Mathematics before Newton

An Inaugural Lecture

GIVEN IN THE UNIVERSITY COLLEGE OF RHODESIA AND NYASALAND

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by

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MATHEMATICS BEFORE NEWTON

INTRODUCTION

A LITTLE less than 300 years ago Isaac Newton, having graduated under Barrow, Lucasian Professor of Mathematics in the University of Cambridge, but without, so we are told [1], showing any particular signs of originality, was compelled by an outbreak of plague to retire to his country home in Lincolnshire. In the winter of 1664–5 Newton discovered the method of infinite series and a little later the particular case of the binomial series. In the next year he computed an area under a hyperbolic arc to fifty-two decimal places, and following this invented the method of differentiation. In May 1666 he found the method of integration as the inverse of differentiation. 'In the same year', Newton wrote,

> I began to think of Gravity extending to ye orb of the Moon, & (having found out how to estimate the force with which a globe revolving within a sphere presses the surface of the sphere) from Kepler's rule . . . I deduced that the forces which keep the Planets in their Orbs must (be) reciprocally as the squares of their distances from the centers about which they revolve.

In his own phrase, 'I was in the prime of my age for invention'; as one of his biographers remarks, 'there are no other examples of achievements in the history of science to compare with that of Newton during those two golden years' [2]. He succeeded Barrow as Professor of Mathematics in 1669 and until 1684 his reputation if considerable was limited to professional and academic circles. In 1684 he was encouraged by the Astronomer Royal, Edmond Halley, to begin his famous *Principia*. This work was

finished in the spring of 1686 and published in 1687 under the full title *The Mathematical Principles of Natural Philosophy*. Newton's tremendous fame, which was almost immediate and continues to this day, rests largely on this work. As far as his early and brilliant discoveries in the calculus are concerned, he clearly owed much to his predecessors of the preceding century and a half and to his contemporaries Barrow and Wallis. Moreover, many of these discoveries were made independently by Leibniz who founded a fine school of mathematics which included two of the Bernoullis.

The significance of Newton's mechanics is of a quite different order. Some 400 years before, the medieval thinker Roger Bacon [3] (1214–94) had rightly stressed the fundamental place of mathematics in physical science, but mathematics in his day was a very primitive art. Galileo struggled against and eventually broke through the web of misleading verbal analogies which characterized the 'physics' of Aristotle and his medieval followers to lay the foundations, but little more, of a rational mechanics. It was Newton who, inspired by both the pure mathematics of his day and the needs of physics at that time, at once invented the tool, the infinitesimal calculus, and applied it to write a marvellous first chapter in a new subject, Mathematical Physics. The scope of Newton's contribution is rarely appreciated, many mathematical students, and at least one professor of mathematics once known to me, taking it for granted that he merely formulated the laws of motion and applied them to Kepler motion, that is the motion of a particle in an ellipse under a force to the focus. These contributions would not, of course, be trivial, for they lead immediately to information of the greatest scientific value. For example, the universality of the law of gravitation

throughout the solar system and the ratio of the mass of the sun to that of the earth or the mass of any planet having a satellite can at once be deduced from the observed values of relative distances and times of rotation. However, Newton was able to deduce much more from his laws [4].[1] From the sun's mass followed the height of the solar tide and from the heights of the spring and neap tides he estimated the mass of the moon. From the dynamics of a rotating gravitating body he showed that the earth was not an exact sphere but slightly flattened, an oblate spheroid and that, conversely, the length of the day for the earth or any other planet could be deduced from the observed shape of the planet. Rather more spectacular was his discovery that comets were also subject to the law of gravitation, in fact law-abiding if highly eccentric members of the solar system. All these conclusions were based on the sort of mathematics which, original and difficult as it was in his day, has now been relegated to college courses or to detailed computations in astronomical theory. The discussion of the motion of the moon under the combined attractions of the earth and the sun was recognized by Newton as much more difficult, in fact he admitted that it gave him a headache, and no wonder, for this so-called 'problem of three bodies' remains an important research topic to this day.

Newton did not by any means limit himself to mathematics and mathematical physics. His contributions to numerology, what Roger Bacon would have called 'the application of mathematics to sacred subjects' (cf. the *Opus Majus* [3] quoted below), seem to have been considerable although probably suppressed by his executors who

[1] The following account of Newton's work is given in Bell, reference 4, and is derived from the *Amer. Math. Monthly*, vol. xlix, pp. 561–2, 1942.

marked many of Newton's manuscripts 'not fit to be printed'. His ability and interest in public affairs were typically English, and his final achievements such that had his face not appeared on the coinage in recognition of his scientific fame, he could clearly have ordered its inscription according to his civil powers. Such felicity is rarely the lot of a university don who abandons scholarship for the paths of administration.

It is indeed hard to escape from the influence of Isaac Newton. During the eighteenth century his mathematical physics underwent a vast development at the hands of such masters as Laplace (1749–1827) and Lagrange (1736–1813). Both the calculus and Newtonian mechanics of continuous bodies were developed in very great detail by Euler (1707–83) in some 530 books and papers published during his lifetime, and another 241 published posthumously within forty-seven years of his death. Lagrange's *Mécanique Analytique* (1788) gave a purely analytical account of Newton's mechanics and Lagrange has been described as the first analyst. Laplace's *Mécanique Céleste* (1799–1825) represented the culmination of the classical Newtonian theory. It dealt with the figure of the earth, the theory of the moon, the three-body problem, perturbations of the planets, and the stability of the solar system. In this connexion it was shown by Hill and Poincaré in quite recent times that had the initial conditions of the problem, that is to say the speeds and relative distances of the earth and the moon, been chosen only slightly differently, then the moon would have travelled round the earth not in a circle but in an elongated path having a small loop at each end. It is clear from a sketch of this sort of motion that there would have to be six half-moons between two full moons. One historian of science, recalling their obsession with

circular motions, makes the comment that many philosophers would even have regarded the six half-moons as a logical necessity. The problem of three bodies also inspired Poincaré (1854–1912) to state a famous problem in higher analysis, his so-called 'last theorem', a term used in imitation of another famous problem, Fermat's Last Theorem, proposed in the seventeenth century and unsolved to this day. Poincaré's problem arises as a fixed-point theorem equivalent to the restricted problem of three bodies in dynamical astronomy. It was solved by Birkhoff (1884–1944), the leading mathematician of early twentieth-century America, and within a year of Poincaré's death. Like Newton both Poincaré and Birkhoff were making notable developments towards a new branch of pure mathematics, with Newton the calculus, in their case *topology*, in order to deal with problems arising in dynamics. Poincaré himself admitted that theoretical physics not only provided the problems of mathematics but often suggested how they ought to be attacked. On a more prosaic level, Newton's dynamics remains an essential part of our university teaching, unchanged in content after nearly 300 years. Euclid's geometry had, it is true, lasted nearly two thousand years when Newton was writing his *Principia* and remains to this day substantially unaltered, but 'Euclid' as we shall see was certainly not the work of one man but the final conclusions of a school of thinkers. Undergraduate mechanics is to a very large extent the mechanics of Newton[1] and until recently it seems that students were expected to study Newton's work in its original form. For example, I have in my possession Books I, II, III edited by Percival Frost and presented to a Miss Emily Wilkins on 9 December 1904 as the Upper VIth Mathematics Prize [5].

[1] The first real novelty arises in Lagrange's equations of motion.

The preface makes it only too plain that students were expected to study and be examined on the contents.

The professional mathematician needs no convincing of the excellence of Newton's thought. Not only is his work admired in its historical context (the same is true of his contemporaries Wallis and Barrow), but his methods remain of significance today, in at least one instance at the 'research' level.[1] Any history of the development of scientific thought during the last three centuries must contain a very large chapter on Newton's ideas. To discuss this subject in any detail would require a considerable background of scientific knowledge. I shall attempt to convey some impression of the achievement of Isaac Newton by describing the state of mathematics (as I have said there was little mathematical physics) before his time.

MATHEMATICS IN ANCIENT EGYPT AND BABYLONIA [2, 6, 7]

Our knowledge of early mathematics of the second millennium B.C. is based on quite recent archaeological researches, for example Egyptian hieroglyphics were fully deciphered as late as 1822. Two outstanding mathematical papyri have been found. The first, the Papyrus Rhind, has eighty-five mathematical problems. It was discovered in 1858 and dates from about 1650 B.C. although the material may be older. The other is the Moscow Papyrus which has twenty-five problems. It appears from a study of these that the Egyptians used a decimal system written on the same lines as Roman numerals, that is *without* the place system as known to us with units in one column, tens in the next

[1] I am indebted to Professor R. Wilson for the remark that the well-known 'Newton's triangle' is also used in the modern theory of the classification of power series.

and so on. They did multiplication by the easiest method yet found by man, that is by repeated doubling. The same method is used even today in that much glamorized invention the electronic automatic computer, our so-called 'electronic brain'. This is in reality a nicely contrived circuit with valves which can emit up to two impulses for the price of one together with a number of other components which record the passage of impulses on a magnetic tape, just like the tape recorder which some of us have on our desks. The most remarkable feature of Egyptian mathematics is not their lazy method of multiplication but the truly extraordinary things they did with simple fractions. They worked on the beautiful if arbitrary principle that all fractions should be written as the sum of natural fractions whose numerator is unity. Having arrived at a set of fractions of this type they rested from their labours. If on the other hand they found two or more fractions of the same form in the answer of the addition sum or if, again, they were doing a multiplication sum by the method of doubling, it would clearly become necessary to express unnatural fractions having two in the numerator as the sum of the natural fractions having a mere unit above the line. Fortunately there were provided tables for the decomposition of such fractions into natural fractions, for example

$$\frac{2}{97} = \frac{1}{56} + \frac{1}{679} + \frac{1}{776}$$

and similar decompositions are given in the Papyrus Rhind for all values of odd denominators on the left between 5 and 331. These methods do not appear to have been improved upon in Egypt even as late as Roman times, and this is as it should be, for they represent a certain perfection of futility which has not been achieved elsewhere. One question only remains to be settled. How could the

Egyptians, using such grotesque methods, have found the table of decompositions in the first place? A most interesting reconstruction has been made by O. Neugebauer who is at pains to give a proof which is in accord with known Egyptian practices and modes of thought. For example, an Egyptian who wanted to find $\frac{1}{3}$ of 3 would first point out that $\frac{2}{3}$ of 3 is 2 after which he divided by 2 showing that $\frac{1}{3}$ of 3 is unity. We proceed to Neugebauer's discussion of the proposition that

$$2(\tfrac{1}{5}) = \tfrac{1}{15}+\tfrac{1}{3}.$$

This important identity, however found, certainly underlies the decomposition of all the fractions in the Papyrus Rhind for which the left-hand denominator lies between 3 and 101 and is divisible by 5. Let us start with the natural fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and the special one $\frac{2}{3}$ which was also admitted. We may try various possibilities and after a few trials we might split the 2 into $\frac{1}{3}$ and $1\frac{2}{3}$. The product $\frac{1}{3}$ by $\frac{1}{5}$ is a natural fraction, namely $\frac{1}{15}$. The difficult question remains of finding the product $1\frac{2}{3}$ times $\frac{1}{5}$. We write

1 and $\frac{2}{3}$ (in BLACK),

and

3 and 2 (in RED),

this being an Egyptian custom when dealing with such operations, and the sum is $\frac{5}{3}$. It is now clear that $\frac{1}{5}$ of $\frac{5}{3}$ is $\frac{1}{3}$ and the proposition has been proved.

The Egyptians knew of geometrical progressions, for they had a problem about 7 houses each with 7 cats and each cat watching 7 mice. They had simple formulae for areas and volumes and took π as $\frac{256}{81} \simeq 3{\cdot}16$. The main scope of their mathematics may be inferred from a papyrus of the New Kingdom,[1] written for school purposes:

[1] The text has been slightly restored; see Neugebauer [6].

Another topic. Behold you come and fill me with your office. I will cause you to know how matters stand with you, when you say 'I am the scribe who issues commands to the army'.

You are given a lake to dig. You come to me to inquire concerning the rations for the soldiers, and you say 'reckon it out'. You are deserting your office, and the task of teaching you to perform it falls on my shoulders.

Come, that I may tell you more than you have said: I cause you to be abashed (?) when I disclose to you a command of your lord, you, who are his Royal Scribe, when you are led beneath the window (of the palace where the king issues orders) in respect of any goodly (?) work, when the mountains are disgorging great monuments for Horus (the king), the lord of the Two Lands (Upper and Lower Egypt). For see, you are the clever scribe who is at the head of the troops. A (building-) ramp is to be constructed, 730 cubits long, 55 cubits wide, containing 120 compartments, and filled with reeds and beams; 60 cubits high of its summit, 30 cubits in the middle, with a batter of twice 15 cubits and its pavement 5 cubits. The quantity of bricks needed for it is asked of the generals, and the scribes are all asked together, without one of them knowing anything. They all put their trust in you and say, 'You are the clever scribe, my friend! Decide for us quickly! Behold your name is famous; let none be found in this place to magnify the other thirty! Do not let it be said of you that there are things which even you do not know. Answer us how many bricks are needed for it?

'See, its measurements (?) are before you. Each one of its compartments is 30 cubits and is 7 cubits broad.'

The history of Egypt is written in stone and on papyrus. In Mesopotamia on the other hand there is hardly any stone and the ancient Babylonians built with river clay. As the Bible says, with complete accuracy:

And it came to pass, as they journeyed from the east, that they found a plain in the land of Shinar; and they dwelt there. And they said one to another, Go to, let us make brick, and burn them thoroughly. And they had brick for stone, and slime had they for morter.

They wrote on the clay, too [7], and it is from their baked clay tablets which date from the third millennium B.C. that, following on the decipherment of the Babylonian *cuneiform* script, research workers have quite recently unravelled the mysteries of Babylonian mathematics.

It appears that even during the Sumerian period, computation in a system of numerals based on the unit 60 together with the place-value system had been well developed. The sexagesimal system survives to this day with the unit 60 occurring in the measurement of time in seconds, minutes, and hours, also in our angular measure. We should note, however, that the sexagesimal system in arithmetical work occurs many centuries before its use in Babylonian astronomy. The mathematical texts fall into two distinct groups. The first, the 'Old Babylonian', belongs to the Hammurapi dynasty (*c.* 1800–1600 B.C.) which arose after the highly civilized Sumerians were conquered by the Amorites. This period also saw a great increase in general literary activity. The second period is the 'Seleucid' one, that is of Hellenistic times or the Greek period after Alexander the Great, which shows no particular advance save that the symbol for zero is used more consistently.

The ancient Babylonians constructed long multiplication tables and tables of reciprocals. They must have known the theorem of Pythagoras, for on one clay tablet appears a square with a diagonal, across which is written a number in the sexagesimal notation. After decipherment and reduction to the decimal system we find

$$\sqrt{2} = 1{\cdot}414213 \ldots$$

which has an error of only one in the last place. We may well marvel at such numerical skill and compare it with the muddle in Europe some 2,000 years later. However, it is by no means certain as to whether any geometrical arguments, let alone a rigorous treatment on the lines of Greek geometry, had been advanced for the proof of the general theorem of Pythagoras. The interest of the Babylonians in the result appears to be largely numerical and they constructed tables of Pythagorean 'triples' such as

$$45^2+60^2 = 75^2, \quad 65^2+72^2 = 97^2,$$

and many others. The second example is probably new to most of us. Neugebauer thinks that this table may have been reached so systematically from the solution of quadratics as to be equivalent to our known general solution. It must be understood that a use of algebraic formulae involving a general representation of numbers by letters is as recent as the sixteenth century A.D. On the other hand, the definite procedures laid down by the Babylonians are equally effective with the solution by algebraic formulae even if they read like a cookery recipe or, perhaps, a knitting pattern. As an example of a problem depending on a quadratic equation we have: An area A, consisting of the sum of two squares, is 1,000. The side of one square is $\frac{2}{3}$ of the side of the other square, diminished by 10. What are the sides of the square? This was solved during the Hammurapi dynasty. Approximate solutions of cubic and even higher equations, arithmetical progressions and problems in compound interest, also an estimate $\pi \simeq 3\frac{1}{8}$, have been found but, in the words of Neugebauer, 'Babylonian mathematics never transgressed the threshold of pre-scientific

thought.' In particular, accurately as they calculated $\sqrt{2}$, they completely failed to comprehend the nature of an irrational number. In the sphere of mathematics proper, if no worse than most people in our civilization, they quite failed to reach the understanding of the Greeks. It appears that Babylonian mathematics was devised for practical ends because it was useful in business and trade. Nor do we find any reference to the individual workers; the only vindication of the Babylonian mathematician is that after several thousand years his efforts were eventually communicated to the learned world through the combined efforts of archaeologists and mathematicians. There are perhaps few safer and more harmless ways of seeking to be remembered by posterity than by recording one's thoughts on wet clay.

GREEK MATHEMATICS [2, 8, 9]

The history of the Greek mathematicians begins, traditionally, with Thales of Miletus (*c.* 624–547 B.C.) and lasts until Pappus of Alexandria, whose *Collection* is dated A.D. 300. The first stage extends from the early Ionian thinkers in Asia Minor to the Golden Age of Greece when Athens was the leader of the Greek Empire. Three first-rate mathematicians, Archytis (*fl.* 400–365 B.C.), Theaetetos (d. 369 B.C.), and Eudoxus (*c.* 408–355 B.C.), were connected with Plato's Academy. After the death of Alexander the Great, one of whose tutors was Menaechmus the discoverer of conic sections, mathematics flourished in Alexandria. This became the intellectual centre of the Hellenistic world and there were also schools of mathematics in Athens and Syracuse, the home of Archimedes (*c.* 287–212 B.C.). Euclid (*c.* 300 B.C.) lived in Alexandria, and Apollonius (*c.* 260–170 B.C.) of Perga taught there a little

later. After the Roman domination of Greece and the Orient, Alexandria continued as a centre of learning up to at least the third century A.D. From Hellenistic times onwards the Babylonian influence is apparent, particularly in the computations of Archimedes, but if the flowering of Greek mathematics was in Alexandria its roots were certainly in Ancient Greece.

It is a curiosity of history that whereas we have the original writings, clay tablets of Babylon, which have been dated from early in the second millennium B.C. until the beginning of the Christian era, our primary sources of Greek mathematics are of Christian or Arabic times. By means of classical scholarship the texts have been reliably restored as far back as the fourth century B.C. which includes the important works of Euclid, Archimedes, and Apollonius, famous for his treatise on conic sections. Our picture of mathematics at that time is very clearly that of a well-developed science. As for the previous two centuries our information has had to be built up by careful comparisons of fragments and of remarks made by non-mathematical writers such as Plato and Aristotle. Even so we have arrived at quite consistent views of this early Greek mathematics. In particular one surviving fragment, the work of the Ionian philosopher, Hippocrates of Chios (*fl.* 450–430 B.C.), an island off the coast of Asia Minor, has been held as strong evidence for the complete maturity of this early Greek mathematics.

Before giving details of the Ionian school it seems proper to point out that the very conscientious historian, Neugebauer, although ready to credit Miletus of the eighth century B.C. with being the birthplace of the Greek alphabetic number system, regards the traditional accounts of Greek theoretical geometry as unproved and, following

a remark of Archytas, as even dubious. However, according to tradition, Thales of Miletus was the first Greek geometer. He was said to be a merchant, possibly of Phoenician descent, who travelling to Egypt learned Geometry (literally the science of land measurement) from the priests of Isis and Osiris. Apparently he was a very apt pupil because, says Hieronymus, 'he determined the height of the Great Pyramid by measuring the shadow it cast at an hour when a man's shadow was equal to his height'. This involves nothing beyond an easy application of the method of similar triangles by which means Thales is supposed also to have measured the distance of ships at sea. Again, another historian says of Thales that 'he endowed geometry with rigour and founded it on congruence and similitude'. On such indirect evidence it has been argued strongly by T. Dantzig that the first proof of the so-called Theorem of Pythagoras was probably due to Thales and depended on the method of similar triangles. It seems quite probable that Pythagoras and his school at Crotona merely speculated on the 'Pythagorean' triples which had been known to the Babylonians much earlier along with other ideas, of a sometimes mystical character, concerning relations between whole numbers. On this view the importance of the Pythagoreans is that they stressed the idea that whole numbers are the only logical basis for mathematics. This quickly led them to the discovery that the ratio length of the diagonal of a square to the side is not a rational number. It seems that they discovered the possibility of completely filling a plane area with equilateral triangles and squares or regular hexagons, and tried to imagine both lines and space as the sum of 'points'. When Zeno (*c.* 450 B.C.) presented his famous paradoxes he may well have been arguing against Pythagorean ideas.

Both Thales and Pythagoras were Ionians although the latter set up his school in southern Italy because of political troubles at home. The only complete mathematical fragment which has survived from the fifth century B.C. deals with the crescents of Hippocrates, another Ionian. According to legend he was a merchant of the island of Chios who, after being indoctrinated by the Pythagoreans in Athens, decided to become a teacher. The aim of his investigation was to discover when areas bounded by circular arcs of different radii can be expressed rationally in terms of the two diameters. The subject itself implies a considerable degree of geometrical knowledge, both notion of rational numbers and of areas bounded by a curve being involved in the mere statement of the problem. The problem was analysed by Claussen in 1840 who proved that there were five quadrable crescents, of which Hippocrates had found three. Later work by Landau 1903, Tchebotarev 1934, and Dorodnov 1947, showed that there are *only* five solutions of the problem as formulated by Hippocrates. In addition to his famous crescents Hippocrates has been credited with writing an 'Elements' (Gk. *stoicheia*) which would have preceded Euclid by a century.

We have it on the authority of Pappus, who wrote his *Collection* in A.D. 300, that Hippias of Elis, a sophist and a contemporary of Hippocrates, had invented a curve by means of which the circle could be squared. The solution was supposed to have been restored by Dinostratus, a member of Plato's Academy and a brother of Menaechmus, who discovered conic sections. In his book *The Bequest of the Greeks* T. Dantzig has given a very simple and convincing explanation as to how this 'quadratrix' might have been used to solve the problem. He shows that this yields the same limiting process for π as one of the, much later,

formulae of Archimedes. However, some critics claimed that Hippias merely begged the question. It must be admitted that Sophists had generally a pretty bad name. Hippias in particular was said to be a braggart who attended the Olympic games in festive attire made entirely by his own hands and offered to lecture on every subject from astronomy to ancient history.[1]

Whatever we may think of the evidence that they had already found the *solutions* it does seem quite certain that at an early date the Greeks had already singled out three particular problems:

(i) the trisection of an angle;

(ii) the duplication of a cube;

(iii) the squaring of the circle;

and that only very well-grounded geometers would have been able to appreciate the special feature common to these three problems; none of them can be solved by ruler and compass or, as we would say, by Euclidean methods. As we shall see below the Greeks did find solutions eventually, but the impossibility of the solution by ruler and compass could not be proved until the nineteenth century when mathematics had been greatly advanced.

We now come to a very important period in Greek mathematics and in Greek history as a whole. It may be noted that the crisis in Greek mathematics, when on account of the difficulty of dealing with incommensurables it seemed doubtful if mathematics could be regarded as an exact science, coincides with the latter part of the Peloponnesian war which ended with the fall of Athens. Of the three great mathematicians of Plato's Academy which was established soon afterwards, Archytis of

[1] Benjamin Farrington, *Greek Science*, p. 87.

Tarentum is supposed to have discovered the eclipse as the oblique section of a circular cylinder and to have duplicated the cube. Theaetetos is credited with the theory of irrationals, and Eudoxus of Cnidus with geometry on the torus, duplication of the cube by means of oval curves and the axiomatic definition of proportion which is given in Euclid Book V. This section of Euclid has been greatly admired in modern times as giving the same abstract treatment of incommensurable quantities in geometry as Dedekind's arithmetical theory of the nineteenth century.

Of Euclid himself, the author of the most famous textbook in all history, we know little except that he lived in Alexandria which, after the death of Alexander in 323 B.C., became the capital of the Hellenistic world. He probably lived under Ptolemy I. His elements are in part familiar to most of us as school geometry. The proof of the theorem of Pythagoras (or Thales!) which we learn is that given by Euclid. The simple and almost 'obvious' argument from similar triangles involves difficulties in the theory of proportion which Euclid quite properly left over until later. And so it comes about that although we learn our algebra and calculus in a quite non-rigorous way (the worst horror is our early introduction of logarithms as a mere table of numbers) our geometry at a vital point is arranged to satisfy all the rigour of the 'Dedekind' section! Those who have struggled with Euclid's first proof of the theorem of Pythagoras may console themselves with the comment of the German philosopher Schopenhauer that this was not an argument but a mouse-trap. Besides his geometry, leading to the study of the regular solids, Euclid included the theory of the Greatest Common Divisor, a proof that the primes are infinite in number, the sum of a geometrical series, and the proof that of all

rectangles of a given perimeter the square has the largest area.

Archimedes of Syracuse (287–212 B.C.) is the most famous mathematician of all antiquity. His discoveries of formulae for the areas of a parabolic segment, the surface of a sphere and of a body of revolution, also his result for the volume of a sphere, form the first chapter in the Integral Calculus. Unlike Newton he did not devise a method of differentiation and so he found no general method of integration as we know it, that is to say involving indefinite integrals. In his excellence as a computer —for example, he found π as a number between two fractions equivalent to

$$3\tfrac{10}{71} \text{ and } 3\tfrac{10}{70}$$

—he reveals the Babylonian influence which seems to have been absent until Hellenistic times. On the other hand he was still a Greek rigorist who insisted on strict proofs of results found by tentative (as we say nowadays, heuristic) arguments. It will be recalled that Newton, having found his results by the calculus, then prepared Euclidean proofs. Archimedes understood quite clearly our modern notion of infinity as that which has no bound. He wrote as follows in the introduction to *The Sand Reckoner*:

> There are some, King Gelon, who think that the number of the sand is infinite in multitude . . . but I will try to show you by means of geometrical proofs, which you will be able to follow, that, of the numbers named by me and given in the work which I sent to Zeuxippus, some exceed not only the number of the mass of sand equal in magnitude to the earth filled up in the way described, but also that of a mass equal in magnitude to the universe.

Archimedes was also an able applied mathematician and

his name will always be linked with one of the first results in this subject, the 'principle' for floating bodies. An instructive and amusing passage occurs in Plutarch's life of Marcellus [9]. The latter is attempting most unsuccessfully to capture Syracuse whose main defence, according to Plutarch, was the machinery devised by Archimedes. Plutarch was a Greek who lived about A.D. 50, but the sources of his work were in Flavian Rome so we may assume that he would not be unduly flattering.

But Archimedes made light accompt of all his devises, as in deede they were nothinge comparable to the engines him selfe had invented. . . . For this inventive arte to frame instruments and engines (which are called mechanicall, or organicall, so highly commended and esteemed by all sortes of people) were first set forth by Architas, and by Eudoxus: partly to beawtifie a litle the science of Geometry by this finenes, and partely to prove and confirme by materiall examples and sencible instruments, certeine Geometrical conclusions, whereof a man can not finde out the conceivable demonstrations, by enforced reasons and proofes [i.e. by *Euclidean methods of construction*]. As that conclusion which instructeth one to search out two lynes *meane* proportionall [*the geometrical equivalent of extracting a cube root*] which can not be proved by reason demonstrative, and yet notwithstandinge is a principall and an accepted grounde, for many thinges which are conteined in the arte of portraiture [*a reference to the doubling of a cube*]. Both of them have facioned it to the workemanship of certeine instruments, called Mesolabes or Mesographes, which serve to finde these meane lines proportionall, by drawing certeine curve lines, and overthwart and oblike sextions [i.e. *hyperbolae and ellipses which can quite readily be constructed by means of fairly simple linkages*] . . . since that time I say, handy craft, or the arte of engines, came to be separated from Geometry, and being long time despised by the Philosophers, it came to be one of the warlike artes.

Later on Plutarch pays tribute to Archimedes the geometer in the following terms:

Notwithstanding, Archimedes had such a great minde, and was so profoundly learned, having hidden in him the onely treasure and secrets of Geometricall inventions [*possibly a reference to the 'Method' discussed in the letter from Archimedes to Eratosthenes which came to light as recently as 1906*]. . . . For in all Geometry are not to be founde more profounde and difficulte matters wrytten, in more plaine and simple tearmes, and by more easie principles, than those which he hath invented.

After which Plutarch adds a humorous touch on the question as to whether this remarkable clarity was a natural gift or merely the result of taking extreme pains:

And therfore that me thinks is like enough to be true, which they write of him: that he was so ravished and dronke with the swete instysements of this Sirene, which as it were lay continually with him, as he forgate his meate and drinke, and was careles otherwise of him selfe, that oftentimes his servants got him against his will to the bathes, to wash and annoynt him: and yet being there, he would ever be drawing out of the Geometricall figures, even in the very imbers of the chimney. And while they were annointing of him with oyles and swete savors, with his fingers he did draw lines upon his naked body: so farre was he taken from himself, and brought into an extasy or traunse, with the delite he had in the study of Geometry, and truly ravished with the love of the Muses.

The nature of mathematical invention has often appeared mysterious, and not only to laymen. If Plutarch was indulging in a little poetic licence we find a quite serious reference to these matters by Poincaré. The language has been brought up to date, but the notions remain. In his well-known book *The Psychology of Invention in the Mathematical Field* Hadamard [10], the famous French analyst, quotes Poincaré as follows: '(the unconscious self) is not

Later on Plutarch pays tribute to Archimedes the geometer in the following terms:

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purely automatic; it is capable of discernment; it has tact, delicacy, it knows how to choose, to divine. What do I say? It knows better how to divine than the conscious self, since it succeeds where that has failed.'[1]

Thus did one of the best creative minds in the history of mathematics expand on the notion that if one sleeps on a problem the solution sometimes appears without further conscious effort. Not all thinkers have been so fortunate in their experiences as Archimedes and Poincaré; Hadamard quotes also the case of the unhappy philosopher von Hartmann who regarded the unconscious as a 'universal force', a 'specifically evil one', for whom the only cure was 'cosmic suicide', the immediate destruction of the whole planet. But, lest we should be discouraged in creative work even to the extent of refusing to attempt geometrical 'riders', let me add that Hadamard himself comments drily, having himself made notable mathematical discoveries: 'The true mystery lies in the existence of any thoughts, of any mental processes whatever, these mental processes being connected . . . with the functioning of some of our brain cells. The existence of several such kinds of processes is hardly more mysterious than the existence of one kind of them.' Or, to complete the discussion, the apparent non-existence of any process at all.

We return to the more prosaic aspects of geometry, the third great mathematician of Hellenistic times being Apollonius who wrote several books on conic sections. He discovered certain properties of areas which in our co-ordinate notation would be writen as

$y^2 = px$ (parabola) $y^2 = px \pm px^2/d$ (hyperbola, ellipse).

These forms give us our names for the curves. Here parabola means 'application', ellipse 'application with deficiency', and hyperbola 'application with excess'. It seems

[1] H. Poincaré, *Science and Method*, Part I, ch. iii.

an easy step to us from the discussion of areas to the use of the co-ordinates but, strictly speaking, this is not so. If we are to set up co-ordinates rigorously we must first define the system of real numbers, that is both rationals and irrationals. The Greeks had taken great pains to achieve a rigorous account for areas, which was founded on the theory of proportions of Eudoxus, and we may suppose that Apollonius deliberately rejected the use of the co-ordinates themselves. We refer to this matter again in connexion with Diophantus of Alexandria.

The history of mathematics has always been closely connected with that of astronomy. Aided by the climatic conditions—the skies of Mesopotamia are cloudless[1] from April to October—the Babylonians accumulated astronomical observations extending over many centuries. Systematic reports to the Assyrian court were made about 700 B.C. and Ptolemy states that eclipse records were available from Nabonassar (747 B.C.). Unlike their mathematicians who remain anonymous there are two famous Babylonian astronomers, Nabu-rimanni *c.* 500 B.C. and Kidinnu 367 B.C. The latter, the Assyriologist Chiera ranks with Copernicus, Kepler, and Galileo. They observed eclipses of the sun, moon, and the stars and so carefully that, using our own highly developed mathematical astronomy, we have some check on the dates of their tablets. They discovered the movements due to the precession of the equinoxes, i.e. the wandering of the pole due to the attraction of the sun and the moon on the earth's equatorial bulge which exhibits itself as a change of longitude in star positions. It is, however, a far cry from

[1] Chiera says so, but Neugebauer claims that this is merely a literary cliché. There are similar discrepancies in the interpretation of the Rhodesian climate, also the problem of dust.

these ancient Babylonian observations to their rational explanation by Newton. In spite of their love of computation and their great skill at it—for example they used arithmetic progressions to describe periodic quantities and were able to predict lunar phenomena with an accuracy of a few minutes—it was not a Babylonian but a Greek, the geometer Eudoxus, who first attempted an *explanation* of the motion of the planets. He assumed them to be attached to four concentric spheres each with its own axis of rotation. The idea of circular motions persisted until Kepler and, as Struik remarks, the same principle is involved today when we apply trigonometrical series to compute the motion of a planet. The Greek astronomer Hipparchus (*c.* 150 B.C.) is supposed to have made use of the Babylonian star calendars when he wrote his treatise. This is now lost, but his method of eccentric circles and epicycles was probably that given in the famous *Almagest*, the 'Great Collection' of Ptolemy (*c.* A.D. 150), who belongs to the second Alexandrian period. The *Almagest* contains the first trigonometrical tables known to us, a table of chords for various angles equivalent to sines of angles from 0° to 90° proceeding by half degrees. It also gives, in effect, the addition formulae for sines and cosines, and Ptolemy's theorem for a cyclic quadrilateral appears as a *lemma* in connexion with the proofs. Ptolemy knew also the method of stereographic projection.

The last two famous Alexandrian mathematicians were Diophantos (*c.* A.D. 250), and Pappus (*c.* A.D. 300). The former is renowned as the originator of Diophantine equations, that is to say indeterminate equations to be solved in terms of whole numbers. It was a translation of his work which inspired the famous researches of Fermat in the seventeenth century, who laid the foundation of

modern 'number theory'. In discussing arithmetical questions in an algebraic manner Diophantos is in the Babylonian tradition, but the Greek influence is apparent in his acceptance of positive rational solutions. His rejection of irrational solutions as impossible is evidence in support of the view that Apollonius consciously rejected the idea of co-ordinate geometry in which the co-ordinates would, of course, be generally irrational. The introduction of Diophantine analysis in the modern sense, that is with the solutions restricted to whole numbers, is due to the Hindu mathematician Brahmagupta (*c.* A.D. 625) who will be mentioned again. The *Collection* of Pappus is chiefly of value on account of the references to original works which have been lost, and the same is true of the *Commentary on the First Book of Euclid* by Proclos (A.D. 410–85).

THE INDIAN AND ARABIC CONTRIBUTIONS TO MATHEMATICS [2, 11]

The history of mathematics in the next millennium takes us through India, Mesopotamia, the Byzantine Empire, and finally by way of Islam to the West. With the exception of the Hindu contribution mathematics was not much developed during this period. The earliest Indian scientific works, the Siddhāntās, which date from about the fourth century A.D., deal with astronomy treated by means of epicycles as in Ptolemy's *Almagest*, and the calculations are performed in sexagesimal fractions. Trigonometrical tables appear with 'sines' instead of the 'chords' of the *Almagest*. The first well-known Hindu mathematicians were Āryabhata (b. A.D. 476) and Brahmagupta (b. A.D. 598). The latter initiated the theory of Diophantine equations. The linear case was completed by H. J. S. Smith (1826–83) and for equations of higher degree a general theory has yet to

be achieved. They studied mainly arithmetic and algebra. In addition to developments of those methods of the Babylonians and the Greeks which we have already noted, we find apparently for the first time the idea of permutations and combinations.

This suggests an interesting criticism of Greek mathematics which has probably occurred to many schoolboys. We are told, and the evidence is convincing, that the Greeks rejected algebra on account of conceptual difficulties connected with irrational numbers. There was nothing of this sort to hinder them from dealing with *permutations*, even the classification of permutations according to their symmetry which would today come under the heading of the 'theory of finite groups'. Indeed, the ideas were available long before Greek civilization, the notion of symmetry in decorative patterns being much older than the written word. It is precisely along these lines that one would have expected the Pythagoreans to have developed their notions of whole number relations and space-filling properties of regular figures. However, they did nothing in this direction, neither for that matter was the idea of a permutation developed any further by their discoverers the Hindus. As for the claim that the Hindus first invented the symbol for zero and the decimal place system we are assured by the Assyriologist E. Chiera that the sign for zero existed in Sumeria before 2000 B.C. and that it was certainly employed by Nabu-rimanni (*c.* 500 B.C.) in his astronomical tables for calculating the new moon and eclipses. O. Neugebauer is more guarded in his views. He is sure that it was not in existence before 1500 B.C. but says that it was in full use from 300 B.C. onwards. Both the decimal notation and the place system belong to Ancient Babylon, but for actual computation the Babylonians,

possibly because they were such skilful computers, preferred to combine the sexagesimal notation with the place system. No doubt the reduction of the unit from 60 to 10 was a convenience for most of us; we should not like to have to learn a 60 times 60 table for multiplication, but it hardly ranks as a profound mathematical step. We do not as yet have a full history of Indian mathematics, particularly for pre-Christian times. It is not clear to what extent their work derived from Babylonia either directly or through Alexandria. Indian mathematics shows both skill and originality, but it marks no fundamental advance in ideas. Historically, however, the Indian period is of some importance because it was by way of India of this time that knowledge of algebra came to Western Europe, being transmitted through the Arabic world of the Middle Ages.

Later, the centre of learning shifted from India to Mesopotamia under the Sassasians, a revival of the old Persian kingdom which was made famous in the legend of *The Thousand and One Nights*. After the Arabic conquest of A.D. 641 the Caliphs of Bagdad encouraged learning. During this period Muhammad ibn Mūsā al-Kwārizmī wrote a book, under the title *Hisab al-jabr wal-muq abala* (The Science of Reduction and Cancellation). The *Al-jabr* became in Latin translation our 'algebra', a latinization of the author's name, our 'algorithm'. He showed how to solve linear and quadratic equations but used no symbolic notation. The Babylonians did as much. The famous Omar Khayyam (*c.* 1000), author of the Rubaiyat, astronomer and philosopher, found time to write an algebra in which he gave the solution of *cubics* along Greek lines, that is by the intersection of conics. It seems that the main interest in mathematics was practical. Indeed in the work cited al-Kwārizmī limits himself 'to what is easiest

and most useful in arithmetic, such as men constantly require in cases of inheritance, legacies, partition, law-suits and trade and in all their dealings with one another, or where the measuring of lands, the digging of canals, geometrical computation, and other objects of various sorts and kinds are concerned'. Cf. [11]. The Arabs did not make any fundamental contributions to mathematics, although this does not hold for other branches of science, but, as we have said, it was through them that a knowledge of both Greek and Hindu (or Babylonian) mathematics first came to Western Christendom. In the East the Byzantine Empire preserved some Greek culture for many centuries. The Arabs did, however, improve the instruments for astronomical observation and they constructed more accurate tables. In optics Alkindi (d. *c.* 873) and Alhazen (*c.* 965–1039) made significant advances on the works of the Greeks and discovered the properties of parabolic and spherical reflecting surfaces. Alhazen was also a good geometer and he is remembered for the problem of constructing a reflected ray from a point A to a spherical surface and passing also through another given point B. The introduction by the Arabs of the decimal place system was ultimately of great practical importance for Europe. It is remarkable how long other systems survived. Roman numerals were used in Italy until the middle of the sixteenth century, the bankers of Florence being forbidden by a statute of 1299 to use the Arabic notation on the grounds that it made their books hard to read.

EARLY EUROPEAN MATHEMATICS AND THE INVENTION OF DYNAMICS [2, 3, 11, 12]

For many centuries only the most rudimentary mathematics was known in the West. The Romans were not

particularly interested in science and Western Christendom had no need for mathematics except of the simplest character. This was provided by the writings of Boetius (sixth century), based on Euclid, the arithmetic of Nicomachus, an Alexandrian mathematician, and Ptolemy, but containing only fragments. It was enough to deal with surveying, the adding up of accounts and the calculation of the date of Easter. There was a temporary revival of learning under Charlemagne, but its basis was only such knowledge of the Latin encyclopaedists, e.g. Pliny and Boetius, as had been preserved in the monastery schools. Alcuin of Britain contributed to Charlemagne's court his *Problems for the Quickening of the Mind* containing the well-known riddle: 'A wolf, a goat and a cabbage must be moved across a river in a boat holding only one besides the ferryman. How must he carry them across so that the goat shall not eat the cabbage, nor the wolf the goat?' One might explain the sorry state of mathematics as a result of the troubled political times, the Roman Empire disintegrating under the attacks of Goths, Vandals, Franks, and, later, Norsemen from the north and under the advance of Islam in the Mediterranean. The matter probably goes deeper. The original motive of Greek philosophy had been the pagan search for the good in life. Geometry had an almost religious appeal. As Plutarch says [12]: 'Geometry draws us away from the realm of the senses. From there it directs our attention to Nature, everlasting and only perceptible by the mind, the sight of which is the aim of philosophy, just as the highest grade of initiation is the aim of the Eleusinian ritual.' The distinction between 'arithmetica' in the sense of mathematics and 'logistics' or practical computation had become part of their language and thought. On the other hand the great thinkers

of the West did not always turn their backs on secular learning. For example, St. Clement of Alexandria (third century A.D.) had likened the fear of pagan philosophy to a child's fear of goblins. His pupil Origen denied the existence of the infinite, but St. Augustine, who had discussed the rational basis of the faith, accepted the set of integers 1, 2, 3, . . . , as an actual infinity. As for the infinitesimal, William of Ockham wrote as follows: 'in the whole universe there are no more parts than in one bean because in a bean there are an infinite number of parts'. It seems certain that both the founders of the infinitesimal calculus in the seventeenth century and the students of the transfinite in the nineteenth were to some extent aware of the work of the scholastic philosophers. In fact, G. Cantor, one of the pioneers of the 'transfinite' paid tribute to the ideas of St. Augustine. In spite of this, one feature of these times was a disinclination to develop mathematics, and as far as this subject is concerned we need not apologize for the designation the Dark Ages: T. Dantzig very unkindly uses the alternative 'unpenetrable darkness'.[1] Mathematics remained in a bad way for many centuries and this is brought home to us rather painfully in the work of Roger Bacon (*c.* 1214–92), a remarkable personality of the early Middle Ages.

Bacon studied in the University of Oxford until 1240, received the degree of Doctor of Theology in the University of Paris and, later, in the year 1247, became a Franciscan. He is famous for his almost prophetic view of the possibilities of applied science: in the *Epistola de Secretis Operibus* he envisaged both the submarine and the motor-car. In his important work the *Opus Majus* he aimed to explain the unity of science and its importance in the life of the Chris-

[1] From a remark made by E. Galois.

tian Church. In this noble endeavour he chose to devote some 300 pages to 'mathematics' and succeeds chiefly in revealing the poverty of learning at that time. He was at pains to prove by 'authority' that every science requires mathematics, quoting from the prologue to the *Arithmetic* of Boetius. 'If an enquirer lacks the four parts of mathematics he has very little ability to discover truth.' Before Bacon's time the 'geometry' of Boetius had deteriorated to a description of the abacus and methods of practical surveying although the *Arithmetic* still contained some shreds of knowledge. The references to Euclid in the *Opus Majus* are probably based on the translation from the Arabic by Adelard of Bath in the twelfth century. Things had reached a level in the preceding century where one geometer could try to square the circle by the 'infant school' method of cutting up cardboard, and Raimbaud of Cologne corresponded with Radulfus of Liège, each trying to outdo the other in proving that the sum of the angles of a triangle equals two right angles. Having made his peace with authority Bacon proceeded to the 'proof by reason' and aimed at showing that mathematics is not only fundamental to science but innate: 'but this science is the easiest . . . for the people at large and those wholly illiterate know how to draw figures[1] and compute and sing,[2] all of which are mathematical operations'. And again: 'a man by listening once or twice can learn more about this science with certainty and reality than he can by listening ten times about the other parts of philosophy, as is clear to one making the experiment'. Whatever we think of this, his main thesis that mathe-

[1] Compare the 'arte of portraiture' in Plutarch's digression on Archimedes.

[2] Musical notation with time values originated in the twelth century. The six-part round 'Sumer is icumen in' appeared towards the end of Bacon's life.

matics is necessary for the sciences may be allowed to stand. Later on, having repeated the Pythagorean result for the irrationality of the diagonal of a square, he concluded that the atomic hypothesis of Democritus is invalid and added his own contribution to the effect that not only would the diagonal be incommensurable but, by counting the 'atoms', it would be equal in length to the side. This is perhaps an error of physics rather than of mathematics. Bacon on mechanics, with a view to applications in optics, shows a spark of intuition (although his notion of the component of a force may derive from the statics of Jordanus Nemorarius) and a certain British common sense. He notes that by geometry, the perpendicular on a line is the shortest distance: 'wherefore, the force coming along it will act more strongly . . . and if a man falls from a height perpendicularly he is hurt more: for should any one divert from the perpendicular path a man falling from a height, he would not be injured, provided he was near the ground. If this help is not given he will die from the perpendicular fall, and will be shattered completely.' He understood the curvature of the earth which he inferred from an observation of ships at sea, and, very cleverly, deduced from the curvature that a vessel placed in a low position contains more water than one in a higher position, which is quite true at least as far as Newtonian mechanics is concerned. Finally he devoted 224 pages to the application of mathematics to sacred subjects. His fourth point is rather revealing: 'One must have an excellent knowledge of the methods of computation because of the corruption of numbers in the Scripture, since they are corrupted in almost endless ways; for almost all the numbers have been corrupted. This corruption cannot be reduced to truth except by means of the ability to compute in every way, both in

fractions and in whole numbers.' In fact, multiplication as used by the Hindus and Arabs has been described as 'uncertain' and it was only during the thirteenth and fourteenth centuries that modern methods of multiplication and division brought the latter operation from the province of skilled mathematics to the routine of the counting-house. Roger Bacon was quite aware of the decay of learning in his day and he, apparently, wrote his *Opus Majus* and later works with the object of stirring up the ecclesiastical authorities. In this he was partly successful. He was first placed under restraint in the Paris house of the Franciscans and at a later date brought to trial and imprisoned for fourteen years, from 1278 until shortly before his death in 1292. The ideas of the *Opus Majus* as well as his polemical *Compendium Studii Philosophiae* constituted his crime.

THE RISE OF EUROPEAN MATHEMATICAL THOUGHT [1, 2, 11]

From this very low point in the history of mathematics and indeed of science as a whole, we can now trace a slow but steady improvement. Historians have not reached final conclusions as to the causes. Against the generally poor intellectual quality of European life through the 'Dark Ages' and until the later Middle Ages it is only fair to set the practical achievements in civilized life, particularly in mechanical inventions. A. C. Crombie points out that the improvements in the harnessing of horses and oxen, together with the use of windmills and water power, which were seen in the eleventh and twelfth centuries had as great an effect on everyday life as did the invention of the steam-engine in the nineteenth century. Certainly the progress in mathematics during this period appears to be

closely connected with mechanics which also had significance in Gothic architecture. We have already mentioned the school of Jordanus Nemorarius (*fl. c.* 1180–1237), in connexion with the *Opus Majus*. The 'axiom of Jordanus' to the effect that the motive power which can lift a given weight a certain height can lift a weight k times heavier to $1/k$ times the height, is practically our 'principle of virtual work'. Jordanus also studied the movement of a body along an oblique trajectory and made the important step of dissociating the motive power by which the body was moved into two parts, the gravity downwards and a 'violent', horizontal force of projection. He also had some notion of the component of gravity along the trajectory. The term 'violent' referring to 'unnatural' motion as distinct from natural motion under gravity and celestial movements, reminds us that Aristotle's ideas still held sway in mechanics. In his philosophy, motion in everyday life, being clearly a 'change of state', must be regarded as a 'process' by which a 'potentiality' towards motion becomes an 'actuality'. In simpler language, it is self-evident that a body cannot continue to move of its own accord and so its motion must be due to the continual action of the air surrounding it. This was an hypothesis in mechanics whose absurdity, which must be evident to any long-jumper, had been pointed out long before by Philoponus of Alexandria (sixth century A.D.). Jordanus, whose knowledge of the works of Archimedes was probably slight, had followed Aristotle in this matter and his contribution must be described as mainly kinematical, although the term kinematical meant little at his time. Eventually Buridan (1295–1358), following Philoponus, accepted the idea that motion was impressed on the *body* rather than on the air and invented the notion of impetus, analogous to momen-

tum in Newtonian mechanics. By the use of this term he satisfied Aristotle's principle that motion was a *process* maintained by a motive power, namely the 'impetus'. A mere mathematician who is no philosopher might be forgiven for describing this as a very elegant solution of the difficulty, the removal of a verbal illusion by the choice of a better 'notation'. Furthermore, having observed, it would seem, the actual behaviour of falling bodies, he was led to remark that 'impetus' increased in falling bodies and that force was something that *altered* motion. Here we have the germ of Newton's first two 'laws' of motion, the notion of *acceleration*.

Following Buridan, Albert of Saxony said that the motion of a projectile was divided into three parts: (i) horizontal motion under the impetus which annihilated natural gravity, (ii) a curved motion which was both 'violent' and 'natural', and (iii) a free vertical motion, the 'impetus' having been completely overcome by the action of both gravity and air resistance. It was a bold step in (ii) to contradict Aristotle who had denied the possibility of a 'substance' having two contradictory attributes and perhaps this is why Buridan failed to observe that state (i), however greatly desired by the philosophers, does not occur in nature. If we may believe Plutarch's tales, the old master Archimedes had much more accurate ideas about trajectories and knew how to bombard the enemy both near and far.

However, the notion of acceleration[1] had been introduced into mechanics which was a significant advance. It was not long before various people succeeded in proving that the *distance* travelled by a body in uniformly accelerated

[1] The phrase was 'uniformly difform motion' or as we would say 'uniform acceleration'.

motion could be found by multiplying the whole time by the mean speed. This was due to Nicole Oresme (1323–82), Bishop of Liseaux in Normandy and a master of Paris University, who used a graphical method, and slightly earlier by some scholars of Merton College, Oxford, by names John of Dumbarton, William of Heytesbury, and Richard Swineshead (called *Calculator*) who used arithmetical arguments. They appear very feeble, but eventually something quite new emerged to play a considerable part not only in 'mathematical physics' but in the history of mathematics as a whole. The long acceptance of Aristotle's evidently unsound notions on dynamics was surely due to the difficulty of formulating something satisfactory to replace them. The notion that freely falling bodies move with uniform acceleration was due to Albert of Saxony, but it was not until 1572 that the Spaniard Dominico Soto gave the formula for distance, and then without proof. All the early investigators found great difficulty in arriving at those elementary notions in mechanics which we now take for granted. The notion of acceleration appears very early, but the other concepts of dynamics were not formulated until Galileo some two centuries later. And yet, Jordanus and Oresme were mathematicians of merit. The former seems to have discovered linear and quadratic equations independently of Arabic influences and he used letters in the place of numbers, a generalization which was not fully developed until the sixteenth century. The latter invented fractional powers and devised 'co-ordinates' and 'graphs' in connexion with the kinematical problem already cited.

Another current of the mathematical development of this age begins with Leonardo of Pisa ('Fibonacci') (*c.* 1170–1250), an Italian merchant who travelled both in the Orient

and to Barbary in North Africa where he acquired a knowledge of Arabic mathematics. Leonardo's journeys remind us of Thales who reached Egypt during the sixth century B.C. in similar circumstances. For the second time in the history of mathematics Africa passed on vital ideas to the people in the lands to the north. Leonardo's *Liber Abaci* (1202) gave the first complete account of Hindu numerals which reached Europe. In the *Practice Geometrica* he described Arabic geometry and trigonometry. He invented the series of Fibonacci satisfying the recurrence relation

$$u_{n+1} = u_n + u_{n-1}$$

and proved that there is no Euclidean construction for the roots of the cubic

$$x^3 + 2x^2 + 10x = 20.$$

A discussion of this problem which he based on 'Euclid's theory of irrationals' was a remarkable achievement for the thirteenth century. The reader of Felix Klein's *Elementary Mathematics from an Advanced Standpoint* will recall that this sort of question was still a live issue in the nineteenth century. It was through the Italian cities of the twelfth and thirteenth centuries that, with the growth of trade, a knowledge of Arabic mathematics spread to Western Europe. Much later, after the fall of Constantinople (1453) and with the end of the Byzantine Empire, many Greek scholars fled to the West. Computational mathematics was largely in the hands of the 'reckon masters' who appeared in response to the great interest of the fifteenth- and sixteenth-century burghers in practical mathematics. These reckon masters were quite often independent of the universities. A famous fifteenth-century figure was Johannes Müller of Königsberg ('Regiomontanus'), who translated

Apollonius, Hero, and Archimedes. His book *De triangulis omnimodus* gave a complete account of trigonometry in a form similar to that in which it exists today. He also computed sines at intervals of one minute and we must remember that this was before there were logarithms. Since ancient Babylon there have always been those who find satisfaction in long and intricate arithmetical operations. Only in our own day have such efforts been rendered partly superfluous by the invention of high-speed calculating machines, and their extended use has given rise to a new type of computational fanatic, the 'programmer'.

We return to the history of mechanics. Leonardo da Vinci (1452–1519) had access to the ideas of Archimedes in the work *On Plane Equilibrium*, basing statics on the principle of the lever. Leonardo also used the statics of Jordanus and added several interesting ideas: a more general idea of the moment of a force, the principle of work in machinery and hydrostatics. He said that a ball rolling down an inclined plane was accelerated uniformly and that its velocity depended on the vertical descent. He accepted the idea of 'impetus', his views on motion remaining in that sense Aristotelian, as did Tartaglia (1500–57), who questioned stage (i) of the hypothetical motion of a projectile on the impetus theory, while Cardano criticized the whole scheme as arbitrary. With the replacement of the cast-iron guns of the fourteenth and fifteenth centuries by bronze cannon having accurately bored barrels the problem of finding the trajectory of a projectile became of greater importance. The approach of Archimedes, the use of deductive reasoning based on axioms or on simple hypotheses which could be tested by experiment was becoming well known in mechanics. For example, Stevin (1548–1620) based hydrostatics and statics on the impossibility of

'perpetual motion', in effect on the method of potential energy. It was Stevin (not Galileo) who dropped two leaden balls, one ten times heavier than the other, on a sounding-board. He concluded, from the observation that they reached the ground simultaneously, that Aristotle's rule, that velocity is proportional to force (weight) divided by air resistance, must be incorrect. In his day Galileo Galilei (1564–1642) was most famous as the astronomer who defended the Copernican system. On a modern estimate his work on the fundamentals of mechanics is of greater importance. Long before Galileo the 'impetus' school had calculated the distance moved by a body in uniformly accelerated motion and Albert of Saxony had conjectured that a *freely falling* body would have uniform acceleration. It is not easy for us to understand the medieval point of view, for the notions of Galileo and Newton have passed into the language and we have largely forgotten many problems which troubled them. However, as far as dynamics is concerned the crucial points seem to be that (*a*) the impetus school had no simple hypothesis to explain why a body fell with uniform acceleration and (*b*) they could not decide whether the instantaneous velocity ought to be proportional to the time or the distance. Even the notion of instantaneous velocity was vague because the only mathematical relations envisaged between variable quantities amounted to little more than simple proportionality. An outstanding problem was the need for a relation connecting the time of fall and the distance traversed. In 1604 Galileo gave the correct result $s \propto t^2$ but by following a muddled argument. He took as his *axiom* that speed was proportional to distance, which is not so, and decided arbitrarily that the 'quantity of velocity' in Oresme's figure would determine the distance. He verified

the 'square law' for balls rolling down an inclined plane. This illustrates the usual approach of Galileo to mechanics. He consciously took Archimedes as his model and was a master of the method of formulating hypotheses which could later be checked by a crucial experiment. It was much later that after considerable thought he decided that velocity being a time rate should itself be measured against time and so took the alternative law—velocity varies linearly with time. Galileo's great contribution to mechanics was the replacement of the notion of 'impetus' by 'inertia' and 'momentum'. The idea of inertia he seems to have reached by considering that a ball accelerates down a plane and is retarded in moving up it, hence it would continue indefinitely on the horizontal. One of his arguments for the concept of momentum was the remark that a balance could oscillate about a position of equilibrium if instead of equal weights in each pan there were suitable unequal weights placed at different distances from the fulcrum. From this easy experiment he concluded that what persists in motion is matter times velocity. By such reasoning Galileo got rid of the unwanted 'impetus' which 'caused' motion and replaced it by 'momentum' which was an effect and a measure of motion. He tried to avoid speaking of causes and his attitude to nature was almost that of Pythagoras. He wrote:

> Philosophy is written in that vast book which stands forever open before our eyes, I mean the universe; but it cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which it is humanly impossible to comprehend a single word.

The casual inquirer might suppose that Galileo was a good

physicist who only struggled with the problem of uniformly accelerated motion for so long because of his poor mathematical ability. On the contrary, Galileo was seeking the elusive concepts of rational mechanics, the bare analysis being contained in Oresme's construction. Having arrived at a good, though not by any means a perfect, understanding of the basis of mechanics he was able to treat projectile motion most successfully. He combined the uniform horizontal motion (the idea of inertia) with the uniformly accelerated vertical motion and discovered for the first time that the path was a parabola. Moreover, Galileo is credited with having anticipated Cantor of the nineteenth century in a property of infinite sets. His dialogue between Salviati and Simplicio in the *Two Principal System* contains the observations that the infinite class of all natural numbers can clearly be put in one—one correspondence with the sub-class of the integers which are squares. He may, of course, have regarded this as a good point to confuse his opponents. His ironic style is illustrated by a discussion as to 'cause and effect' which occurs in the *Il Saggiatore*.

If Sarsi wishes me to believe, on the word of Suidas, that the Babylonians cooked eggs by whirling them in a sling . . . we have no lack of eggs, nor of slings, nor of stout fellows to whirl them, and yet they will not cook . . . and, since nothing is wanting to us save to be Babylonians, it follows that the fact of being Babylonians and not the attrition of the air is the cause of the eggs becoming hard-boiled, which is what I wish to prove.

Galileo was a man who liked to argue, and from his student days onwards he was particularly set against Aristotle whom he regarded as practically an ignoramus. Quite early on he was forced on account of these views[1] to quit the

[1] To speak against philosophy was merely 'bad form'. His later views on astronomy ranked much more serious.

University of Pisa and go to Padua. There is a certain historical irony in this because it was to Padua that the Averroists had fled in *c.* 1277. Their crime had been to follow Aristotle too closely; Galileo offended in the opposite direction. Like Roger Bacon he came into conflict with the ecclesiastical authorities although Galileo was a layman. The humility of Roger Bacon is not apparent in many of the utterances and actions of this robust Florentine, but in the realm of natural philosophy he taught his students to say 'I do not know'. Perhaps it was lack of scientific humility more than anything else which had created the 'philosophy' of the Middle Ages with its glaring inconsistencies between theories and facts. In one important matter Galileo himself reveals great inconsistency. Aware of centrifugal force in terrestrial motion he quite failed to extend his idea to the motion of the planets. This is remarkable, for in defending the system of Copernicus (1473–1543)[1] Galileo explicitly criticized the old Aristotelian view that the heavens and earth are distinct in their 'motions' and 'natures'. In spite of which, he not only accepted that the planets moved in those same perfect circular motions as had been inherited from his old opponent Aristotle, but seems to go back on his own ideas by questioning the significance of *motion in a straight line.* He remarked that, as to rectilinear motion, 'at the most that can be said for it, is assigned by nature to its bodies, and their parts, at such a time as they shall be out of their proper places, constituted in a depraved disposition, and for that cause needing to be reduced by the shortest way to their natural state'. And worse, in spite of his own parabolic theory of projectiles as given in the *Discourses*, he even

[1] Copernicus said that the planets revolved about the sun rather than about the earth, as was the orthodox view until Galileo.

tried to prove that terrestrial motions should be circular. Such arguments have a very Aristotelian flavour and, apart from the grave mechanical error, we do observe a certain tendency on the part of Galileo to attack the followers of Aristotle with their own chief weapon, a mass of words, rather than by laborious calculations. For these reasons, while noting the signal services which he rendered to mathematical science, we regretfully leave him in the company of the Middle Ages.

MATHEMATICS JUST BEFORE NEWTON: OR THE PRELUDE TO THE *PRINCIPIA* [1, 2, 13, 14]

Contemporary with Galileo, in friendly correspondence with him and yet apparently unaware of the real connexion between their theories, the great mathematician and astronomer Kepler (1571–1630) spent many laborious years in correcting the idea that planets moved in 'perfect' circular paths by showing that, in actual fact, they moved along ellipses. His calculations were based on the excellent astronomical observations of Tycho Brahe (1546–1601) whose accuracy with the available instruments approached the finest limits possible to the human eye, that is unaided by the invention of the telescope. Kepler knew by stern experience the difficulty of not only reading the vast book in which philosophy is written, as did Galileo, but of writing only one small chapter. As to the reading, he commented:

> Most hard today is the condition of those who write mathematical works, especially astronomical treatises. For unless you make use of genuine subtlety in the propositions, instructions, demonstrations and conclusions, the book will not be mathematical; if you do use it, however, reading will be made very disagreeable, particularly in the Latin language, which lacks

articles and the grace of Greek. And also today there are extremely few qualified readers, the rest commonly reject (such books). How many mathematicians are there, who would toil through the *Conics* of Apollonius of Perga? Yet that material is of a kind that is far more easily expressed in figures and lines, than is Astronomy.

To bring the matter up to date, how many applied mathematicians in England have understood the Leray–Schauder theory for partial differential equations and how many persons, other than the triad, 'author, referee, and reviewer', have followed out Lipman Bers's theorem on the high-speed motion of air past a simple closed cylinder?[1]

Kepler's reputation suffered not only from the abstruseness of his mathematics but from his Pythagorean mysticism. A. R. Hall, however, regards Kepler's rather curious reference to forces seated in the sun and his 'appetites of matter' as the forerunner of Newton's idea of universal gravitation. Galileo had taken away the celestial spheres, and said nothing about the matter, but Kepler went much further. Having observed that there is no visible evidence for the earth being attached to its 'orb' by either chains or harness—it is in fact apparently surrounded by air—he wrote: 'Certainly many will not fear to doubt that there are in general any of these Adamantine orbs in the sky, that the stars are transported through space and the aetherial air, free from these fetters of the orbs, by a certain divine virtue regulating their courses by the understanding of geometrical proportions.' In retrospect, at least, a nice poetic description of Newtonian gravitational theory even if Kepler was thinking more of the Pythagorean whole-number properties of the families of orbits

[1] The author has done neither, but he has hopes that he may one day give an 'elementary' proof of the latter result.

than the detailed geometrical investigations of the sort to be used in the *Principia*.

Before such a work as the latter could conceivably appear the science, or perhaps art, of pure mathematics needed to be considerably advanced. Some hint of the changes which were to take place may be found from a study of the evolution of mathematical notations. Before François Viète (1540–1603), a magistrate, lawyer, and privy councillor who also distinguished himself in algebra, a cubic expression such as $1+3x+3x^2+x^3$ would have been written [13] as $1+3N+3Q+C$. Viète improved this to read $1+3x+3x$ (quadratus) $+x$ (cubus). In 1585 Stevinus would have written 1⓪ +3 ① +3 ② +③ which suggested to him the idea of *fractional* indices already visualized by Oresme. To Harriott (1560–1621) it would have been $1+3x+3xx+xxx$ and eventually Descartes in 1637 introduced $1+3x+3xx+x^3$ which is practically our modern notation.[1] Later, Wallis in 1659 suggested the possibility of negative as well as fractional indices and finally Newton in 1668–9 was able to give a formula for such quantities by means of his invention of the general binomial series.

The main centre of mathematics of the sixteenth century was in Italy, particularly the University of Bologna. This was one of the largest and most famous in Europe. Not only did it draw students from all over Europe but it attracted the same kind of people who nowadays attend certain forms of organized sport: in the case of Bologna the *public disputation*. Having no doubt these matters well in mind the discoverer of the general solution of the cubic equation, Professor Scipio Del Ferro, who seems to have been a very retiring sort of person, did not publish his work

[1] Whether x^2 or xx, there are still only two letters!

or even mention the solution, except to a few chosen friends. However, after his death the Venetian 'reckon master' Fontana (1500–57), whose nickname was 'Tartaglia' (The Stammerer), rediscovered the solution and proceeded to give a public demonstration. He gave only the results and presumably the verification but kept the method a secret. Eventually he could apparently stand it no longer and, under an oath of secrecy, revealed the proof to a learned Milanese doctor Hieronimo Cardano (1501–76) who is usually regarded as the greatest scoundrel in all the history of mathematics. His life scandalized even the Italians of the Renaissance and he had trouble with his family. One of his sons killed his own wife and Cardano saw fit to lop the ears of another. He himself divided the time between intensive study and intensive debauchery. He is remembered by mathematicians for having published Tartaglia's solution of the cubic and by lawyers, one would imagine, for having covered himself by adding an acknowledgement. Another bad character was Ferrari (1522–65) who was once Cardano's servant but has been variously described as a young gentleman scholar and as an adventurer of vile temper and morals. He discovered the solution of the general biquadratic by the reduction to a cubic and is thought to have died from poison administered by his own sister. No such scandal seems to attach to the name of the last famous mathematician of the Bolognese school, Raffael Bombelli, whose *Algebra* (1572) gave the first full account of 'imaginary' numbers. These were first introduced for cubics because, as distinct from the case of quadratic equations, it may well happen that the real-number solutions appear in terms of conjugate complex numbers. The *Algebra* of Bombelli was of historic importance, being known to both Leibniz and Euler.

The final stage leading to the mathematics of Newton was the growth of ideas about the 'calculus'. The roots of these are in the fourteenth century when Oresme noted in passing that the rate of increase or decrease of a quantity is slowest in the neighbourhood of a maximum or minimum of a function. The same idea was due to Kepler who did not however develop it any further. Kepler's astronomical work suggested ideas concerning the infinitesimal and he wrote a little book on the Integral Calculus entitled *Stereometrica doliorum vinorum* (solid geometry of wine barrels), 1615, an investigation into their cubic capacity. Although he acknowledged his debt to Archimedes he explicitly left aside the rigorous proof as demanded by the Greek: such a proof was not forthcoming until the nineteenth century. As for Galileo, his discussion of motion in the *Discourses* implies differentiation, for only by this operation can one define accurately the basic ideas of velocity and acceleration. However, a full account of such an operation did not appear until his pupil Cavalieri (1598–1647), now a professor at Bologna, published the *Geometria indivisibilium* (1635). By means of the well-known argument that solids of equal altitudes and with equal cross-sectional areas at corresponding heights are of equal volume Cavalieri succeeded in getting results equivalent to the integration of polynomials. The emergence of the infinitesimal calculus as later formulated by Newton and Leibniz was surely connected with the increasing use of algebra in geometry. The invention of co-ordinate geometry is usually credited to Descartes who, in his *Géométrie* published as part of his *Discourse on Method* (1637), did certainly introduce the idea of co-ordinates x and y connected by a relation $y = f(x)$. Descartes also extended the use of the new algebraic knowledge to

geometry and succeeded in ridding mathematics of the notion that linear, square, and cubic quantities can refer only to length, area, and volume, respectively. It was Fermat, a lawyer of Toulouse, whose discoveries in the theory of numbers place him as the most gifted amateur in the history of mathematics, who first wrote, in effect, the 'cartesian' equations for lines and conics. Fermat also showed how to find the maxima and minima of functions by seeking what we should now call a vanishing differential coefficient. He did not, however, define the latter quantity. The historical lineage of the invention of the differential calculus is probably much as follows: the Greek method of *geometrical* reasoning can be traced through Cavalieri and Torricelli, who was another pupil of Galileo, to Isaac Barrow who was Newton's teacher at Cambridge. On the other hand the influence of the new Italian algebra, surpassing by so much that of the ancient Greeks and Babylonians, appears in the work of Fermat, Descartes, and John Wallis of Oxford. Wallis published an important work, the *Arithmetica Infinitorum* (1655) and so set the fashion in Oxford that the holder of the Savilian chair of geometry may write good work on analysis. His work is remarkable not only for the novelty of the method but for the brilliant discoveries made by the use of infinite processes. In particular he found the infinite product

$$= \frac{2.\ 2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 8}{1.\ 3.\ 3.\ 5.\ 5.\ 7.\ 7.\ 9.\ 9}.$$

The *Arithmetica Infinitorum* is said to have directly led Newton to his discovery of the general binomial theorem. It is remarkable that for all this Wallis was never quite happy about Newton's calculus. Wallis lived in difficult times but he survived. He made his fortune during the

Civil War by deciphering codes for the parliamentary party and when it was over Oliver Cromwell appointed him to the Oxford chair. As for Barrow, Newton's tutor, he undoubtedly influenced the latter through his own interests in geometrical optics and mathematics. Indeed, the principle that differentiation and integration are inverse operations is essentially contained in Barrow's *Lectiones Geometricae*, Lecture X, No. 11. However, leaving aside the possibility of Barrow having been influenced by his pupil, whose genius he recognized to the highest degree possible,[1] it is a pretty fair view that none of Newton's predecessors had fully discovered the calculus. Hadamard points out that this involves not only the use of the method in special cases but an appreciation of its full scope and generality. Even in the case of Barrow the full meaning of the principle was obscured by his definition of the tangent to a curve.

Before proceeding to a more general description of mathematical science in the seventeenth century one other special topic should first be mentioned, the invention in 1614 of a form of logarithms by John Napier, a Scottish laird. Of this invention Laplace remarked that it doubled the lifetime of astronomers. It was certainly appreciated by Kepler and it led to the happy situation that Henry Briggs (1556–1631), Professor of Geometry at Oxford, travelled all the way to Scotland to do homage. Arriving there he delivered himself as follows: 'My lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help in astronomy, viz. the logarithms; but, my lord, being by you found out, I

[1] He gave up his chair to Newton: he may also have been influenced by his desire to devote himself to theology.

wonder nobody found it out before, when now known it is so easy.' An Englishman is never at a loss for the right comment in any situation.

The seventeenth century contains too many brilliant scientific figures for us to be able to attempt a worthwhile account of Newton's great contemporaries, such as Hooke the physicist, Descartes the mathematician and philosopher, or Pascal the great mathematician. There is however one very sympathetic figure, the Dutch physicist, mathematician, and astronomer, Christian Huygens (1629–95). Some fourteen years Newton's senior his life seems to illustrate most admirably many features of the seventeenth century. Perhaps a layman can learn more of the status of science at any period of history by studying the life and actual words of some leading figure[1] than by perusing the catalogue of inventions mentioned in a formal scientific history. As we have seen, granted some slight knowledge of the terms employed, even the gossiping reminiscences of Plutarch can reveal in a few well-chosen lines the brilliant story of Greek theoretical geometry. The mathematical muddle of the Middle Ages can scarcely be doubted by a reader who has spent ten minutes with Roger Bacon's *Opus Majus*, nor did we find much improvement in the time of Galileo, except that the latter had firmly decided where the trouble lay and refused to be turned aside by the apologists.

Christian Huygens was the brilliant son of a distinguished father who studied at Oxford and was a well-known man of culture; a poet, natural philosopher, classical scholar, and diplomat. All this was possible in the

[1] We might have selected Omar Khayyam to illustrate the last Persian period under Islam, but he was a poet and his ideas have been very well known since 1859.

days before specialization. At the University of Leyden which Christian entered at the age of 16 he would have learnt mathematics including Stevin's mechanics. He also had the misfortune at this tender age to acquire the ideas of another *Principia*, that of Descartes, of whose philosophy of science A. R. Hall doubts if 'it ever produced a single useful thought save in the mind of its originator'. Again, Andrade has remarked that the Cartesian scheme was 'easy, pictorial, general: the Newtonian difficult, mathematical, precise'. Descartes did realize that a new philosophy was needed. It would have been difficult not to do so after the blows struck against Aristotelianism by Galileo's observations of the planets and their satellites, but he tried to put the clock back by rejecting Galileo's idealization from experiment as an error in physics. Descartes wanted to replace it by *a priori* reasoning. Andrade puts it very neatly when he says that 'what Newton ignores is what Aristotle and Descartes tried to start with'. However, Huygens's gifts in mathematics and physics were such that he grew up to do excellent work and, in particular, to point out that not only did Descartes's laws of impact disagree with any experiments but the fifth law conflicted with the second, thus removing both the physical and mathematical content and leaving only the philosophical. Huygens was a modest man and he did not communicate these results to other scientists for some twelve years. His earliest publication at the age of 22, the *Cyclometriae*, was devoted to exposing the error in a work of one Gregory de St. Vincent who claimed to be able to square the circle and in four different ways! Some three years later he published his *De Circuli Magnitudine Inventa* which was received with acclamation and led to flattering comparison of Huygens with the Greeks Apollonius and Pappus. He

had succeeded in giving algebraic proofs of some of the purely geometrical work of Archimedes and other Greek geometers.

In the years 1630–70 Paris was the centre of science in Europe and to this city, in spite of many delays due to the troubled political times, the brilliant Dutchman eventually made his way. Huygens spent many years in Paris being under the patronage of Louis XIV in spite of his protestantism and connexion with the House of Orange. He was a close friend of the founders of the French Académie Royale des Sciences of whom the best known is perhaps Fr. Mersenne the famous scientific correspondent. The Académie was founded on 1 June 1666 and previous to this Huygens had paid two visits to the London scientific group originally known as Gresham College. It is reported by Pepys that Charles II 'mightily laughed at Gresham College for spending time only in weighing of ayre and doing nothing else since they sat'. However, the 'Merry Monarch' eventually gave them their Royal Charter in 1662 some four years before the foundation of the Académie. It seems that Huygens's second visit to England was made in order to study the organization of the Royal Society. His contacts seem to have been very happy. Later in 1670 when very ill in Paris, he decided to send his unpublished works on mechanics to London, entrusting them to the secretary to the English ambassador who wrote as follows: 'hee fell into a discourse concerning the Royal Society in England wich hee said was an assembly of the Choicest Witts in Christendome & of the finest Parts: hee said hee chose rather to depositt those little labours of his wich God had blest . . . sooner than in any else.' As for the French Académie with whom he apparently had some differences he recalled that at one time 'hee judged the Seat of Science

to bee fixed there & that the members of it did embrace and promote Philosophy not for interest, not through ambition or a vanity of excelling others . . . but out of naturall principles of generosity, inclination to Learning & a sincere Respect and love for the truth', whereas now, 'hee said hee did forsee the dissolution of this academie because it was mixt with tinctures of Envy because it was supported upon suppositions of profitt because it wholly depended upon the Humour of a Prince and the favour of a minister, either of wich coming toe relent in their Passions the whole frame & Project of their assembly cometh to Perdition'. But afterwards he recovered and even on his last visit to Paris, between 1678 and his return to the Hague in 1681 to convalesce after another illness, he was widely regarded as the true head of the Académie.

It has been remarked that in 1600 the ideas of an educated person in Europe were more than half medieval, by 1660 more than half modern. Huygens, a Dutchman who resided in Paris during its scientific ascendancy, who later headed the Académie and who had frequent contacts with the increasingly important Royal Society of London, exhibits these changes of the early seventeenth century. It should be appreciated that the new scientific societies with their devotion to scientific truth were in a sense opposed to the universities in which Aristotelianism still held its ground. Huygens wrote sharply on this matter on several occasions, for example when Père Fabri put up a wholly artificial explanation of the observed form of the planet Saturn. On another occasion Huygens had cause to remark to the author of another scientific work,[1] 'I notice that in many places you dispute the opinions of Aristotle. That is always worth doing.' Huygens, in the words of Mach,

[1] Boulliau's treatise on light.

shared with Galileo 'a noble unsurpassable and complete uprightness' and seems to have been remarkably free of any undue professionalism in his approach to science. Space does not allow a detailed discussion of his numerous works. His excellence as a geometer has already been noted and this aroused the admiration of Newton himself. Regarding Huygens's important work, the *Horologium Oscillatorium* (Treatise on the Pendulum Clock) which included the theory of oscillations on a cycloidal arc, evolutes and the measurement of curves, and the oscillations of a compound pendulum, Newton wrote to the secretary of the Royal Society that he had found it 'full of very subtile and usefull speculations very worthy of ye Author'. Newton particularly praised him as a geometer, calling him 'the most elegant writer of modern times' and 'the most just imitator of the ancients'. Regretting that he had not applied himself to geometry before starting on the algebraic analysis of his early years Newton was to make amends in the *Principia*. Its fully geometrical exposition was of the form which mathematicians from Eudoxus to Hilbert in the nineteenth century would have regarded as entirely rigorous. Again, as to mechanics proper, Newton paid tribute to Huygens's work on centrifugal force with the remark that 'what Mr. Huygens has published since about centrifugal force I suppose he had before me'. It is, however, a happy feature of their relationship that after Newton's great legal tussle with the Fellows of King's College for the post of Provost, it was Huygens who accompanied him when Newton set off to petition the king. His failure in this ambition much more than his strenuous mathematical labours would seem to be the cause of his nervous breakdown. It was another Newton who later gave covert assistance to those small minds who desired, for

their own ends, to prove that Leibniz had made use of correspondence with Newton in the former's quite independent invention of the infinitesimal calculus. In one important matter Huygens could not go the whole way with Newton, in his greatest discovery the law of universal gravitation. Five years after the appearance of the *Principia* he wrote: 'I esteem his understanding and subtlety highly, but I consider that they have been put to ill use in the greater part of this work, where the author studies things of little use or when he builds on the improbable principle of attraction.' He said, indeed, that the idea of universal attraction 'appears to me absurd'. Similar difficulties have been encountered in more recent times by several great physicists who could not accept the theory of relativity while Einstein himself was quite unhappy about certain implications of modern quantum theory.

With Huygens's comments on the law of gravity we come to an end of our survey. Apart from recounting a few of the many steps which led to modern mathematical science we have endeavoured to indicate something of the attitudes of the thinkers who lived in other times. It was Neugebauer who concluded one most careful treatise on the history of science with the comment that what stood finally revealed, like the fabulous unicorn of medieval paintings, may be little more than the creature of our own imagination. Also, as a 'secrétaire perpetuel' of that same Académie which Huygens helped found, the famous mathematician D'Alembert puts it, 'It is not that history repeats itself but that historians repeat each other.' This we fear is only too true of this account.

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