepsilon > 0$ is associated with the range of the function but it was associated in this case with the domain of f. Another illustration of mechanical use of symbols was the case of Cortney who considered the solution $n^2 > \frac{-5}{4\varepsilon}$ to be valid when she tried to determine the natural number $N(\varepsilon)$ that would ensure that the sequence converges. The proof behaviours just described indicate the dominance of external conviction symbolic proof scheme among the student teacher informants.

Second, the students' proof behaviour can be described as *chaotic* or *messy* since it was characterised by violations of conventions of proving statements in mathematics. For example, the conclusion was stated before adducing evidence that is, providing the premises. Other indicators of chaotic student proof behaviour include mix up of concepts e.g., uniform continuity confused with Cauchy sequences or with differentiation, and the conditions $|x - y| < \varepsilon$ instead of $|x - y| < \delta(\varepsilon)$ and $n > N$ (condition for convergence of a sequence) were confused with uniform continuity. The chaos that was a feature of student behaviour in this study can be explained in terms of severe challenges noted in students' micro reasoning abilities as students failed to identify crucial aspects of their reasoning such as the representation system or reference theory of the proof task. For instance, solutions such as $i\sqrt{\frac{5}{4\varepsilon} + \frac{6}{4}} < n$ that are senseless were considered to be valid even though the reference theory is real sequences where n should be a natural number.

Third, student teachers' proof behaviour shown was also characterised by ontological oscillations, meaning that while students showed a preference for formal deductive argumentation, that is, the use of definitions and axioms during proof attempts, they could slide down the proof scheme ladder from the higher level analytic proof scheme to the lower level empirical proof scheme and vice versa. The ontological oscillations were caused by student teachers' difficulty with structural mode of thinking when attempting the proof tasks. For instance, from Table 5.16 part of Tina's written response to the task: *Define a sequence* (x_n) *inductively by* $x_1 = 1$ *and* $x_{n+1} = \frac{2x_n+3}{4}$. *Prove that* (x_n) *is a bounded monotone sequence is bounded and hence determine its limit,* is now reproduced.

$x_1 = 1 \quad x_2 = \frac{2x_1+3}{4} = \frac{2(1)+3}{4} = \frac{5}{4} = 1\frac{1}{2}$

For a monotonic increasing $(x_{n+1} - x_n) > 0 \ \left(\frac{2x_n+3}{4} - x_n\right) > 0 \qquad \frac{2x_n+3-4x_n}{4} > 0$

$\frac{3-x_n}{4} > 0$. As $n \to \infty \qquad \frac{3-x_n}{4} < 0$, *which means there is a contradiction, so* x_n *is bounded*

Given $\varepsilon > 0$, *there exist a number which is a natural number n such that* $n > N$, *therefore* $x_n - L < \varepsilon$

We know that $x_{n+1} = \frac{2x_n+3}{4}$, *then* $x_n = \frac{4x_{n+1}-3}{4} \qquad x_1 = 1 \qquad x_2 = 1/2$

As $n \to \infty$ $x_n \to 0$, *so* 0 *is the limit of* x_n

It can be seen that Tina started by using inductive explorations and then switched to the structural mode of thought which was then shelved and he finally resorted to use of particular instantiations. The chaotic proof behaviour by Tina can also be accounted by the fact that students do not have coherent connections of mathematical resources to enact on proof tasks. For example, his claim of a "contradiction" does not show how this led to the conclusion that the sequence (x_n) is bounded. Further, Tina did not reflect on numerical tests done since he had two different values of "$x_2 = \frac{5}{4} = 1\frac{1}{2}$" and "$x_2 = \frac{1}{2}$", in the same proof attempt. Tina could not uncover the underlying property for monotone sequences that could have formed the crux of the proof since the quantitative evaluations were inaccurate. Hence, resources were enacted without an awareness of their essence or relevance to the proof situation (Michner, 1978).

From Table 5.19 Getrude's proof attempt to the task: *Prove that the sequence defined by* $(u_n) = \frac{n^2-1}{2n^2+3}$ *converges,* serves to reinforce the point about students' chaotic proof behaviour. For example, Getrude wrote "$n \leq N(\varepsilon)$" instead of $n > N(\varepsilon)$ for convergence of a sequence indicating Getrude had a fragile grasp of pertinent ideas involved in the concept of convergence of sequences of real numbers, indicating the dominance of the external conviction symbolic proof scheme.

Finally, contradictory behavioural tendencies were also characteristic of student teachers' proof attempts as they traversed the formal deductive-counter-argumentation continuum when solving proof tasks. Students displayed inconsistencies in their formal rhetoric aspects of proof in the following manner. On one hand, tasks that required proof by refutation were tackled using structural or axiomatic reasoning, e.g., use of order axioms to solve the task: *Determine whether the following statement is true or false. For all real numbers* a *and* $b, a - b > 0 \Rightarrow a^2 - b^2 > 0$. On the other hand, students used particular instantiations for tasks that demanded proof by analytic deductive means.

6.2 Discussion of Research Question Two Results

6.2.1 How do the undergraduate student teachers develop their proof schemes?

The purpose of posing such a question was to identify possible trajectories that exist regarding the emergence of the mathematical object and to account for how the proof schemes emerge among the student teachers. A realist process approach was used in this case study to explore the development of proof schemes from students' descriptions of their proof experiences from pre 'A'-level to undergraduate learning contexts (Maxwell & Mittapali, 2007). According to scientific realism the proof events and processes described by students were treated as real observable phenomena that were causally relevant to the explanation of proof encounters experienced at various scholastic levels.

Table 6.3 shows observations on students' experiences with proving from pre-'A'-level to undergraduate level. Here the focus of the study was on teasing out the status of the proof schemes at the various scholastic levels and then develop a proposition about how the proof scheme evolves.

Table 6.3: Students' experiences with proof from pre-A level to undergraduate level

Aspect	Main Observations	Researcher's Comments
Students' conceptions of mathematical proof	• Logical arguments • Procedural ideas and facts	The dominant conception of mathematical proof was use of proof for verification purposes through step-by-step use of facts, rules, and procedures given and endorsed by an external authoritative source (Varghese, 2009; Weber & Mejia-Ramos, 2011. Proof was not conceived as a tool for discovering new mathematics (Wiest, 2015).
Pre-'A'-level experiences	• No proof exploratory experiences • Use of formula • Applying memorised facts • Empirical proof scheme dominated as can be inferred from drawing and measuring activities.	Student teachers indicated that there was low intensity to no proving activity at all. The empirical-numeric mode of argumentation employed as implied by the words "drawing and measuring." External conviction authoritative proof scheme was dominant as the teacher and textbooks were main sources of "memorised facts" applied (Alcock, 2010). Structural- intuitive (semantic) mode of thought was prominent.
'A'-level experiences	• Applying facts • Few proofs • Solving equations	Emphasis at 'A'-level was on how immensely applicable were the memorised facts (formulas and identities) in solving equations and identities that is, importance was given to usefulness of theorems (Wiest, 2015). Low intensity of proving activities. There were external authoritative sources of memorised facts, so external conviction authoritative proof scheme was dominant. That is, authoritative sources of applied facts were dominant (Weber & Mejia-Ramos, 2011).
Undergraduate	• Difficult/challenging • Use of axioms, definitions and lemmas • Justification skills • Time • High intensity of proving activities	Students described that proving at undergraduate was difficult and challenging and a time consuming exercise. Absorbing a piece of mathematics is a painful time consuming exercise (Davis & Hersh, 1981). Truths of the mathematics discipline no longer taken for granted that is, not just memorized but they were justified. Undergraduate student use substantial time reading and writing proofs in lectures and textbooks (Inglis & Alcock, 2012). That is, there is a high intensity of proof construction (Stylianides, 2009). Syntactic mode of argumentation became prominent.
Distinctive features of undergraduate proof experiences	• Level of detail • Justification • Challenging	Mathematical rigour became pronounced as student teacher engaged in structural thinking (Alcock, 2010) to justify mathematical proposition. Proving became challenging because it was introduced abruptly at this level (Stylianides, Stylianides & Philippou, 2007). There was a sudden and abrupt shift from empirical-numeric (use of imagistic and visual modes e.g., graphs) to structural thinking (Dawkins, 2012, p. 39).
Ways of attaining conviction	• Logical presentation • Use of theories • Use of examples • Methods/stages calculation	Students mentioned that axioms and definitions are given indicating the prominence of the external conviction authoritative proof scheme. Particular instantiations were also suggested as means of attaining conviction for proof contexts that demanded search for counter examples.

Summative content analysis revealed that the dominant meaning of mathematical proving held by the undergraduate students was seen as one in which proof serves the verification purpose that is, establish the truth or falsity of a proposition by means of logical presentation. This mode of thought about mathematical proving is well aligned to Varghese (2009) where proving was rarely conceived as a tool for explaining or discovering new mathematics. In other words, proof was rarely thought of as a vehicle for illuminating fundamental properties of focal ideas to the proof task

(Wiest, 2015). One's concept usage (Moore, 1994) is determined by one's definition of a particular concept that in turn provides insights as to how a proof scheme emerges because definitions are complete descriptions of the mathematical structure or behaviour of a mathematical idea (Wilkerson-Jerde & Wilensky, 2011). Hence, I considered it vital to explore students' conceptions of mathematical proof in order to relate it to how students' proof schemes develop.

Regarding undergraduate student teachers' experiences at pre-'A'-level period, most responses pointed to the fact there was no exploratory proof experiences as emphasis was on applying formulas and facts given and endorsed by external authoritative sources. So the authoritative warrant type (external conviction proof scheme) dominated proving activities at the pre-'A'-level period. Thus, the dominant proving activity at pre-'A'-level consisted of applying memorised formulas from an external source to obtain answers (product oriented rather process driven) as was shown by Tanya's interview excerpt. Thus, the student teachers engaged with technical symbolic manipulations of given mathematical ideas without getting a sense of the underlying ideas and consequently students at pre-'A'-level did not develop beneficial behavioural schemas with respect to the object of mathematical proof (Selden & Selden, 2011). Tanya's utterances from the results section, such as, "then you will just be given …" and "we didn't really concern ourselves with the part of proving," support the inference that the student did not develop a network of mathematical resources around the concept of mathematical proof. So according to Duffin and Simpson's (2000) categorisation of mathematical understanding, during the pre-'A'-level period students failed to develop a network of mathematical resources. Emphasis was on calculations using given formulas as described by Tanya.

Inductively derived codes indicate that at 'A'-level proof activity had a low intensity because "Applying facts" emerged as a dominant inductive code. Further, the facts applied by the students at 'A'-level were given, so the authoritative warrant type (Weber & Mejia-Ramos, 2011) was a prominent feature at A-level. For example, Getrude mentioned that: "there were some facts given […] applying given facts." Such utterances support the inference that the external conviction authoritative proof scheme was dominant at 'A'-level. There was low intensity of proving activity at 'A'- level as can be seen in utterances, such as, "there are not many proofs […], it didn't have many proofs […] there are just a few proofs at 'A'-level." Cortney's utterance that "the only proof I remember is induction," reinforces the idea that the activity of proving at 'A'-level had a low intensity. The major activity at 'A'-level was on applying formulas and identities to some equations and expressions. For instance, Taku mentioned "Proof of trig ratio not part of syllabus but solution of trig equations and identities." It can thus be concluded that no justification skills were fostered among student teachers who concentrated on memorising facts as seen in the following interview excerpt.

> **Tafa**: […] I do not remember any real proof […] may be you were asked to differentiate.
> **Researcher**: So how were you learning the double angle identities?
> **Tafa**: Some of them were just given.

It can thus be inferred from the interview extract that students engaged in technical symbolic manipulations without getting a sense (G) of the underlying ideas of the notion of a mathematical proof (Sandefur et al., 2013). In other words, at 'A'-level the instrumental perspective of mathematics was dominant with respect the concept of mathematical proof.

Emphasis at 'A'- level was on application of procedural steps without establishing contact with the structural relationships of focal mathematical ideas pertinent to the processes engaged in (Hanna & Mason, 2014). For instance, from Tafa's interview excerpt it can be seen that he engaged in processes such as differentiation in a mechanical manner as revealed by utterances such as "I do not remember any real proof." It can therefore be inferred that according to Selden and Selden's (2011) theory of actions during proof constructions the student teachers did not develop beneficial behavioural schemas of concepts dealt with as emphasis was on instrumental ideas. In terms of Koichu's (2012) notion of an intellectual need it can be inferred that student teachers engaged in problematic situations involving equations and identities and generated answers (pieces of knowledge) without seeing how those pieces of knowledge generated resolved the problematic situations.

Regarding students' undergraduate proof experiences the inductively derived code, 'Difficult'/challenging' had the highest frequency during summative content analysis. Other inductive codes derived from content analysis of textual data are 'high intensity,' 'axioms and definitions' and 'time.' It was noted that mathematical proof at undergraduate level was difficult and challenging for the students to comprehend. For instance, Bea explained that she failed to understand the cut property in $\mathbb{R}$ (*If an ordered pair (A, B) of non-empty subsets of* $\mathbb{R}$ *form a cut then there exists a unique element* $\varepsilon \in \mathbb{R}$ *such* $a \leq \varepsilon, \forall, a \in A$ *and* $\varepsilon \leq b, \forall\, b \in B$*)* even after reading her lecture notes several times. For Bea the concept of a mathematical cut was so difficult for her that she expressed the sentiments that it was impossible for a learner at her level to generate a proof by her own as can be noted from the following extract.

> **Researcher**: Can you generate a proof?
> **Bea**: I do not think it's possible. At my level. Not without reference to some source.

Bea had experienced serious discomfort with the cut property when reviewing lecture notes to a point where she doubted the feasibility of producing a mathematical proof autonomously. Hence, the authoritative warrant type emerged as a dominant leading belief as shown by the utterance "Not without reference to some source." Tino's interview extract also illuminates challenges students face with proofs at undergraduate level.

> **Tino**: Proofs were challenging
> **Researcher**: What was challenging?
> **Tino**: Trying to argue logically to convince.

Tino had challenges in putting together arguments in a logical manner. In other words formal deductive modes of argumentation posed serious challenges to students. The difficulties that Tino and Bea faced with proof at undergraduate level can be explained by the fact that proof had not been part of the student's culture as indicated by no exploratory proof experiences at pre-A level or low intensity of the proof activity at 'A'- level.

At undergraduate level there is an abrupt introduction of the concept of proof where undergraduate students are expected to spend substantial learning time reading and writing proofs in lectures and from textbooks (Alcock, 2010; Stylianides, Styliandies & Philippou, 2007). Challenges faced by student teachers in constructing proofs forced them to resort to what Tendai called her 'own understanding' as revealed in the extract of the interview.

> **Tendai**: […] now maybe the theorem, if you did not understand the theorem and its difficult, maybe you can use your own understanding.
> **Researcher**: What do you mean by your own understanding?
> **Tendai:** If say you want to give may be an ordered field theorem then, you use, eee, if you want to say $a > b$ then I can fix, write $2 > 1$. I can use numbers.

From Tendai's interview excerpt, it can be noted that deductive reasoning can sometimes cause serious discomfort to students that can force them to slide down the proof scheme ladder from the higher level analytic proof scheme to lower level empirical-numeric proof scheme. This is a typical example of an ontological oscillation, a term used to describe a shift from one proof scheme category to another

"High intensity" of proving activity was another category that emerged at undergraduate level. There is pronounced proving activity at undergraduate level that is characterised by strides to foster justification skills among learners. Therefore according to Koichu (2012), as students engaged in problematic proof tasks they should have seen the need to generate pieces of knowledge that resolve the proof problems. In other words, students should provide an epistemological justification as to how the pieces of knowledge generated are solutions to the proof tasks. This requirement is in stark contrast to the 'A'-level and pre-'A'-level scenarios where the prominent proving activity was application of given facts and formulas to get answers by engaging in calculations. Derivation or proving of the "tools" given was not emphasised. The focus at undergraduate level became different, with an emphasis on justifying how "tools" come into being, that is, their ontology. In Selden and Selden's (2011) terms undergraduate students should have developed some beneficial behavioural schemas. For instance, Tino's utterance points to the need for fostering of justification skills at undergraduate level.

> **Tino**: If you look at 'O' and 'A' level Mathematics you have to know rules and procedures, the steps involved in solving. But now at this level [undergraduate level] I think you have to understand the reasons behind what you are doing [...] depth of conceptual understanding is much deeper.

Finally, "Time" also emerged as factor that characterised undergraduate mathematics proof learning. Tina mentioned, "You take your time on an aspect. Proofs take time to comprehend." Tina's comments are aligned with Davis and Hersh's (1981) description that absorbing a piece of mathematical knowledge is such a painful and time consuming act.

6.2.2 Main observations to research question two

From the discussion of summative content analysis results the three main trajectories that emerged are now presented. First, the lateral shift within the lower cognitive level external conviction proof scheme category is shown in Figure 7.

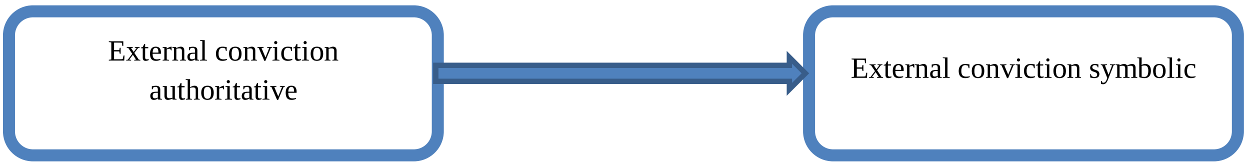

Figure 7: Emergent proof scheme trajectory one

This trajectory was dominant in the study since student teachers indicated that as they moved from pre-'A'-level through to undergraduate mathematics level they engaged in technical symbolic manipulations without establishing contact with structural relationships of pertinent ideas involved. From the discussion this trajectory consisted of student teachers who were given *facts* at pre- 'A'-level and 'A'- level to apply in doing calculations. At undergraduate level they engaged in technical symbolic manipulations without getting a sense of the underlying ideas of the symbols involved. Further, the lateral shift within the main external conviction proof scheme category indicates that there was no cognitive growth with respect to the students' conceptualisations of mathematical proof. For instance, the case of Tanya:

> **Tanya**: Aaaa, I would like to believe I did not deal with them so much, maybe at the end I would just be given the result.
>
> **Researcher**: Ok.

Tanya: Then just use the result.
Researcher: To do what?
Tanya: Maybe, like we have the quadratic formula where we want to solve for the roots of the equation. Then you will just be given how to go about and come out with the proof. I didn't know that much.

What is central from my exchange with Tanya is that she lacked conceptual understanding of the concept of proving. Further, the influence of the authoritative warrant was evident as implied by the term "just given the result." Tanya's extract reveals that the categories of proof schemes were not mutually exclusive but rather interrelated as can be seen by the phrase "solve for the roots," which implies that this trajectory also encompassed the empirical-numeric proof scheme at Pre-'A'-level and 'A'-level scholastic levels. Further, the trajectory indicates loss in sense making because during pre-university mathematics learning the students had a grasp of the essence of calculations done as indicated by Tanya's utterance, "like we have the quadratic formula where we want to solve for the roots of the equation." However, when they moved to university mathematics there was loss of sense making shown by manipulating symbols without getting a sense of the symbols.

Second, a proof scheme trajectory that shows a scenario where students' proof schemes regressed is shown in Figure 8.

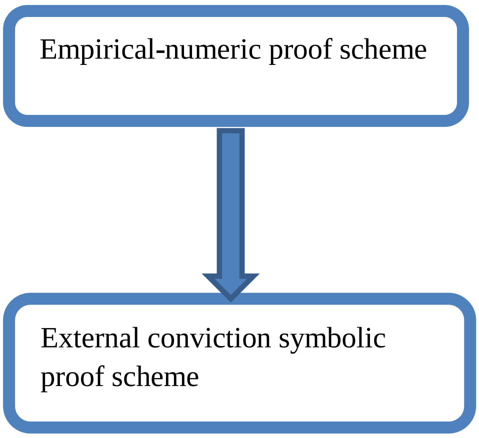

Figure 8: Emergent proof scheme trajectory two

Figure 8 shows that there was a decrease in student teachers' cognitive level with respect to their conceptualisation of mathematical proof. This scenario is contrary to the expected gain in mathematical proficiency that should be typical of the transition from a lower to higher level of mathematics learning— pre-university to undergraduate level of mathematics learning. For example, Bea, describing her pre-'A'-level encounters with proofs, mentioned that "Proofs not challenging and could easily follow steps." The steps were presumably for doing calculations as was confirmed by Taku: "measuring and calculations, no axioms". This trajectory indicates that students did not engage in structural or axiomatic reasoning but manipulated symbols and notation in an instrumental way so they attained a lower level proof scheme— the external conviction symbolic proof scheme. The dominance of external conviction was also revealed by students' proof attempts, for example, the cases of students' proof attempts discussed in section 6.1.13.

Finally, summative content analysis also gave rise the inductively inferred trajectory of proof scheme shown in Figure 9.

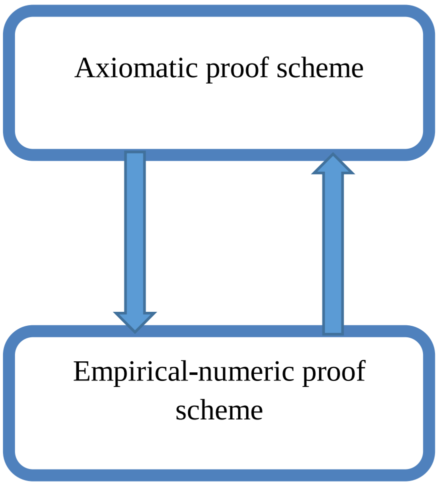

Figure 9: Emergent proof scheme trajectory three

The third trajectory of proof schemes has arrows that are opposing each other. The arrow directed upwards shows a vertical upward shift from a lower cognitive level empirical-numeric to a higher cognitive level axiomatic proof scheme. The arrow pointing downwards indicates that proof schemes regressed to the empirical proof scheme. This scenario reflects lack of stability in the axiomatic proof scheme that sometimes forces students move back to the lower level empirical proof scheme. The point about lack of stability in axiomatic proof scheme can be seen in the interview extract that involved Tendai.

> **Tendai**: […] now maybe the theorem. If you did not understand the theorem and its difficult, maybe you can use your own understanding.
> **Researcher**: What do you mean by your own understanding?
> **Tendai**: If you say you want to give may be an ordered field theorem, then, you use if you want to say $a > b$ then I can fix, write $2 > 1$, I can use numbers.

Tendai's utterances revealed that when she faced challenges in arguing in a formal deductive manner, in her case by using order axioms, she could then turn to particular instantiations of the order properties. The ontological oscillation which points to lack of stability in the analytic proof scheme among the undergraduate students was revealed by her moving back to empirical proof scheme.

My attempt to get a more revealing picture about how proof schemes emerge by combining the 3 trajectories gave the schematic representation shown in Figure 10

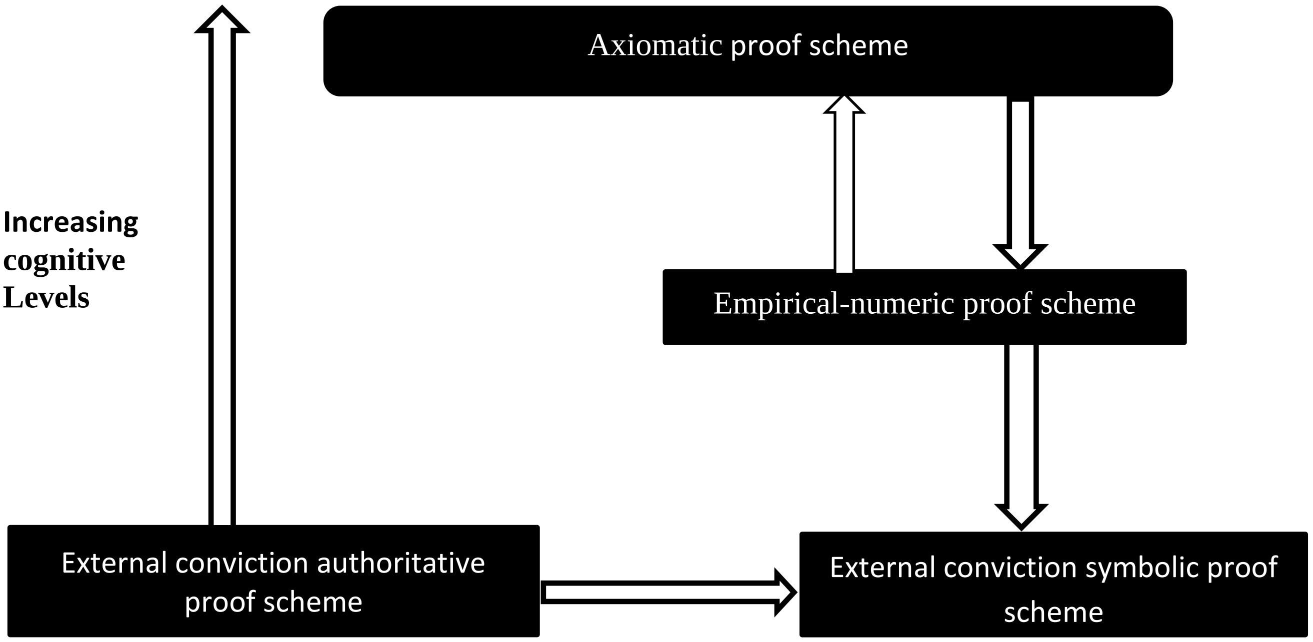

Figure 10: Three trajectories in one schematic representation

Overall, Figure 10 shows the dominance of lower cognitive level proof schemes. The external conviction symbolic proof scheme was dominant as it featured in two trajectories as shown by the two arrows converging to it. The dominance of the external conviction symbolic proof scheme indicates the indiscriminate use of symbols by the student teacher informants as they manipulated the symbols without establishing their essence. Another salient feature of schematic representation of the emergent trajectories of proof schemes is a decrease in cognitive level indicated by arrows directed downwards. This is an interesting scenario in the sense that during pre-university mathematics learning the students used to handle numeric tests (empirical inductive explorations) with meaning (sense making) but when they shifted to the external conviction symbolic proof scheme during undergraduate studies, there was indiscriminate use of symbols without drawing meaning from the symbol manipulations. The lateral shifts indicated by horizontal arrows and the lack of stability shown by arrows pointing in opposite directions in Figure 10 count as other salient features unravelled by this study.

A proposed proof scheme trajectory is shown in Figure 11.

Figure 11: Expected proof scheme trajectory

The crucial experiment proof scheme category by Balacheff (1998) is one in which a specific example is chosen on the basis of some rationale such as being an appropriate counterexample or the instantiation is selected from the proper subset because of its capacity to illuminate the underlying property that forms the crux of the proof (Fukawa-Conelly, 2012; Stylianides, 2011). The axiomatic proof scheme by Harel and Sowder (1998, 2007) employs arbitrary mathematical objects (axioms, definitions) in syntactic derivations which provide complete and conclusive evidence about the truth-value of a conjecture. Hence, the crucial experiment proof scheme category would be strategic when a prover wants to refute false mathematical assertions. On the other hand, the axiomatic proof scheme would be strategic when dealing with true mathematical propositions. The explanation just given justifies the suggestion that the given trajectory could be ideal. Another feature of the suggested trajectory shown in Figure 11 is that arrows shown at both ends should indicate the crucial interplay or interaction that should exist between the two categories of proof schemes. The arrows need not signify an emotional response to a cognitive challenge as was uncovered by this study.

The expected trajectory of proof schemes proposed here would be difficult to attain in our Zimbabwean mathematics curriculum because of the undesirable effects of the authoritative warrant type uncovered by this study where learners insist on being given the tools and theories needed for composing proofs. Further, the low intensity of the structural mode of argumentation in pre-university mathematics reported in this study also requires some attention in order to attain the expected trajectory in mathematics learning. Finally, the proof method by refutation should gain some emphasis if we are to make strides towards attaining the expected trajectory. I now state the main conclusion to research question 2.

6.2.3 Overall conclusion to research question two

Undergraduate student teachers' proof schemes revealed some lateral shifts within the external conviction proof scheme category (authoritative to symbolic) and in other cases proof scheme states regressed to the external conviction symbolic proof scheme from the empirical proof scheme. In other cases, proof schemes evolved from the empirical-inductive proof scheme to the axiomatic proof scheme which showed instability and fragility.

Let me attempt to account for the nature of the shifts in students' proof schemes. The central ideas are the lack of stability in the desired (highest level) analytic proof scheme and the lack of cognitive growth in students' schemes of argumentation shown by a lateral shift within the external conviction proof scheme category. Another central idea revealed by the trajectories is the decrease in cognitive level of the proof schemes. The main cause of instability in the inductively developed categories is the discomfort experienced by students when using the structural mode of thinking. Informants indicated that they were unfamiliar with proof tasks that involved formal deductive schemes of argumentation which were not part of their earlier learning experiences. The category "difficult/challenging" had the highest frequency (4 out of 13). Typical exemplifications are:

> **Bea:** […] challenging because it's something I am not familiar with. Taking the whole course with proofs, failing to understand.
> **Tino**: Proofs were challenging
> **Researcher:** What was challenging**?**
> **Tino:** Trying to argue logically to convince.

Another reason for the lack of stability and lateral shifts in students' schemes of argumentation relates to the volume of work as implied in Bea's response "Taking the whole course with proofs, failing to understand". When students fail to understand as indicated by Bea, they resort to particular examples which were regarded as part of students' culture. For instance, Tafa's extract:

> **Tafa**: using numbers […] closer home.

The use of particular instantiations is a culture cultivated through pre-university curricular practices. The empirical proof scheme implied by use of particular instantiations and external conviction symbolic proof schemes are lower cognitive proof schemes which indicate that there was no cognitive growth as students engaged with the concept of mathematical proof at different scholastic levels. Besides lack of stability reported about the trajectory of proof schemes, other additional observations made concern the fall in cognitive level and the lateral shifts in proof scheme categories as the student teachers moved from pre-university to undergraduate mathematics learning. The decrease in cognitive level can be explained in terms of overwhelming symbolism that characterised university mathematics learning. Contrary to undergraduate mathematics learning, at pre-university level the student teachers could draw meaning of the inductive explorations (numeric tests). Further, the lateral shifts are interesting observations as they are inconsistent with the expected vertical upward shift in schemes of argumentation as the student teachers took up undergraduate mathematics.

6.3 Overall conclusion to main research question

Overall, the study has revealed that mathematical proof was conceptualised in terms of logical steps and procedures through which axioms, definitions, and symbols are handled in order to validate mathematical statements. The manner in which the undergraduate student teachers handled axioms and definitions revealed the dominance of the external conviction authoritative and symbolic proof schemes. Trajectories of proof schemes revealed lateral and vertical downward *shifts in proof scheme states. Furthermore, a vertical upward shift from empirical-numeric proof scheme to axiomatic proof scheme was observed — that however was unstable and fragile.*

Overall, student teachers' written responses, chalkboard demonstrations and utterances from reflective interviews revealed that mathematical proof was thought of in terms of logical steps and procedures through which axioms, definitions, and symbols are used to prove mathematical conjectures. Therefore, the structural mode of thought about mathematical proof was shown to be a strong leading belief. Let me point out that, although students thought of mathematical proof largely in terms of steps and procedures that involve use of axioms, they encountered serious challenges with syntactic proving. However, despite facing severe challenges with the structural mode of reasoning, there was an overemphasis on the construction of formal deductive arguments. This caused students to use axioms in situations that require proof by refutation. Such proof behaviour was caused by students' lack of appreciation of counter-argumentation. Typical exemplification of students' fragile grasp of the notion of a counter example can be seen from Cortney's interview excerpt below.

> **Cortney**: […] you will always think that what if I just come up with a counter example […] I have to make use of axioms so that the answer [proof] gets balanced.

Further, the students' descriptions revealed that the axioms, definitions and symbols should be given and endorsed by instructors indicating the dominance of the external conviction authoritative proof scheme. The study also uncovered that while mathematical proof was conceived principally in terms of axiomatic argumentation the evocation of particular instantiations (empirical-numeric proof scheme) was caused by lack of in-depth knowledge about the role axioms and definitions pertinent to the proof task students sought to resolve. Weak command of structural mode of thought (axiomatic proof scheme) was caused by pre-university mathematics curricula practices that were characterised by low intensity of formal proof.

Mathematical proof was also thought of in terms of manipulating symbols. The external conviction symbolic proof scheme emerged as a robust proof scheme. For example, from Table 5.16 Tina wrote "$\frac{2x_{n-1}+3}{4} - L < \varepsilon,$" for a task that involved sequence (a_n) revealing that symbol manipulations were not consistent with the notation involved. Further, symbols were manipulated in a mechanical way without reflecting on ideas embedded in those symbols. For instance, from Table 5.11, Tina regarded the radius $\delta(\varepsilon) > 0$ as a natural number and the condition for uniform continuity was stated as $|x - y| < \varepsilon$ instead of $|x - y| < \delta(\varepsilon)$. Therefore lack of in-depth knowledge about meaning of axioms and definitions intertwined with instrumental handling of symbols support the inference that the external conviction symbolic scheme emerged robust as a proof scheme among the undergraduate student teachers.

While some findings from the current study have shown some consistency with ideas in existing literature, the following could count as new insights this study has generated. First, the study has unravelled inconsistencies in students' formal rhetoric aspects in the following manner. The contradictory student proof behaviour shown through the use of particular instantiations in deductive proof tasks and through the use of syntactic derivations to resolve tasks that require proof by refutation as the students traversed the inductive-deductive proof scheme continuum can be argued to be an additional insight the study has generated with regard to the learning of mathematical proof. Second, this study has uncovered some ontological oscillations in students' schemes of argumentation which I now try to explain. An ontological oscillation refers to a switch from one scheme of argumentation to another when resolving a proof task at hand. The lack of stability and fragility of the axiomatic proof scheme can be explained by the fact that deductive reasoning by means of definitions and axioms caused serious discomfort to students that forced them to slide down the proof scheme ladder from the higher level axiomatic proof scheme to the lower level empirical-numeric proof scheme their comfort zone. Thus although the axiomatic proof scheme emerged as a strong leading belief students' struggles with structural mode of thought induced some fragility and lack of stability in the mathematical entity.

Third, another interesting and unexpected observation was loss of sense making during mathematics learning at undergraduate level revealed by the trajectory shown in Figure 8. This trajectory does not only reveal a decrease in cognitive level in the student' schemes of argumentation but also shows that the student teachers could not draw meaning out of symbols handled. On the contrary, at pre-university level the same student teachers got the essence of inductive explorations or numeric tests done.

Finally, the tenacity of the external conviction authoritative proof scheme was also uncovered by this study in which student teachers' leading belief was that theories and tools (axioms, definitions, symbols) needed to validate mathematical statements should be given or endorsed by external sources rather than be generated by self through axiomatic argumentation. While the tenacity of the authoritative warrant type is not new this study has proposed an account of this tenacity in terms of huge disparities between pre-university and undergraduate curricula practices with regard to the learning of mathematical proof. The tenacity of the authoritative warrant can also be explained in terms of impasses reached during proof construction by the student teachers who would then insist that proofs should be given to them by instructors.

This study has also unearthed some lateral shifts in students' schemes of argumentation represented by the trajectory shown in Figure 7. This trajectory indicates lack of cognitive growth in students' schemes of argumentation which is inconsistent with the expected cognitive growth during the transition from pre-university to undergraduate mathematics. Therefore, the leading belief (authoritative warrant type) acts as a stumbling block in efforts to promote autonomous proof productions and promote vertical upward shifts in students' schemes of argumentation which can then lead to the attainment of higher cognitive level proof scheme states among the student teachers. Hence, an explication of the undesirable effects of the external conviction authoritative proof scheme in Zimbabwean undergraduate mathematics education students' proof schemes presented here could also count as an insight generated by this current study.

6.4 Implications for Theory

This study has developed an explanatory theory that characterise the structure of the nature of proof scheme states among mathematics education undergraduate students. I now state the theory:

Proof scheme states are characterised by an unstable and fragile axiomatic proof scheme. Lack of stability in the axiomatic proof scheme state induces some regression to the empirical-numeric proof scheme state. Furthermore, lateral shifts occur within the external conviction proof category, from the authoritative to the symbolic proof scheme.

Salient features of changes in proof scheme states are;

- Lack of cognitive growth anticipated as students negotiate the transition from pre-university to undergraduate mathematics learning.
- Dominance of lower cognitive level proof scheme states such as external conviction symbolic and authoritative proof schemes
- Loss of sense making during the transition from pre-university to undergraduate levels of mathematics learning.

I conclude on this section by making the following remarks. According to Harel and Sowder's (1998) taxonomy of proof schemes it is possible for one to hold more than one proof scheme during the same encounter. However, the taxonomy is silent about the nature of the interrelatedness in a person's proof schemes. This study has not only explicated the possible trajectories that can exist but has also uncovered causal links that informed students' leading belief during the shifts from one proof scheme to another. These causal links include among others, discomforts experienced with use of axioms, and definitions that forced students to resort to the empirical proof scheme as well an overemphasis placed on use of axioms and definitions that resulted in use of syntactic approach in proof tasks that required proof method by refutation. Hence, explicating the structure of the nature of proof scheme states and inductively uncovering reasons for possible shifts in student teachers' proof schemes could count as new observations this study has made to the discipline of mathematics education.

6.5 Limitations of the study

First, the results of this study are limited to proof schemes, that is, persistent characteristics revealed in proof attempts of ten undergraduate mathematics education students involved in this study at one high ranking state university in Zimbabwe. Consequently, these findings about the kinds of proof schemes held by student teachers and the emergence of the proof schemes among undergraduate students cannot be automatically transferred to all mathematics education undergraduates in Zimbabwe.

Second, the use of *think aloud* protocols during reflective interview phase was some form of threat to the credibility of the study. With respect to research question two data were based on 'after the effect self-reports' by student teachers (Kidron & Dreyfus, 2014, p. 298) and so could have suffered from some possible omissions in students' self- reports. However, I believe that these data were sufficient for serving the purpose of determining how proof schemes evolve among undergraduate student teachers because I used the split-half method, (Lewis, 2009) and triangulation for the purpose of generating rich data that could give a revealing picture about students' thinking processes about mathematical proof.

Closely related to use of interview protocols is the notion of a mathematical representation. Herlina and Batusangkar (2015) define a mathematical representation as an expression of mathematical concept (definition, mathematical proposition) that is used by a student to communicate outcomes of the student's interpretation of that concept. In the context of this study a mathematical representation is an outcome that relates to the students' effort to search a solution to a proof task at hand. Mathematical representations include a combination of ideas expressed through written responses and an array of ideas which were constructed in students' minds and expressed verbally during chalkboard demonstrations of students' proof attempts. A limitation noted that relates to the concept of a representation as explained is that outcomes of students' interpretations expressed were surface results of the functioning of the deep mind structures that did not depend on the actual awareness of the student teachers (Piaget, 1967 in Duval, 2006). It is certainly quite possible that important thinking processes not part of the students' evoked concept images during those moments of chalkboard presentations and writing sessions were omitted. However, I retain confidence in the validity of the data because data were elicited from different sources (triangulation) and this contributes to credibility of the findings.

Third, the study could have also suffered from threats related to the researcher-lecturer dual role in the following manner. When probed about their level of conviction during reflective interview sessions during both Mid-instruction and End-of-instruction data collection phases, the students stated that they had absolute conviction in their proofs. It is quite possible that these students had relative conviction in their arguments, that is, the subjective probability that they had attributed to arguments constructed had a certain threshold to provide a warrant for further efforts to verify their accuracy (Weber & Mejia-Ramos, 2015). However, the student teachers could not express their relative conviction because of the possibility of a multitude of factors such as: the student's mood and lecturer-student relationship shared. So the interpretations of the students' utterances during reflective interviews and chalkboard demonstrations were done with my full recognition of this conversational constraint. However, I retain confidence in the plausibility of the findings because of the consistency of results of data analyses from the three sources: written responses, chalkboard demonstrations and students' evaluations of their proof efforts during reflective interviews.

Fourth, the study suffered from threats related to time. Student teachers spent an average 1 hour 35 minutes working on proof tasks during Mid-instruction data collection phase and about 1 hour 50 minutes during End-of-instruction data collection. In the actual practice of research mathematicians, proof construction may span over several weeks or months or even years, especially when we take cognizance of the fact that the tasks were novel to the students (Weber, 2008, p. 449). Hence, if student teachers had been allowed longer periods of engagement with the proof tasks it might be possible that alternative patterns in the kinds of proof schemes could have emerged. Herlina and Batusangkar (2015) commenting on time as a significant factor in proof construction say "when a learner mulls over a problem at some point he/she will gain flash of insight and invitation, a vision of the solution he was seeking" (p.129). Herlina and Batusangkar further suggest that the vision of the solution presents itself as a "bunch of intuitive threads that have to be woven together and some of the early ones present themselves visually" (p. 129). Hence, although no time restrictions were imposed as students worked on the proof tasks, the fact that students worked on the tasks on a single day implies that there could be many aspects involved in proof constructions this study failed to capture.

Nonetheless it is reasonable to claim that the study was able to identify and describe persistent characteristics, that is, proof schemes from the student teachers proof attempts because data were analysed within the scientific realistic analytic framework using a process theory approach (Harel & Rabin, 2010; Maxwell, 2004; Maxwell & Mittapali, 2007). Critical realism employed in this study recognizes and accepts the importance and validity of causal explanations even in single events and case studies (Maxwell, 2004). Furthermore, concepts about the kinds of proof schemes held and how they evolve, such as, the *mess* or *chaotic* nature of student conceptualisations of proof schemes and lack of stability in higher level proof schemes were abstracted from data within a realist analytic framework. I can therefore, claim with some degree of confidence that the results from the case study activities represent typical Zimbabweean undergraduate students' thinking about the notion of mathematical proof.

I summarize major limitations of the study. First**,** I acknowledge that the limitation that the small sample used (10 student teachers) and study context limit transferability of findings. Second, while results reported were based on actual *voices* of student teachers and I employed member checking technique to validate inferences of written responses with the participants, it is possible that the study could have suffered from some conversational constraints. For example, being their lecturer the students might have agreed with my interpretation of proof attempts to impress their teacher, that is, reactivity might have adduced limitation. Finally, use of verbal protocol in chalkboard demonstrations and reflective interviews had a limitation that the students could only articulate concept images evoked during interviews.

6.6 Recommendations from the study

Directed content analysis of textual data that focused on research question one and summative content analysis for data used to address research question two had implications for mathematics educational practice and implications for mathematics educational research.

6.6.1 Implications for mathematics educational practice

The goal of research on the concept of mathematical proof is to bring students' proof competency to secure levels. In other words, the intent of research on mathematical proof is to promote and foster thinking habits among student teachers that are as much as possible close to mathematicians' conceptions of proof (Weber & Mejia-Ramos, 2015). The results of this research have contributed towards such efforts by uncovering valuable information about the terms in which student teachers think about proof.

First, one of the key findings of the current study is that student teachers conceptualise mathematical proof in terms of steps and procedures through which axioms, definitions and symbols are handled for the purpose of establishing the truth or falsity of a mathematical proposition. These students' thoughts about mathematical proving reveal some inadequacy because mathematicians conceptualise proof in terms of judicious use of mathematical objects where counter-argumentation is used to refute false conjectures and similarly axioms and definitions are employed to validate true mathematical propositions. Hence, proof method by refutation had not been grasped by student teachers. In other words, proving was understood in terms syntactic derivations to a point where any proving attempt that did not involve use of definitions and axioms was not considered being a proof. A typical example of this point can be seen from the verbatim transcription of Taku's End-of-instruction reflective interview.

> **Taku**: You need axioms particularly when dealing with analysis. Mathematically when you use an example to testify a proof then its wrong […] You need axioms practically when you are dealing with Analysis you do not have to use an example.

Taku's utterance reveals that the student lacked an appreciation for a counter example. Hence, the study has uncovered a lack of comprehensive or analytic conception of mathematical proof where formal deductive arguments receive more prominence than proof by refutation. Hence, these are important insights for the learning of Real Analysis concepts at undergraduate level because understanding what proof is and composing proofs contribute to deep learning (Ersen, 2016).

Second, the study has uncovered lack of the crucial interplay between the technical handles and conceptual insights in the manner in which students deploy mathematical resources (axioms, definitions and previously established theorems) during proving in the following manner. In some cases, access to the structural relationships of ideas involved in the proof tasks was not accompanied by a strategic access to the relevant technical handles. For example, from Table 5.16 Taku stated the crucial relationship $x_{n+1} - x_n \geq 0$ for a monotone increasing sequence but failed to access procedural steps of proof method by mathematical induction to accomplish that the sequence (x_n) is monotone increasing and bounded. Conversely, in the majority of the cases symbols were manipulated without getting the essence of ideas embedded in those symbols. It is therefore, recommended that mathematics education research explore ways of fostering the crucial interaction between technical handles and conceptual insights.

Third, the study has illuminated trajectories that characterise the emergence of proof schemes among student teachers which in turn gave a pictorial view of the temporary nature of axiomatic proof scheme that forced students to revert to use of empirical verifications. Previous research studies (Mejia-Ramos & Inglis, 2009; Varghese, 2009) have expressed concern about lack of knowledge about processes involved in composing proofs as well as impasses experienced by students during proving. This study has contributed towards efforts to address these concerns by explicating causes of students' discomfort with syntactic proving which include low intensity of proving activities in pre-university curricular, and the leading belief that axioms, definitions and steps and procedures needed to validate propositions must be given and endorsed by an authoritative source such as a teacher. Hence, the study has revealed the dominance of the external conviction authoritative proof scheme— a low cognitive level proof scheme. In addition, the study has also uncovered lack of stability in the deductive proof scheme. This is valuable knowledge that could inform the design of tasks that support students in their efforts to produce proofs autonomously thereby eliminating the undesirable effects of the external conviction authoritative proof scheme such as insisting on being given the proof. The argument is that students can learn more concepts in proof if they compose proofs by themselves. These insights could inform instruction on proof and serve as important strides towards autonomous proof production by students.

Fourth, another key finding of the study is that prospective high school mathematics teachers demonstrated lack of intellectual need as they engage with the proof tasks (Koichu, 2012). The point is, although students generated solutions to proof tasks assigned they did not see how the pieces of knowledge constructed (solutions) resolved the given proof tasks. In other words, the students failed to provide an epistemological justification, that is, interpret how the solutions generated were answers to tasks. Hence, the students failed to develop a rational understanding of concepts pertinent to the task but rather demonstrated mechanical understanding. The student teachers lacked rational understanding because they did not know the mathematical concepts meaningfully as these were not accompanied with justifications (Maya & Sumarmo, 2011, p. 232). Therefore student teachers' fragile understanding of mathematical proof revealed in this study suggest that to meet their future students' needs undergraduate student teachers should have rich experiences with mathematical proof in order to frame their mathematics instruction around fundamental aspects of the notion of proof. It can therefore, be recommended that proof oriented instructors need to realign their teaching methods. For instance, mathematics educators should employ strategies that would induce habits of self-questioning and reflection (i.e., encouraging meta-mathematising) that allow students to realise ideas critical for successful proof constructions. Such efforts may help the students to establish connection between their solutions and demands stipulated by proof tasks. Thus, professional development efforts must be directed towards creating and fostering proving events among student teachers.

Fifth, Real Analysis concepts students learn at undergraduate level are built on earlier student experiences. This study has revealed pronounced discrepancies between pre-university and university curricula with respect to the notion of mathematical proof. Previous learning encounters are essential in effective learning (Tall 2005 in Ersen, 2016). Student proof behaviour could be described as *chaotic* and *messy* and was also characterised by some ontological oscillation and rampant violations of proof conventions. Data analysis revealed lack of continuity between pre-university and university mathematics practices in the learning of proof. This finding leads to the recommendation that curriculum developers at undergraduate level should be sensitive to the fact that learning requires a student to build on his or her previous learning experiences in a manner that may show a fit between the intended learning experiences and student's met-befores (Tall, 2005 in Ersen, 2016). Hence, it is recommended that mathematics instructors should devise means of developing argumentation skills at lower scholastic levels so that discomforts associated with abrupt changes in mathematical proof styles from use of facts and formulas before undergraduate mathematics to formal deductive argumentation at undergraduate level are eased.

Finally, the current study has also uncovered inconsistencies in students' proof behaviour demonstrated by the proclivity by students to use formal deductive reasoning in proof situations that call for proof method by refutation and vice versa. An explanation has been proposed for contradictory behavioural tendencies in terms of over-emphasis on use of axioms, definitions and symbols that caused student teachers to look down upon proof efforts that do not involve axioms and definitions. Conversely, the evocation of empirical-numeric proof scheme was caused by discomforts students experienced with the structural mode of reasoning when proving but mathematical proof was largely conceptualised in terms of logical steps and procedures in which axioms, definitions and symbols are employed in validating conjectures. These findings are fresh insights that could be an important consideration in understanding the problem of mathematical proof among student teachers. Hence, it can therefore be recommended that mathematics educators consider these insights in their efforts enable students to overcome the problem of constructing proofs with meaning so that proof can be seen as a viable means for producing explanations (Liu & Manouchehri, 2013, p. 19).

6.6.2 Implications for mathematics educational research methodology

The empirical analysis of the processes involved in students' proof construction attempts have unravelled the terms in which students think about mathematical proving. Data analysis has also revealed how students' thinking about proving can be characterised based on students' actual productions as opposed to students' evaluations of arguments supplied by researchers. The process of reflective writing (written responses) played a central role in illuminating the kinds of proof schemes held by undergraduate student teachers as they engaged with the proof tasks. For instance, the student teachers' fragile understanding of mathematical proof was revealed through students' written responses. Reflective writing technique was applied with both task-based interviews and the chalkboard demonstrations. Although reflective writing technique is not new, the way reflective writing was applied was different. With previous studies reflective writing was applied to task-based interviews only or to chalkboard demonstrations only. The act of applying it to the two data sources helped in strengthening inferences by checking for consistency of results coming from two sources. Hence, use of reflective writing with the two data sources could count as a fresh idea about data generation in research.

After reflective writing, I then tried to determine student teachers' levels of conviction in their arguments. In other words, measures were taken to check whether student teachers had absolute or relative conviction in solutions to the proof tasks. Hence, in the reflective interviews I strengthened inferences made about the kinds of proof schemes held with follow up questions, such as, are you convinced that the statement is true? Why is the statement true? Do you have any doubts at all that the proof you have just constructed is true? A more revealing picture grounded in students' own perspectives was therefore, made about student teachers' thoughts on proving— a research strategy which is compatible with the realistic analytic framework. The use of three data sources in pursuit of terms in which the student teachers think about mathematical proof could also serve as a contribution of this study to research methodology. I can retain confidence in this claim because of the following reasons. With most previous studies examined there was use of written responses followed by reflective interviews and in other cases informants expressed their level of conviction in arguments supplied by the researchers. Hence, the activity of writing when students were constructing arguments to validate proof tasks which were considered to be novel to the students has an edge over evaluating a supplied argument. Further, the act probing students during reflective interviews helped to determine whether students had absolute on relative conviction in arguments produced. Hence, data generation techniques employed in this study could have serve as efficient triangulation techniques that can be utilised in future studies in mathematics education.

6.7 Further research

Following Lakatos' (1976) dialectic of proofs and refutations, Cirillo and Herbst (2012) have proposed that student teachers should not only prove statements given to them but rather student teachers should come up with those statements by engaging in conjecturing. The current study focused on students' abilities to resolve proof-related tasks supplied by the researcher without the students having to ponder about the source of the statements or tasks they had to prove. Therefore, further research into ways of fostering means by which student teachers can do more than just producing an argument through conjecturing can help to enrich students' proof experiences. The call for further investigation into students' conjecturing activities has also been suggested by Lampert (1992) in Cirillo and Herbst (2012) who asserted that conjecturing about mathematical relationships is an important process of mathematical practice because conjecturing allows students to participate in socio-mathematical norms of the discipline in ways that are as close as possible to those of research mathematicians who compose proofs. Hence, testing the plausibility of these hypotheses through further research will contribute significantly towards efforts to reduce discrepancies between research mathematicians and student teachers with regard to proof construction competences.

As concluding remarks on this section it is recommended that future studies focus on finding ways of fostering and developing key competences that were identified to be lacking among students involved in this study. Areas that need future research attention include ways of raising micro reasoning (i.e., issues at meta-mathematical level) when doing proofs. Mathematics education researchers need to identify ways encouraging deep engagement with underlying ideas of mathematical proof in order to reduce chaotic behaviour uncovered among student teachers in this study, especially by focusing particularly on conjecturing. Research aimed at promoting stability within the axiomatic proof scheme once it has been attained by the students so that they do not move back to low level cognitive proof scheme categories is suggested as an area for further research.

6.8 Personal Reflections

My initial thoughts about the research problem were hardly coherent, characterised with lack of clarity on whether the focus was on learning or teaching issues in mathematics education. Coaching from my supervisors and reflecting on comments that followed presentations at departmental level helped to shape the incoherent thoughts into a learning problem with the title: *Undergraduate student teachers' conceptualisations of mathematical proof.* During problem refinement process after provisional registration I read about a branch of metaphysics called ontology whose ideas influenced my thinking about undergraduate students' conceptions of mathematical proof and my working thesis title then changed to: *Ontology of proof schemes in Zimbabweean undergraduate student teachers' conceptualisations of mathematical proof.* Central ideas of the research involved determining the student teachers' proof schemes states and how those proof schemes come into being. Hence, there was confusion over whether the focus was on ontology or epistemology (process of knowing) of the proof schemes. My personal reflections on this apparent lack of clarity and comments from conferences and mentoring by my supervisors influenced me to revert to the initial thesis title at the thesis writing stage after observing that it adequately addresses issues pertinent to students' thinking processes about mathematical proof.

My Master Degree dissertation explored secondary school students' interest levels in contexts for learning mathematics. To develop a sense of students' preferences for contexts for learning mathematics I employed a survey research design that elicited largely quantitative data that were analysed using crosstabulations in which descriptive statistics, Kendall's W mean ranks and Chi-square tests were determined using the SPSS facility. The current study employed a case study that yielded thick data that were overwhelming when I fractured them to begin the sense making process. Hence, a paradigm shift from predominantly quantitative to qualitative research skills was called for. Consequently, my data analysis was marked by prolonged periods of frustrations and confusion about how to draw meaning from the data. My personal reading of related studies from journal articles and primary sources on qualitative research skills (e.g., Corbin & Strauss, 2008 Miles & Huberman, 1994, Punch, 1998, 2005; Yin, 2009) helped greatly to improve on qualitative research methods. Reflecting on suggestions by peers and experts that followed my presentation at the Southern African Association for Research in Science, Mathematics, and Technology Education (SAARSMTE) annual conference in January 2015 in Maputo, helped me to incorporate reflective interview audits and member checking as strategies to ensure theoretical saturation in my data collection procedures (Corbin & Strauss, 2018).

Mentoring efforts by my supervisors, especially through parallel research projects I worked on together with them complemented my personal efforts to ensure rigor and precision of the research process. For example, the project on Continuing Professional Development (CPD) of mathematics teachers that took place in 2015 came at an opportune moment when I was developing my research instruments for the pilot study during the same year. The research study on CPD of mathematics teachers involved intensive research techniques ranging from interviewing skills, data capturing, validity and reliability to transcribing and analysis of data. The study also involved research report writing and culminated in a published paper in 2015. These parallel projects helped shape my qualitative research methods as I benefited from rub-on effects from experts involved.

Being a part-time PhD student with full employment obligations meant that I had to deal with heavy workloads that sometimes interfered with my studies. My PhD work rate accelerated significantly when I was granted Sabbatical Leave by my employers. I spent 8 months at the study site devoting many hours to my studies. I accomplished the following activities during this period: data collection, data analysis and by the end of my Sabbatical stay I had produced a full first thesis draft —something I had failed to achieve during the first three years of study.

Feedback from reviewers of manuscripts prepared for publication and advice from supervisors contributed significantly in refining the thesis draft. For example, the need to justify inferences drawn from qualitative data with in-vivo codes and interpret and account for findings inconsistent with existing literature was embraced during thesis refining. Pondering over comments that followed conference and Faculty Higher Degrees Committee (FHDC) presentations helped greatly in refining the thesis draft. For example, most recently during the SAARMSTE annual conference in January 2018 in Gaborone a member of the audience during my parallel session presentation suggested that I find an alternative word to the word "terms" in my main research question*: In what terms do undergraduate student teachers think of mathematical proof?* Another member suggested that I consider errors and misconceptions in my findings. I considered these suggestions seriously and decided that I should not include errors and misconceptions as these were outside the scope of the study. After thinking through the use of the word "terms" and reflecting on literature my decision was to retain the word "terms" in the main research question. At the same conference I got a flash of insight into the nature of horizontal and vertical shifts in proof scheme trajectories— a crucial observation I had overlooked during data analysis and thesis writing phases.

Overall, conducting this research provided me with a new perspective of theoretical and practical considerations in the manner in which undergraduate student teachers think of mathematical proof and the different pathways through which students' thinking come into being. For instance, a key practical consideration for mathematics educators and researchers pertains to the need to attain the proposed trajectory of proof scheme. The whole research journey has enhanced me as both a mathematics educator and upcoming researcher.

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Appendices

Appendix A: Mid-instruction assessment data collection task sheet

Task Sheet 1

Instructions

Answer all questions

Justify your solutions as much as possible

Answer all questions. Justify your answers as much as possible.

1. Determine whether the following statement is true or false.
   For all real numbers a and $b, a - b > 0 \Rightarrow a^2 - b^2 > 0$.
2. Determine whether the statement is true or false. If x is an integer, then $x^2 - x$ is even. Justify your answer.
3. Determine whether the statement is true or false For all real values of x,

   $$f(x) \equiv \ 2x^2 + 7x - 4 \Rightarrow f(x) > 0.$$

4. Prove that the sequence defined by $(u_n) = \frac{n^2-1}{2n^2+3}$ converges.

Appendix B: End of instruction assessment data collection task sheet

Task Sheet 2

Instructions

Answer all questions

Justify your solutions as much as possible

1. Define a sequence (x_n) inductively by $x_1 = 1$ and $x_{n+1} = \frac{2x_n+3}{4}$. Prove that (x_n) is a bounded monotone sequence and hence determine its limit.

2. A sequence (a_n) of real numbers is defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$.

Prove that (a_n) converges and find its limit.

3. Prove that $f(x) = x^2 + 2x - 5$ is uniformly continuous on $[0, 3]$.

4. Use the definition of appropriate limit to prove that

(a). $\lim_{x\to\infty} \frac{\sqrt{3x^2 +4}}{x} = \sqrt{3}$.

(b). $\lim_{x\to 1} \left(\frac{x^3}{x-1} - \frac{1}{x-1}\right) = 3.$

Appendix C: Reflective Interview Guide

Entry phase

(i) Do you enjoy doing proofs?
(ii) How often do you engage in proving?
(iii) What is a mathematical proof? [If a student were to ask you what a proof is, what would be your answer?]
(iv) How do you usually construct proofs? [If a student were to ask you how a proof is constructed, what would be your answer?]

Reflective phase

Questioning was interactive in which probing was used to focus on points of change in the students' (ways, manner/practices of proving) and levels of conviction. In other words interviewing was directed at eliciting students' modes of arguments employed. Stimulated recall questions were used to evoke students' earlier and current proof construction experiences. Interview instructions and questions for third and final data collection process are reproduced. t

Please provide a historical account of your experience with mathematical proof at:

Pre A level

……
………………………………………………………………………………………

A level

……
…………………………………………………………………………………………

Undergraduate level (student's current Rea Analysis experiences)

……
……………………………………………………………………………………….

Describe any differences you noted as you went through the above phases of learning about proving in mathematics.

……
……………………………………………………………………………………….

Appendix D: Sample of mid-instruction reflective interview transcriptions

Student teacher: Tina

Phase 1: reflective interviews

Date: 05/04/2016 Venue: Mathematics Lecture room

Researcher:[…]. Thank you for being one of the participants to this study. […] we have been doing proofs in Fundamentals of Analysis. What do you think is a mathematical proof ?

Tina: Aaa, mathematical proof is something that can be used to come up with various formula or something that can be used to prove, maybe event, maybe possibilities. (Laughing). Because, eee, in statistics we have analysis but as far as this analysis is concerned actually it is not related to probability but it is on its own related to pure maths, sort of.

Researcher: […] when do we say one has produced a mathematical proof?

Tina: When one has produced a mathematical proof, actually he/she has to convince the learners or listeners to come with something you have to follow certain procedures. And those procedures must reach a conclusion whereby the learner is satisfied that that is correct. Through the channels or steps reaching a certain answer or stage.

Researcher: What are you referring to as steps?

Tina: Actually, I am referring to, eee. Ok let's say you are given a problem and you are asked to prove it you cannot just write the problem and then say the proof which is given is the answer. Between the problem and the answer there are mathematical steps required which are understandable so that you reach to […] (Aaa) sometimes they have to be logically connected but sometimes not for example, when there is a negative operation followed by a positive operation, someone can vice versa start with the positive, go to the negative but the operations might be the same.

Researcher: Ok, thank you. [….] Next, I want you to reflect on your earlier experiences with this idea of a proof. […] at these phases, pre-A'level, A'level experiences and undergraduate experiences. So we will start with your pre A'level experiences say from Grade one up to Form four. Describe your experience with proofs?

Tina: Aaaa, actually I didn't experience much about proofs but, aaa, the mathematical concept which were actually used to like the formula was already provided. You could be provided with the sort of formula then you could just substitute the required numbers then you come up with an answer. At that stage I had no much experience as far as proofs were concerned.

Researcher: So, but I understand you were using at primary, the area is LxW and the perimeter is $2(L+W)$. At primary how did you come to understand these […] (area and perimeter).

Tina: Ok, eee, those actually the demonstrations which could be done by the teacher made us to understand those formula.

Researcher: Oh, so teacher demonstrations.

Tina: Yes, the teacher could demonstrate on the board, draw a rectangle there then label this breadth, length or width then he would tell you that area is length x width. Yes. After telling he then demonstrates by labelling those two oh! 4 sides with actual numbers. Then he would demonstrate by substituting [.....] Yes like that. Then just reach to the answer. […] The emphasis was on calculations as well as identifying sides.

Researcher: So the proof was at the level of identifying sides, appropriate formula and then applying.

Tina: Yes

Tina: How about secondary level experiences?

Tina: Secondary level, at A' level that's when I started to experience how to prove. At O' level, (huuu), proofs – [silent]

Researcher: But how were you learning the sine rule […] and Circle Geometry […].

Tina: Yes, yaa. On the circle geometry actually, you could be given a theorem and then you were told that this is the angle at the centre is twice that it subtends in any remaining part of the circumference by just drawing a circle on board. No meaning, no no any idea of proofs were involved in circle geometry. There were just theorems. Some of which you could apply (eee) the concepts of triangle from form one. But as proofs were concerned not much was being done.

Researcher: How about proving?

Tina: In terms of proving maybe it was out of the syllabus but..

Researcher: And then at A' level now?

Tina: Now at A' level actually that's when I started to make proofs, like the formula for finding area of a triangle lets say $\frac{1}{2}\ absin\theta$. Then that's when I started to apply the half base times height concept at A level to come up with $\frac{1}{2}\ absin\theta$, whereas at O level we were being given the formula $\frac{1}{2}bh$ or $\frac{1}{2}\ absin\theta$ without proving where they came from but at A level you could actually develop the $\frac{1}{2}bh$ into $\frac{1}{2}\ absin\theta$

Researcher: How about the double angle identities in trig. The compound angle identities […] were they proved?

Tina: Yes, these were now prove, they could be proved using the concept of cos20=… (stuck). We actually went through the proofs but …the

Researcher: How about the $sin\ (A + B)[...]$?

Tina: Yes we could prove that [repeated]. We could prove that in the interest of the teacher. Actually the proofs were not part of our syllabus. The teacher could tell you that the proofs of trig ratios is not in your syllabus, what is in your syllabus is the solving of equations involving trig ratios.

Researcher: Equations and identities.

Tina: Equations involving trig.

Researcher: […]what are your current [undergraduate level] experiences with the idea of a proof.

Tina: Aaaa, at this level, it's a little bit better as far you can take your time on a certain aspect. Proofs at this level they take much time to comprehend whereas at A' level college level (aaa) you could accomplish to comprehend the proofs on a shorter space of time supply because they were few. […] here we deal with greater items.

Researcher: Ok. Reflecting on those various phases; the pre A' level, the A' level and the university. Have you noted differences in the way the proofs are done?

Tina: Yes I have noted some difference.

Researcher: Can you describe the differences?

Tina: At this level, aaa! At this level the proofs are done [silent] what can I say. They are done in a certain way which leaves one. Actually satisfied… [silent] eeee

Researcher: […] go on

Tina […], whereas at those levels the college levels, A'level, you could not ask; how did you come with this, whereas at this level you know as a professional person it will be humiliating if you didn't ….[inaudible]. So you to know, so it's a matter of mutual cooperation [….] At university proofs are [silent] you ought to know them whereas at A' level you can even write the exam without knowing them, at the end you get your points. You don't know certain aspects. .

Researcher: Right. Given a mathematical statement, how do you decide on which method of proof to use?

Tina: In terms of what? In terms of numbers or algebra?

Researcher: […] how do you decide on which method to use to prove?

Tina: Actually, to prove the statement, I think a statement, a mathematical statement.

Researcher: Yes […] How do you decide to say I think I need induction here, I need contradiction here ….need direct deduction. […] How do you decide that?

Tina: Ok, there are certain aspects which go hand in hand such that if you use one, you might get to the answer using the appropriate way whereas as there will be the actual way to reach to the answer. Aaa, there is a problem because if you apply the Triangular rule [presumably referring to the Triangle Inequality] where it is not required then just manipulate up to the required answer I don't think it's a good idea at this level.

Researcher: Alright

Tina: Just because you ought to apply the proper formula

Researcher: […] How do you become convinced that I have produced a proof/

Tina: The stages [repeated] I will have gone through convince me that that's the correct answer. There is some kind of cohesion from stage to another. You check for cohesion. Then you check for proper application for certain theorems.

Researcher: And [….], now let me come to your work [referring to task 3] Here on question 3 […] you needed to use examples for question $3, f(1)\, f(..)$ in other words, you decided to pick specific examples to prove this statement here. […] Oh so, it is greater than zero for all $x > 0$ […] this implies $f(x) > 0$ for $x > 0$ and $f(x) < 0$ for values of $x < 0$. So what is your overall conclusion about the statement.

Tina: Haaa

Researcher: It is greater than 0 for some values and less than 0 for other values. So what is your overall conclusion?

Tina: Aaaa. The overall conclusion there might be, there might be just from –∞ to +∞ because if are check for values which are less than 0, aaa!

Researcher: You said for some values it is greater than 0 and for others it is less than 0. This is what you wrote here.

Tina: Yes. So this conclusion is based on what I have, what I had worked above. I could see the trend of the numbers. For all x

Researcher: And in your working here, you established that it is greater than 0 for some values of x and less for other values less than 0?

Tina: Ok. Actually I was referring to the output there. For all values that are greater than 0, I took them as the domain, those numbers which I could substitute in but the output varied, they range from minus…,minus. So I could conclude if I substitute negative numbers you could get negative numbers also up to 0. greater than 0. But for numbers greater than 0, actually they were positive as outputs.

Researcher: I get you but the statement had said: Prove that if $f(x) = 2x^2 + 7x + 4$ then this implies $f(x) > 0$. What is your overall conclusion after doing this?

Tina: [Coughs]

Researcher: Are you accepting this statement or you're saying it's wrong?

Tina: It is a wrong statement because when we are saying […] So it's a wrong statement because some of the real numbers are negative but we are not getting numbers greater

Researcher: […] As you see, you have used specific examples […] how does it lead to $\varepsilon < \frac{3}{x^2}$? Where is it? Yaa. [..] What were you trying to find?

Tina: To tell you the truth, I knew nothing, I didn't read about this but I know when you are testing for convergence it has to do with ε greater than something actually I had not grasped the formula, it's like that $|a_n - L|$ should be less than ε. That's what I was trying to apply but actually I had not grasped the [inaudible]. So what I wrote here was just a matter of writing, I did not know. Actually I was writing for the sake of answering the question.

Researcher: And then question 2 you said if a and b are odd. How did you show that a^2 and b^2 are odd? You made that claim here. If we take a, b and c as odd numbers, then a^2 b^2 and c^2 are still odd.

Tina: Ok, ok, it's like aaaa, iii, I missed a certain statement there I could use the actual numbers like you take 3 for example, 3^2 you get a 9 which is odd. If you square 7 then you get 49

Researcher: Alright. […] I have noticed that where they [students] are supposed to use axioms they use examples and where they are supposed to use examples they use axioms. What do you think causes that among learners?

Tina: Personally, aaa, I think, aaa, as a matter let's say, I am examined […] question, to test you properly they ask you for a certain aspect, application of aspect which seem to be difficult but for my own understanding I can simplify it by using examples. For example, they can ask you to apply a certain axiom yet I don't know that axiom, so I have to use an example.

Researcher: How about [ ] when you are now supposed to look for examples. You are now axioms?

Tina: It's just the same. It depends on the statement that is written on the question. Some of the questions can give examples and they ask for axioms whereas other can […] I actually was trying to make it clearer to anyone who doesn't understand the aspect if he gets hold of the question. He/she can understand as far as use of numbers is concerned.

Appendix E: Sample of end of instruction reflective interview transcription texts

Student teacher: Cortney

Date: 11/05/2016 Venue: Mathematics Lecture room

Researcher: Afternoon [….]

Cortney: Afternoon. How are you sir?

Researcher: Welcome back to the last session of our interviews.[…]. First of all […] I wanted to describe your experience of the whole, eee, [interjection from student teacher]

Cortney: Course! Well, aaa, for starters I can, eee, say that prior to engagement to this course I was one person who just, aaa, used the induction method. Well, maybe it's not precisely the induction method but supposing values I could say, If I am given something I would simply assume that $x = 1, x = 2, x = 3$ and try to go on and on till I get maybe 20 or till I get tired. [Laughs], without really its just generalisation based on numbers. Basing on the examples I wouldn't really prove to say does it really hold for everything

Researcher: And now [referring to analysis experience]

Corney: Now it's a different case because after this course I now know how to go about proofs. I now know that you don't just start by claiming that something is true after working with numbers only. You have also consider induction supposing n is something supposing n is $k + 1$ and then making a conclusion basing on items getting to n or something.

Researcher: Thank you […]. Thank you very much. You have just said that there are cases when you used to use examples. Now can you account for the reason why there is a tendency to use examples by student to prove them [...] even when they require deductive reasoning?

Cortney: [inaudible] especially in my case when you don't know anything there is really a good reason to do that. But if you look at us, like or myself I now know that I cannot generalise things using numbers. I have to use induction or the axioms or something so that I have made my conclusion to the fullest.

Researcher: Oh, thank you […]. Then my next question is now I think you will agree that there are situations where we have to use just a counter example but students have the tendency to use axioms and definitions in those cases where they are supposed to use one example. Can you account for that sort of behaviour?

Cortney: Like someone is supposed to use an example, they use axioms?

Researcher: [...] To take you back for example that problem which says that if $a - b > 0$ then $a^2 - b^2 > 0$ one would just simply look for a counter, but people used axioms and order properties. Have you thought about this scenario […] say, eee, [interrupted by student teacher]

Cortney: Ok fine. I think when you are given a question like this sometimes you wouldn't really know what it wants. So looking at the course that we are doing, you will think that maybe everything it wants me to use axioms. You will trying to attain better marks so you will always think that what if I just end up with a counter example, will it really get all the marks or I have to make use of axioms so that the answer gets balanced or maybe you look at the number of marks.

Researcher: […] so anything that does not have axioms and definitions wouldn't count as proof in your opinion right now?

Cortney: Oh, yaaa. So sometimes if you look at the number of marks like for example, if this question is carrying say 2 marks, I can just do substitution [use examples] but if it is carrying 7 marks, I have to consider the axioms so that I get out the examination knowing that well I did the right thing.

Researcher: Ok, yaaa, it's driven by the need to earn marks.

Cortney: Somewhat, not only that, maybe ok fine we have covered the syllabus we are done so sometimes you just feel that ok fine I have covered this now know how to that what if I just put down what I know on paper as long its really support that. *Kana wanyatsonzwisisa* [if you have really understood] you can actually put stuff on paper.

Researcher: Alright Rose. Now I think you have accounted for the behaviour. Coming back to the previous task that you worked on. What do you understand by the limit of a function $f(x)$ as $x \to \infty$?

Cortney: I know that if I am given this [referring to the definition] I have to come up with the value of X and that X must be greater than, it has to be greater than $\delta(\varepsilon)$.

Researcher: You can illustrate here [referring to answer sheet paper meant for illustrating] if you want. You are free.

Cortney: I hope I still remember

Researcher: Yaa, go on it does not matter

Cortney: So I will have to let $\varepsilon > 0$ be given. Now, aaa, we need to determine this big X s.t. the small x is an element of x and y element of real numbers then there is also a condition that....well I'm going lost but whatever I know that in the end there must be function of x subtracting the limit under the modulus sign there must be less than...but in the end also if I substitute here I am solving for x. Small x, it must be greater than something there and that something the I get I have to set my big X equal to that something.

Researcher: […] How do you usually start proving [...] mathematical statements? Maybe at school or when in this course how do you usually start proving some mathematical statements?

Cortney: Well, prior to engaging in this course, I will just start by plugging in numbers and maybe weeks after you know I then started using the induction processes and all that and as time passed I started using, aaa, definitions of concepts like for example, if I am trying to come up proofs for limits. Whatever that I have to do it must be in line with the definitions of limits. If I have to prove stuff that has to do with uniform continuity I have to come up with the definition for that first. It's a guide like for example, if you do this especially this one on limits [ ] I will know that I will have to substitute for function of x. I have to put something for the limit. I have to put something that Iam given so if you come up with theee, if you start from the definitions it helps you to come up with how to really go about it till you get the answer.

Researcher: Thank you […]. I appreciate. Do you have anything to say about your experiences with proof in this course? Anything that you might want to say?

Cortney: In as much as I have benefited on a lot of concepts, I can say I benefited on something. We were actually debating just yesterday how I worked out number (aaa).... The one that I did on the board. I am not sure what question that was.

Researcher: This one? [referring to task]

Cortney: I thought this was ok to do this and when we were working out even some students thought it was ok to do this because sometimes that’s what we do when working out form 4 stuff but we only get the answers by chance. But the proper thing like we got a negative sign, you can’t square this. This and that you have for as long you have a negative sign [inaudible]. […] I am saying I actually benefited. If it wasn’t for this course I was going to keep on confused.

Researcher: Ok

Cortney: So somewhat I will remember this and I will simply say I got it from this course.

Researcher: Thank you […]. May God bless you.

Appendix F: Sample of Mid-instruction assessment chalkboard demonstration transcription texts

Student teacher: Debra

Date: 29/03/2016 Venue: Mathematics lecture room

Task 1: *Decide whether the following statement is true or false. For all* $a, b \in \mathbb{R}$, $a - b > 0$ *implies that* a^2 -$b^2 > 0$. *Justify your answer.*

[Student reads task instructions]; {Decide whether the statement is true or false, then the statement is..} [Student teacher writes on the chalkboard]

For all $a, b \in \mathbb{R}$, $a - b > 0 \Rightarrow a^2$ -$b^2 > 0$.

{So if a and b are real numbers it means that a and b be can be either positive or negative numbers so it means that a can either positive or negative. But if we square both negative and positive numbers, the result is positive} [Student teacher writes and verbalizes the following]

If a *is –ve* → $(-a).(-a) = a^2$

If a *is +ve* → $(a).(a) = a^2$

{So the result is the same} [referring to squaring of – a and a]

{Then to find b^2, b can also be positive or negative but if we square we also find the same rsults}

[Then the student writes on the chalkboard while verbalizing what she is writing]

If b is *–ve* → $(-b).(-b) = b^2$

If b *is +ve* → $b.b = b^2$

{So this does not affect the sign , so it means that } [Student teacher writes]

a^2 -$b^2 > 0$

{Since the product of negative values of a is the same as the product of positive values of a

And then here [pointing to illustration on *If* b is *–ve* → $(-b).(-b) = b^2$

If b *is +ve* → $b.b = b^2$] the negative product (pause), the product of negative values of b is the same as the product of positive values of b. So this statement is true] [Student writes conclusion on the board]

∴ *the statement is true*

Appendix G: Sample of End-of-instruction assessment chalkboard demonstrations transcription texts

Student teacher: Tanya

Task 4(a): *Use the definition of appropriate limit to prove that*

$lim_{x\to\infty} \frac{\sqrt{3x^2 +4}}{x} = \sqrt{3}$

[Student reads out question] {Use the definition of appropriate limit to prove that } [student writes]

$lim_{x\to\infty} \frac{\sqrt{3x^2 +4}}{x} = \sqrt{3}$ [student is silent after writing statement on chalkboard]

{So we are going to first of all define the limit as $x\to\infty$ of a function } [student writes and verbalizes]

Let $\varepsilon > 0$ *be given, we need to determine* $X \in \mathbb{R}$ *s.t. if* $x > X$ *then* $|f(x) - L| < \varepsilon$

(coughs) {And} [student verbalizes and writes the following] *if* $x > X$ *then* $\left|\frac{\sqrt{3x^2 +4}}{x} - \sqrt{3}.\right| < \varepsilon$ {From the numerator, we are going to factor out x^2, the square root of x^2 so that inside the bracket we have} $\left|\frac{\sqrt{x^2}\sqrt{3+\frac{4}{x^2}}}{x} - \sqrt{3}\right| < \varepsilon$ {If we simplify the numerator $\sqrt{x^2}$ becomes x and x and the denominator [referring to x] , will cancel each other. Then we have } [Student writes]

$\left|\sqrt{3+\frac{4}{x^2}} - \sqrt{3}\right| < \varepsilon$ {We have that identity which states that } [student verbalizes and writes]

$\sqrt{x} - \sqrt{y} = \frac{x-y}{\sqrt{x}+\sqrt{y}}$ {So we are going to apply this identity here. Which means we are going to have}

$\left|\frac{3+\frac{4}{x^2}-3}{\sqrt{3+\frac{4}{x^2}} +\sqrt{3}}\right| < \varepsilon$ {In the numerator, 3 and 3 goes so we are left with}

$\left|\frac{4}{x^2\sqrt{3+\frac{4}{x^2}} +\sqrt{3}}\right| < \varepsilon$ {What we want to do here is that we want to find ,eee, x in terms of ε. This is going to be (coughs)} $\left|\frac{4}{\sqrt{3x^2+4} +\sqrt{3}}\right| < \varepsilon$ $\frac{4}{\sqrt{3x^2+4} +\sqrt{3}}$

[Student makes an effort to simplify, relooks at the working and says] {I don't why…[inaudible]. Iam failing to, uuu, simplify but, but I want, what I want is that I must get x in terms of ε. So we multiply this, this bracket} [silent and staring at the working on the C/B. Student rubs what has been written. Student is apparently stuck i.e. an impasse].

Appendix H: Informed consent form

University of Zimbabwe

Department of science and Mathematics Education-

Informed consent form

Dear BEd Mathematics student

I am doing research in the Department of Science & Mathematics Education. I am pursuing a Doctoral degree entitled: Undergraduate students' conceptualizations of mathematical proof

The aim of this study is:

To explore undergraduate mathematics education student teachers' thinking around the notion of mathematical proof

Through your participation I hope to understand and describe:

- Why you construct proofs of mathematics statements in the manner you do,
- The critical elements of knowledge of the processes involved in proving

Your participation in this study is voluntary. You may refuse to participate or withdraw from the study at any time with no negative consequence. There will be no monetary gain from participating in the study. Confidentiality and anonymity of records identifying you as a participant will be maintained. Please sign below to show your willingness to participate in the study.

Thank you:

Researcher: Mr Zakaria Ndemo

Contact Details: Department of Science and Mathematics Education

zndemo@gmail.com

[0779328070]

I (full name)…XXXXXX………YYYYYY…hereby confirm that I understand the contents of this document and I voluntarily consent to participating in the study.

Signature of participant XXXXXXX Date 29-03-2016

Appendix I: Study Leave supporting documents

DEPARTMENT OF
SCIENCE AND MATHEMATICS EDUCATION

Chairperson
C.K. MUKUNDU
Cert. in Education, B.ED (U.Z), M.Sc. ED (U.Z)

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SCIENCE AND MATHEMATICS EDUCATION UNIVERSITY OF ZIMBABWE

Registrar
Bindura University of Science Education
Bindura

28 October 2015

Dear Registrar

RE: Study leave support for BUSE lecturer and UZ doctoral student Mr Zakaria Ndemo: February – June 2016

Warm greetings

As you are aware, Mr Zakaria Ndemo, your mathematics lecturer at BUSE is concurrently doing a doctoral study (part time) with the Department of Science and Mathematics Education (DSME) here at UZ. He has been working on the pilot phase of his study in the past 5 months, from data gathering with some students at BUSE right through to analysis of the data. He is scheduled to give a Faculty seminar on that pilot study at UZ on 12 November 2015.

We expect him to commence fieldwork for his main study early next year (2016). Mr Ndemo is researching aspects of undergraduate student learning of a subject called mathematical analysis. Fortuitously, there is a rare opportunity for him to generate the sort of data he requires for his main study with our fulltime mathematics students in the DSME here at UZ. The students will be taking a mathematical analysis course in the semester that runs from February to mid-June 2016. While we can facilitate his fieldwork activity here at DSME, we note that he would require fulltime engagement with the activity. If the possibility exists on your side that you could support him with a grant of a study/sabbatical leave that covers the period of his fieldwork [Feb – June 2016], you would contribute substantially to Mr Ndemo's progress with his research to an extent that might bring his expected thesis completion date forward to 2018. The DSME would highly appreciate such support to the student.

With kind regards

D. J. Mtetwa

SCIENCE AND MATHEMATICS EDUCATION
OFFICIAL

D K J Mtetwa
Professor of mathematics education and supervisor for Mr Ndemo
Department of Science and Mathematics Education
University of Zimbabwe
Email: dmtetwa@education.uz.ac.zw Phone: +263 712 877 820; +263 4 303211 ext 16081/82

DSME Chair: Dr S J Mpofu ......The department supports the idea. ...... Mpofu 28/10/15

REGISTRY DEPARTMENT

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BINDURA UNIVERSITY OF SCIENCE EDUCATION

5 April 2016

Mr Zakaria Ndemo
Physics and Mathematics Department
Bindura University of Science Education
P Bag 1020
Bindura

REF: APPLICATION FOR SABBATICAL LEAVE

Reference is made to your application for Sabbatical Leave dated 16 March 2016. You are hereby advised that your application has been approved. You may proceed on Sabbatical leave for 240 days with retrospect effect from 14 March 2016 to 18 November 2016. Please note that you shall not accrue leave days for the period of your Sabbatical leave.

Your next Sabbatical Leave will be due to you after 1 March 2022

I hope and trust that participating in the Sabbatical Leave scheme will assist you to realize both your personal and professional goals.

Yours Sincerely

E Manhando (Mrs)

Cc. Vice Chancellor
Dean, Faculty of Science
Registrar
Bursar
Chairman, Physics and Mathematics
Sabbatical Leave File
Salaries
Personal File

Appendix J: Anti-Plagiarism Report

Plagiarism Checker X Originality Report

Similarity Found: 1%

Date: Thursday, February 22, 2018
Statistics: 1141 words Plagiarized / 136368 Total words
Remarks: Low Plagiarism Detected - Your Document needs Optional Improvement.

Undergraduate student teachers' conceptualisations of mathematical proof By Zakaria Ndemo A thesis Submitted to the Faculty of Education, University of Zimbabwe In fulfilment of the requirement of the Doctor of Philosophy Degree (DPHIL.ED) In mathematics education Department of Science and Mathematics Education Harare 2018 Abstract Students face serious challenges in learning mathematical proofs.

Although many studies have been done with the aim of improving the learning of mathematical proof beyond mere regurgitation of memorized facts, very few studies have been based on students' actual proof attempts. Motivated by the need to develop an understanding of students' thinking grounded in their actual proof attempts the main research question put forward was: In what terms do Zimbabwean undergraduate student teachers think of mathematical proof? The goal was to explore students' schemes of argumentation and how students' thoughts around mathematical proof evolve.

A case study approach guided by the scientific realist philosophy was applied in the context of a teaching experiment that involved 10 undergraduate mathematics education students, 6 female and 4 male. Three tools were used to elicit data: written proof tasks, reflective interviews and think aloud interview protocols. Directed and summative content analysis techniques in which theoretical constructs about proof learning such as the notion of technical handles and conceptual insights, syntactic and semantic modes of argumentation and ideas drawn from Harel and Sowder's taxonomy of proof schemes were applied to the transcription texts of audio and video recorded interview data for the purpose of inferring the kinds of proof schemes held by the students and how students' proof schemes emerge.

Masasu
02/05/2018

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