OPTIMAL STOCHASTIC CONTROL AND RISK MINIMISATION IN INSURANCE

A thesis submitted to the University of Zimbabwe for the degree of Doctor of Philosophy in the Science

By
Eriyoti Chikodza
Mathematics
June 2011
Contents

List of Figures ............................... 5

Abstract ..................................... 7

Declaration ................................. 9

1 Introduction, Preliminary Results and Problem Formulation .......................... 17

1.1 Introduction .............................. 17

1.2 Risk and Dividend Control in Insurance: Review of Existing Results ............ 19

1.2.1 Optimal Dividend Control ............ 20

1.2.2 Optimal Proportional Reinsurance .............................. 22

1.2.3 Optimal Combined Risk and Dividend Control .......................... 23

1.3 The Research Problem .................... 25

1.3.1 Mathematical Formulation of the Problem .................................. 26

1.4 Basic Definitions .......................... 28

2 Combined Singular and Impulse Control in Insurance ................................. 31

2.1 Introduction .............................. 31
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.2</td>
<td>Market Model and Problem Formulation</td>
<td>32</td>
</tr>
<tr>
<td>2.3</td>
<td>Combined Singular and Impulse Control Theory</td>
<td>34</td>
</tr>
<tr>
<td>2.4</td>
<td>Main Result</td>
<td>38</td>
</tr>
<tr>
<td>2.5</td>
<td>Application to Research Problem</td>
<td>43</td>
</tr>
<tr>
<td>3</td>
<td>Combined Regular-Singular Control</td>
<td>61</td>
</tr>
<tr>
<td>3.1</td>
<td>Introduction</td>
<td>61</td>
</tr>
<tr>
<td>3.2</td>
<td>Problem Formulation</td>
<td>62</td>
</tr>
<tr>
<td>3.3</td>
<td>Integro-Variational Inequalities</td>
<td>64</td>
</tr>
<tr>
<td>3.4</td>
<td>Application</td>
<td>67</td>
</tr>
<tr>
<td>4</td>
<td>Optimal Differential Game Strategies in Insurance</td>
<td>75</td>
</tr>
<tr>
<td>4.1</td>
<td>Introduction</td>
<td>75</td>
</tr>
<tr>
<td>4.2</td>
<td>Preliminary Results and Motivation</td>
<td>76</td>
</tr>
<tr>
<td>4.3</td>
<td>Formulation of the General Problem and Passage to the First Main Result</td>
<td>80</td>
</tr>
<tr>
<td>4.3.1</td>
<td>Formulation of the Problem</td>
<td>80</td>
</tr>
<tr>
<td>4.3.2</td>
<td>Passage to the First Main Result</td>
<td>81</td>
</tr>
<tr>
<td>4.4</td>
<td>Passage to the Second Main Result: An HJBIVI for Nash equilibria</td>
<td>91</td>
</tr>
<tr>
<td>5</td>
<td>Risk Minimization in Lévy Markets Using g-expectation</td>
<td>103</td>
</tr>
<tr>
<td>5.1</td>
<td>Introduction</td>
<td>103</td>
</tr>
<tr>
<td>5.2</td>
<td>Preliminary Results</td>
<td>104</td>
</tr>
<tr>
<td>5.3</td>
<td>Problem Formulation</td>
<td>105</td>
</tr>
</tbody>
</table>
5.3.2 The five-step scheme for jump diffusions . . . . . . . . . . 107

5.3.3 Application: Pricing of European call option in the presence of jumps . . . . . . . . . . . . . . . . . . . . . . . . . . . . 114

6 Conclusion 117
List of Figures
Abstract

UNIVERSITY OF ZIMBABWE

ABSTRACT submitted by Eriyoti Chikodza for the Degree of Doctor of Philosophy and entitled OPTIMAL STOCHASTIC CONTROL AND RISK MINIMISATION IN INSURANCE
Month and Year of Submission: June 2011

The thesis examines a generalised problem of optimal control of a firm through reinsurance, dividend policy and convex risk minimisation in the presence of market friction. The major mathematical tool applied is the theory of stochastic control for jump-diffusions. In the absence of intervention the financial reserves of the firm are assumed to evolve according to a stochastic differential equation with a jump component. In the second and third chapters, the objective is to derive reinsurance and dividend policies that maximize the expected total discounted value of a spectrally negative process in incomplete markets. The assumption is that transaction costs are incurred whenever dividends are paid out. Several verification theorems are derived and proved for combined singular and impulse control. The verification theorems are new results which provide a federative approach to the analysis of control problems involving transaction costs in finance and insurance.

Two methodologies are examined for risk minimisation. First, we investigate risk minimisation using zero-sum stochastic differential game theory in the presence of transaction costs. Our major contribution in this direction is that we have investigated, for the first time in the literature, a singular control problem for jump
diffusion stochastic differential games. Hamilton-Jacobi-Bellman-Isaacs variational inequalities (HJBIVI) are formulated and proved for the case of zero-sum stochastic differential games. The notion of HJBIVI is later on extended to the more general case of Nash equilibrium. Minimisation of risk is also studied using $g$-expectation. In this case a five step scheme is formulated. The scheme constitutes a mechanism for solving forward-backward stochastic differential equations. The solution provided by such a scheme minimises risk of terminal wealth of an insurance company. An existence and uniqueness theorem for the solution is provided. Several examples are discussed, throughout the thesis, to illustrate the theory.
Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.
Acknowledgments

I would like to express my gratitude to my supervisor Professor Bernt Øksendal for his comments, advice and role of mentor throughout this thesis. His attitude was a source of inspiration, encouragement and strength at all times.

I am also thankful to my local supervisor, Professor Alastair Stewart, for his support. I benefited immensely from the discussions that I regularly held with him in the Department of Mathematics at the University of Zimbabwe.

The Department of Mathematics, at the University of Zimbabwe, provided a conducive environment for my research. I would like to thank every member in the department for that, particularly the late Chairman Mr. L. Mudehwe.

Finally, I want to express my warmest thanks to my family for constantly supporting me during this work.

The financial support from NUFU is gratefully acknowledged.
Dedication

To My Family, My Mother and My Late Father.
Notation and Symbols

The following symbols and notation are used in this thesis.

- $H^o$ the interior of a set $H$
- $\bar{H}$ the closure of a set $H$
- $\mathbb{R}^n$ n-dimensional Euclidean space equipped with the usual Euclidean norm $\|\cdot\|_2$
- $C(G)$ the set of real-valued continuous functions on the open set $G \subset \mathbb{R}^k$
- $C(\bar{G})$ the set of real-valued continuous functions on the closure of the set $G \subset \mathbb{R}^k$
- $C^2(G)$ the set of real-valued twice continuously differentiable functions on the open set $G \subset \mathbb{R}^k$
- $C^2_0(G)$ the set of functions in $C^2(G)$ with compact support on the open set $G \subset \mathbb{R}^k$.
- $A^c$ denotes the complement of set $A$
- $\Omega$ sample space of all possible scenarios in a given economy or random experiment
- $\mathcal{F}$ $\sigma$-algebra of subsets of $\Omega$
- $\{\mathcal{F}_t\}_{t \geq 0}$ an increasing sequence of sub-$\sigma$-algebras on $\Omega$
- $P$ probabilty measure defined on $\mathcal{F}$
- $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ filtered probability space
Chapter 1

Introduction, Preliminary Results and Problem Formulation

1.1 Introduction

In recent years there has been increased research interest, among mathematicians, actuarial scientists and financial practitioners, in the interplay between insurance and finance. This is attributable, partly, to the fact that insurance companies are finding it imperative to apply control theory, which had previously been exclusive to finance and other disciplines, in their day-to-day business activities. Some of the major control variables that have received significant attention from researchers are dividend policy, new investment, reinsurance policy and premium levels, see for example, [4], [28], [29], [35], [51], [52].

When I embarked on the work that culminated in this thesis, I had the following two major objectives in mind:

(a) To propose an optimal combined risk/dividend control policy for an insurance firm that operates in Lévy markets characterised by friction.

(b) To derive risk minimising portfolios for an insurance firm that invests part of its capital in a generalised Black -Scholes market.

The two objectives where achieved in the thesis.
The importance of control theory in actuarial science and insurance was summarised by K. Borch in his address to the Royal Statistical Society of London in 1971, [51], when he remarked:

”The theory of control processes seems to be tailor-made for the problems which actuaries have struggled to formulate for more than a century. It may be interesting and useful to meditate a little how the theory would have developed, if actuaries and engineers had realised that they were studying the same problems and joined forces over 50 years ago. A little reflection should teach us that a highly-specialised problem may, when given the proper mathematical formulation, be identical to a series of other seemingly unrelated problems”

A classical problem in actuarial mathematics deals with the question of optimal dividend pay-out of an insurance firm. Early publications on this subject include [17] and [38]. In [38] the authors made groundbreaking contribution to dividend research by claiming that dividend policy was irrelevant in frictionless and complete markets because it has no impact on firm value. However, real-life financial markets are not perfect and so, the Miller-Modigliani claim may not hold in such markets. Consequently, considerable research work has been devoted to the problem of optimization of dividend policy under various market imperfections such as taxation, transaction costs, asymmetric information, solvency constraints, etc. in order to demonstrate the relevance of dividend policy and to identify optimal dividend policy, see for example [1], [4], [17], [27], [29], [35], [48], [51], [52] and references therein.

Closely related to the optimal dividend control problem is the question of optimal control of risk exposure of an insurance company. In the optimal risk control problem, management face the challenge of determining how much risk to avoid. This is referred to as the reinsurance problem in the insurance industry. Reinsurance entails controlling the revenue base by channeling a fraction of the incoming claims to another insurance company and thereby reducing its own risk as well as the potential profit. Proportional reinsurance has been treated in [27], [29], [30], and [32]. In all these references the uncontrolled liquid reserves of an insurance company were modelled as a diffusion process. Very little research seems to have been done in the area of stochastic control applied to insurance in the context of Lévy markets, hence this thesis.
In brief, this thesis seeks to investigate the stochastic control problem of optimising risk and dividend policy in the framework of Lévy markets with transaction costs. The rest of the thesis is organised as described in the next paragraph.

The remaining sections of this chapter focus on literature review, formulation of the general problem and definition of some basic concepts. Chapter 2 examines a simplified version of the problem where the firm insures all its claims. In Chapter 3 the more general case is treated. In order to engineer a stable dividend policy the company needs to minimise risk of terminal wealth. Chapters 4 and 5 propose methodologies for minimisation of risk in Lévy markets characterised by friction. Chapter 6 presents a conclusion of the thesis.

1.2 Risk and Dividend Control in Insurance: Review of Existing Results

In order to establish a mathematically rigorous framework we start with a stochastic basis \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) and a standard Brownian motion \(\{B_t\}_{t \geq 0}\) with respect to \(\{\mathcal{F}_t\}_{t \geq 0}\), see [4], [27], [51], [52] and [29]. The filtration \(\mathcal{F}_t\) models the information available to decision makers at time \(t\). The stochastic basis, also known as a complete filtered probabilty space, is a mathematical representation of all the uncertainties associated with a given economy.

Considering \(\{X(t)\}_{t \geq 0}\) to be a diffusion process, that at any time \(t\) denotes the liquid financial resources of an insurance company, researchers have assumed that in the absence of any controls, \(\{X(t)\}_{t \geq 0}\) satisfies the equation

\[
dX(t) = \mu dt + \sigma dB(t); \quad X(0) = x > 0
\]  

(1.1)

where \(\mu\) and \(\sigma\) are positive constants. The reader is referred to [4], [27], [29], [35], [51], [52] and [53] for the motivation of this model as well as its applications and extensions.
1.2.1 Optimal Dividend Control

In [35] and [45] the liquid reserves of an insurance company are modelled by means of equation (1.1) while in [4], [27], [51], [29] and [52] more general models are considered. [1] treats a problem on optimal harvesting with density-dependent prices and solved it using singular control theory for diffusion process. A typical application appearing in these references, considers the dividend flow as a controlled process and the objective is to maximise the expected total discounted dividends that are paid out to shareholders up to the company’s bankruptcy. The time to bankruptcy is a stopping time representing the moment at which the reserve process hits 0.

The plain dividend pay-out problem (without reinsurance) was solved in [4]. In this reference it was shown that a threshold control is optimal. Of special interest to this thesis is Example 3.1 discussed in [1]. The problem is summarized below:

In the absence of interventions the stochastic process $X(t)$, which represents the value of an insurance firm evolves according to equation (1.1).

Now, assume that the investor harvests from the system and, as a result, the evolution of the controlled process $X(\gamma)(t)$ is described by

$$dX(\gamma)(t) = \mu dt + \sigma dB(t) - d\gamma(t); \quad X(\gamma)(0) = x > 0$$

where $\gamma(t)$ is a nondecreasing real-valued function representing the total amount of resources paid out up to time $t$.

In this case the aim is to maximise the performance functional $J^{(\gamma)}(s, x)$, given by

$$J^{(\gamma)}(s, x) := E^{s,x} \left[ \int_0^\tau e^{-\rho(s+t)} g(X^{(\gamma)}(t^-))d\gamma(t) \right]$$

where $g: \mathbb{R} \rightarrow \mathbb{R}$ is a given non-increasing function,

$$\tau = \inf \{t : X^{(\gamma)}(t) \leq 0\},$$

$\rho > 0$ is a discount factor and $E^{s,x}(.)$ denotes expectation with respect to probability law $P$ given that the controlled process $X^{(\gamma)}(t)$ takes the value $x$ when $t = s$.

The case with $g$ constant was examined in [35]. The authors derived the following optimal strategy
1. Do nothing if the resources are below a certain trigger level \( x^* > 0 \),

2. If the resources are above the trigger level \( x^* \) harvest according to a local time of the downward reflected process \( X'(t) \) at \( X'(t) = x^* \).

In [1] Problem (1.3) was generalised to the case \( g(x) = x^{-\frac{1}{2}} \) and the following result was obtained

**Theorem 1.2.1** Let \( X^{(\gamma)}(t) \) and \( \tau \) be given by (1.2) and (1.4), respectively.

1. Assume that
   \[
   \mu^2 \leq 2\rho\sigma^2. \tag{1.5}
   \]

   Then
   \[
   \Phi(s, x) := 2e^{-\rho s}\sqrt{x}; \tag{1.6}
   \]

   where \( \sigma \) and \( \rho \) are defined as before. This value is achieved by applying the policy of immediate chattering to 0.

2. If
   \[
   \mu^2 > 2\rho\sigma^2. \tag{1.7}
   \]

   then the value function has the form
   \[
   \Phi(s, x) = \begin{cases} 
   e^{-\rho s}(e^{\lambda_1 x} - e^{\lambda_2 x}); & \text{for } 0 \leq x < x^* \\
   e^{-\rho s}(2\sqrt{x} - 2\sqrt{x^*} + A) & \text{for } x^* \leq x
   \end{cases} \tag{1.8}
   \]

   for some constants \( C > 0, A > 0 \) and \( x^* > 0 \) where
   \[
   \lambda_1 = \sigma^{-2}[-\mu + \sqrt{\mu^2 + 2\rho\sigma^2}] > 0, \quad \lambda_2 = \sigma^{-2}[-\mu - \sqrt{\mu^2 + 2\rho\sigma^2}] < 0. \tag{1.9}
   \]

   The corresponding optimal policy is the following:

   • If \( x > x^* \) it is optimal to apply immediate chattering from \( x \) down to \( x^* \).
   • If \( 0 < x < x^* \) it is optimal to apply the harvesting equal to the local time of the downward reflected process \( X'(t) \) at \( x^* \).
1.2.2 Optimal Proportional Reinsurance

References [11], [22] and [50] are some of the early works in the area of proportional reinsurance. Diffusion models in proportional reinsurance were applied in [16], [29], [30] and [54].

In [29] the following problem was examined.

Let \( \{X(t)\}_{t \geq 0} \) be a stochastic process which represents the evolution of the value of the liquid (cash) reserves of an insurance company in the absence of controls, such as dividend policy and reinsurance policy. Assume that in such a case of no intervention, \( X(t) \), for \( t \geq 0 \), satisfies the following stochastic differential equation

\[
dX(t) = \mu dt + \sigma dB(t), \quad X(0^-) = x > 0; \tag{1.10}
\]

where \( \mu, \sigma > 0 \), \( B(t) \) is a standard Brownian motion with respect to the filtration \( \mathcal{F}_t \) and \( X(0) = x \) is \( \mathcal{F}_0 \)-measurable.

Now, assume that the management of an insurance company are allowed to control the risk exposure of the firm by applying proportional reinsurance policy \( \pi \) whereby, at any time \( t \) \( \pi \) seeks to retain a fraction \( a(t) \in (0, 1) \) of all incoming claims. This means that the fraction \( 1 - a(t) \) of all incoming claims is reinsured.

We require that \( \{a(t)\}_{t \geq 0} \) be an \( \mathcal{F}_t \)-adapted process. A policy \( \pi \) is said to be admissible if \( a(t) \) is \( \mathcal{F}_t \)-adapted, and \( \Pi \) denotes the family of all admissible policies \( \pi \). When the policy \( \pi \) is applied, we denote by \( \{X^\pi(t)\}_{t \geq 0} \) the controlled reserve process of the insurance firm and in such a case \( X^\pi(t) \) satisfies the following stochastic differential equation

\[
dX^\pi(t) = \mu a^\pi(t) dt + \sigma a^\pi(t) dB(t), \quad X^\pi(0^-) = x > 0. \tag{1.11}
\]

It is considered that every control policy \( \pi \in \Pi \) is associated with a performance functional denoted by \( V_\pi(x) \) and defined as

\[
V_\pi(x) = E\left[ \int_0^{\tau_x^\pi} e^{-\rho t} X^\pi(t) dt \right] \tag{1.12}
\]

where \( \{X^\pi(t)\}_{t \geq 0} \) solves (1.11), \( E[\cdot] \) denotes the expectation with respect to probability measure \( P \), \( \rho \) is a discount factor and

\[
\tau_x^\pi := \inf\{t \geq 0 : X^\pi(t) = 0 \text{ and } X^\pi(0) = x\} \quad \text{(time to bankruptcy)}.
\]

22
The problem is to find the optimal return function $V(x)$ and the optimal policy $\pi^*$ such that

$$V(x) = \sup_{\pi \in \Pi} V_\pi(x) = V_{\pi^*}(x). \quad (1.13)$$

Using stochastic control theory for diffusion processes the authors obtained the following explicit solution for the problem:

1. The return function, $V(x)$, is given by

$$V(x) = \begin{cases} 
\int_0^\tau \frac{1}{G^{-1}\left(\frac{x}{k_1}\right)} dz; & 0 \leq x < x_1 \\
\frac{x}{c} + \frac{\mu}{c} + \beta e^{d_1(x-x_1)} & x > x_1 
\end{cases} \quad (1.14)$$

where $x_1$, $k_1$, $\beta$, $d_1$ are constants determined via exogenous parameters of the problem and $G$ is the cumulative distribution function of the Gamma distribution with known parameters.

2. The optimal policy $a^*(x)$ is given by

$$a(x) = \begin{cases} 
\frac{G^{-1}\left(\frac{x}{k_1}\right) g\left(G^{-1}\left(\frac{x}{k_1}\right)\right)}{\alpha g(\alpha)}; & 0 \leq x < x_1 \\
1 & x > x_1 
\end{cases} \quad (1.15)$$

where $\alpha$ is determined via exogenous parameters and $g$ is the density of $G$.

### 1.2.3 Optimal Combined Risk and Dividend Control

More complicated models involve the simultaneous control of level of risk exposure as well as dividend pay-outs, see for example, [27] and [51].

In [51] the combined risk/dividend control process $\pi(t) := (a_\pi(t), \gamma_\pi(t))$ is introduced to the system described by (1.1). In this case $0 \leq a_\pi(t) \leq 1$ is the proportional reinsurance retention level and $\gamma_\pi(t)$ denotes the cumulative dividends paid out by the company up to time $t$. In this context $\{X_\pi(t)\}_{t \geq 0}$ denotes
the controlled reserve process of the insurance firm. We assume that at any time \( X^\pi(t) \) satisfies the following equation

\[
dX^\pi(t) = \mu a^\pi(t) dt + \sigma a^\pi(t) dB(t) - d\gamma^\pi(t), \quad X^\pi(0^-) = x > 0; \tag{1.16}
\]

where \( a^\pi(t) \) and \( \gamma^\pi(t) \) are as described before. For every admissible control policy \( \pi \) a performance functional \( V_\pi(x) \) is defined by

\[
V_\pi(x) = E\left[ \int_0^{\tau^\pi_{x}} e^{-\rho t} X^\pi(t) dt \right] \tag{1.17}
\]

where \( \{X^\pi(t)\}_{t \geq 0} \) solves (1.16), \( E[\cdot] \), \( \rho \) and \( \tau^\pi_{x} \) are defined in an analogous manner as in (1.17).

The problem in this case is to find the optimal return function \( V(x) \) and the optimal combined policy \( \pi^*(t) := (a^\pi^*(t), \gamma^\pi^*(t)) \) such that

\[
V(x) = \sup_{\pi \in \Pi} V_\pi(x) = V_{\pi^*}(x); \tag{1.18}
\]

where \( \Pi \) is the family of all admissible controls \( \pi \). For the case of bounded dividend rate the return function is given by

\[
V(x) = \begin{cases} 
  C_1 x^\lambda, & 0 \leq x < u \\
  C_2 e^{\theta_2 x} + C_3 e^{\theta_1 x}, & u \leq x < u_1, \\
  \frac{M}{c} + C_4 e^{\hat{\theta} x}, & x \geq u_1 
\end{cases}
\]

\[
\tag{1.19}
\]

where \( u, C_1, C_2, C_3, C_4, \theta_1, \theta_1, \hat{\theta}, u \) and \( u_1 \) are known constants. The reader is referred to [51] for more details.

The optimal policy \( a^\pi^*(x) \) is given by

\[
a^\pi^*(x) = \begin{cases} 
  \frac{x}{u}, & x < u \\
  1, & x \geq u.
\end{cases}
\]

\[
\tag{1.20}
\]

Dividends are paid out when \( x > u_1 > u \) and are paid at maximal rate.
In the case of unrestricted dividend rate the return function is given by:

\[
V(x) = \begin{cases} 
C_1 x^\lambda, & 0 \leq x < u \\
C_2 e^{\theta_2 x} + C_3 e^{\theta_1 x}, & u \leq x < u_1, \\
x + C_4; & x \geq u_1
\end{cases}
\]

The optimal reserve process \( X^*(t) := X^{\pi^*}(t) \) would be reflected at \( u_1 \) with drift \( \mu a_{\pi^*}(x) \) and a diffusion coefficient \( \sigma a_{\pi^*}(x) \). The optimal dividend control functional \( \gamma^* \) is such that \( (X^*(t), \gamma^*(t)) \) constitutes a solution for the Skorohod problem in \((0, u_1]\). In this case

\[
a_{\pi^*}(x) = \frac{x}{u} \wedge 1.
\]

1.3 The Research Problem

As illustrated in Section 1.2, research in the area of risk and dividend control has been driven by the following two major assumptions

(a) No transaction costs are incurred by the insurance company for paying out dividends.

(b) In the absence of control the reserve process of an insurance company is a diffusion process described by equation (1.1).

However, in practical terms a typical insurance company incurs transaction costs in the process of paying out dividends. These transaction costs may be in the form of bank charges or paying employees for effecting the payouts. On the other hand, the diffusion model is not an adequate description of the evolution of the company’s reserve process. Such a model does not cater for abrupt significant changes in the value of a firm due to such circumstances as major shift in policy by monetary authorities, breakout of war, sudden and unpredicted increase in the number of claims by clients.

The aim of this thesis is to generalise the problems and results presented in Subsection 1.2.3 in two directions. First, we consider that each time that dividends
are paid out the insurance company incurs proportional and/or fixed transaction costs. Secondly, we aim at investigating the combined risk/dividend control problem when the reserve process is driven by both a Brownian motion and a Levy process. The objective is to optimise a predetermined performance functional.

The jump diffusion model is a natural extension of the diffusion model. In mathematical finance the jump diffusion model constitutes a more realistic description of the price process of financial assets such as the stock.

We now present the mathematical formulation of the problem investigated in this thesis. At this stage we point out that throughout this thesis we are given a filtered complete probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) which satisfies the usual conditions.

### 1.3.1 Mathematical Formulation of the Problem

Let \(\{X(t)\}_{t \geq 0}\) be a stochastic process which represents the evolution of the value of the liquid reserves of an insurance company in the absence of controls, such as dividend policy and reinsurance policy. Given this scenario of no intervention by management, assume that at any time \(t\), \(X(t)\) satisfies the following equation

\[
dX(t) = \mu dt + \sigma dB(t) + \beta \int_{\mathbb{R}} z \tilde{N}(dt, dz); \quad X(0^-) = x > 0;
\]

where \(X(0^-) = x\) is \(\mathcal{F}_0\)-measurable, \(\mu, \sigma, \beta > 0\) are constants, \(B(t)\) is a standard 1-dimensional Brownian motion with respect to the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) and \(\tilde{N}(dt, dz)\) is a compensated Poisson random measure.

Now, suppose the insurance company applies a combined risk and dividend control policy \(w(t) := (a(t), \psi(t))\), where \(0 \leq a(t) \leq 1\) is the proportional reinsurance retention level and \(\psi(t)\) represents the cumulative dividends paid up to time \(t\). In this case we impose the requirement that for each \(t \geq 0\), \(a(t)\) be \(\mathcal{F}_t\)-adapted and \(\psi(t)\) be an \(\mathcal{F}_t\)-measurable random variable. A policy \(w\) is said to be admissible if \(a(t)\) is \(\mathcal{F}_t\)-adapted and \(\psi(t)\) is \(\mathcal{F}_t\)-measurable. We denote by \(W\) the family of all admissible two-dimensional control policies \(w\).
Consider
\[ 0 < \tau_1 \leq \tau_2 \leq \ldots \leq \tau_j \leq \ldots \]
to be an increasing sequence of stopping times, where \( j = 1, 2, \ldots, M \leq \infty \). Define \( \xi_j \) to be an \( \mathcal{F}_{\tau_j} \)-measurable random variable representing the amount of dividends paid out, at time \( \tau_j \), by the company to the shareholders, for \( j = 1, 2, \ldots, M \leq \infty \). Assume that the payment of dividends amounting to \( \xi_j \), at stopping time \( \tau_j \), is associated with a transaction cost that consists of two components: a component proportional to the amount paid out and a fixed cost. Let \( \delta \xi_j \) be the proportional cost component and \( c \) be the fixed part of the total cost where \( \delta \in (0, 1) \) and \( c \geq 0 \).

As an immediate consequence of applying the policy \( w := w(t) \in \Pi \), we denote by \( \{X^w(t)\}_{t \geq 0} \) the controlled reserve process of the firm and it is assumed that \( X^w(t) \) satisfies equations (1.23)-(1.25) given below.

\begin{align*}
X^w(t) &= X(t) \quad \text{if } 0 \leq t < \tau_1; \quad (1.23) \\

\frac{dX^w}{dt} &= \mu a_w(t)dt + \sigma a_w(t)dB(t) + \beta a_w(t) \int_{\mathbb{R}} z\tilde{N}(dt,dz) \\
&\quad - (1 + \delta) d\psi^w(t) - c H(d\psi^w(t)); \quad \text{if } \tau_j \leq t < \tau_{j+1}; \quad j = 1, 2, \ldots \quad (1.24) \\
X^w(\tau_j) &= \tilde{X}^w(\tau_j^-) - (1 + \delta) \xi_j - c. \quad (1.25)
\end{align*}

where \( H(y) := \chi_{(0,\infty)}(y) \) is the indicator function of the set \((0, \infty)\) and \( \tilde{X}^w(\tau_j^-) = X^w(\tau_j^-) + \Delta_N X^w(\tau_j) \) represents the jump in \( X^w(\tau_j) \) which stems from the Poisson random measure \( N(. , .) \).

For each combined control \( w = (a(t), \psi(t)) \) define the performance functional, \( J^w(s, x) \), by
\[
J^w(s, x) := E^{s,x} \left[ \int_s^{\tau_x^w} e^{-\rho(s+t)} (X^w)^\alpha(t) d\psi^w(t) \right] \quad (1.26)
\]
where \( \tau_x^w = \inf \{ t : X^w(t) \leq 0, \text{ and } X^w(s) = x \} \) (time to exhaustion of resources), \( \rho > 0 \) is a discount factor, \( 0 < |\alpha| \leq 1 \), \( E^{s,x}[\cdot] \) denotes the mathematical
expectation with respect to probability law $P$ given that the controlled process $\{X^w(t)\}_{t \geq 0}$ has value $x$ at time $t = s$.

The problem is to find the optimal return function $\Phi(s, x)$ and the optimal admissible control $w^* := (a_{w^*}, \psi_{w^*})$ such that

$$\Phi(s, x) = \sup_{w \in W} J^{(w)}(s, x) = J^{(w^*)}(s, x)$$  \hspace{1cm} (1.27)

We also extend our investigations to the case of convex risk minimization and risk minimisation using g-expectation.

1.4 Basic Definitions

Definition 1.4.1 (Stochastic Process)
A stochastic process is a parametrized collection of random variables $\{X_t\}_{t \in T}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in $\mathbb{R}^n$.

Definition 1.4.2 (Stochastic Basis/Filtered Probability Space)
A stochastic basis is a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$.

A stochastic basis is also called a filtered probability space.

Definition 1.4.3 (Usual conditions/Usual Hypotheses)
A complete stochastic basis is said to satisfy the usual conditions or usual hypotheses if

1. $\mathcal{F}_0$ contains all $P$-null sets of $\mathcal{F}$
2. $\mathcal{F}_t = \bigcap_{u > t} \mathcal{F}_u$ for all $t$, $0 \leq t < \infty$ that is, $\{\mathcal{F}_t\}_{t \geq 0}$ is right-continuous.

Definition 1.4.4 (Stochastic Process)
A stochastic process is a parametrized collection of random variables $\{X_t\}_{t \in T}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in $\mathbb{R}^n$.  

28
Definition 1.4.5 (Adapted Process)
A stochastic process \( \{X_t\}_{t \geq 0} \) is said to be adapted to the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) if \( X_t \) is \( \mathcal{F}_t \)-measurable for all \( t \).

Definition 1.4.6 (Stopping time)
A mapping \( \tau : \Omega \to [0, \infty) \) is called \( \{\mathcal{F}_t\}_{t \geq 0} \)-stopping time if
\[
(\tau \leq t) := \{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0.
\]

Definition 1.4.7 (Levy Process)
Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) be a filtered probability space. An \( \{\mathcal{F}_t\} \)-adapted process \( \{\eta(t)\}_{t \geq 0} \subset \mathbb{R} \) with \( \eta(0) = 0 \) a.s. is called a Levy process if \( \eta(t) \) has stationary, independent increments.

Definition 1.4.8 (Brownian Motion)
A Brownian motion on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) is a Levy process with normally distributed increments.

Definition 1.4.9 (Poisson Random measure; Lévy Measure; Compensated Poisson Measure)

1. Let \( B_0 \) be the family of Borel sets \( U \subset \mathbb{R} \) whose closure \( \bar{U} \) does not contain 0. For \( U \in B_0 \), the Poisson random measure for the Levy process \( \{\eta(t)\} \), denoted by \( N(t, U) \), is defined by
\[
N(t, U) = N(t, U, \omega) = \sum_{s \in \mathbb{R}^+ : 0 < s \leq t} \chi_U(\Delta \eta_s)
\]
where \( \chi_U \) is the indicator function of set \( U \).

2. The set function
\[
\nu(U) = E[N(1; U)]
\]
where \( E = E_P \) denotes expectation with respect to \( P \), defines a \( \sigma \)-finite measure on \( \mathcal{B} \), is called the Levy measure of \( \{\eta(t)\} \).

3. \( \tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)(dt) \) is called the compensated Poisson random measure of \( \eta(.) \).

29
Chapter 2

Combined Singular and Impulse Control in Insurance

2.1 Introduction

This chapter examines a special case of the general control problem presented in subsection 1.3.1 of Chapter 1. Specifically, we investigate the case where the insurance firm insures all incoming claims. This implies that the only control variable investigated is dividend policy. Both fixed and proportional transaction costs are considered in the market model.

The market model presented in equations (2.2)-(2.3) and Theorem 2.1 are our major contributions in this chapter. Theorem 2.1 is a new verification theorem for the generalised combined singular and impulse control for Lévy processes. Reference [36] may be cited as earlier work on combined singular and impulse control for diffusions. The methodology and results in [36] are not general enough to be applied to the case of jump diffusions. The theory of combined singular and impulse control is applied, for the first time in the literature, to solve the dividend control problem in insurance where the evolution of the risk process is described by a jump-diffusion in the absence of dividend payouts.

1Publication [13] is based on this Chapter.
The rest of this chapter is organised as follows. In Section 2.2 the market model and problem formulation are presented. The general combined singular and impulse control problem is formulated in Section 2.3. In Section 2.4 the verification theorem and its proof are presented. The verification theorem constitutes the main result of this chapter. An example on the application of the theory of combined singular and impulse control for jump diffusions is discussed in Section 2.5. In this example we take both proportional and fixed transaction costs into account.

2.2 Market Model and Problem Formulation.

As already stated in Section 2.1, in this Chapter we focus on the case where the retention level of the insurance company is 1.

Now, let \( \{X(t)\}_{t \geq 0} \) be a stochastic process which represents the evolution of the reserves of an insurance company in the absence of controls. In such a case of no intervention or controls by management, assume that at any time \( t \), the dynamics of \( X(t) \) is described by the following equation

\[
dX(t) = \mu dt + \sigma dB(t) + \beta \int_{\mathbb{R}} z \tilde{N}(dt, dz); \quad X(0^-) = x > 0;
\]

(2.1)

where \( X(0^-) = x \) is \( \mathcal{F}_0 \)-measurable, \( \mu, \sigma, \beta > 0 \) are constants, \( B(t) \) is a standard 1-dimensional Brownian motion with respect to the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) and \( \tilde{N}(dt, dz) \) is a compensated Poisson random measure.

Consider a sequence of stopping times \( (\tau_1, \tau_2, ..., \tau_j, ...)_{j \leq M} \) for \( M \leq \infty \), and a sequence of random variables \( (\xi_1, \xi_2, ..., \xi_j, ...) \) such that \( \xi_j \) is \( \mathcal{F}_{\tau_j} \)-measurable and \( 0 \leq \tau_1 \leq \tau_2 \leq ... \leq \tau_j \leq .... \). Assume that at any time \( \tau_j \), the management of the firm are allowed to pay out a lump sum dividend represented by the random variable \( \xi_j \), from \( X(t) \). We now state the following definition of an impulse control.

**Definition 2.2.1** An impulse control, for the reserves of the firm, is a possibly finite double sequence denoted by \( v \) and given by

\[
v := (\tau_1, \tau_2, ..., \tau_j; \xi_1, \xi_2, ..., \xi_j, ...)_{j \leq M} \quad \text{for} \quad M \leq \infty,
\]
where the sequences \((\tau_1, \tau_2, \ldots, \tau_j, \ldots)\) and \((\xi_1, \xi_2, \ldots, \xi_j, \ldots)\) are described as before.

Also suppose that at times \(t_n \in [\tau_j, \tau_{j+1}]\) for \(j = 1, 2, \ldots\) the firm is allowed to intervene and apply, whenever it is profitable to do so, a singular control \(\psi\) for \(n = 1, 2, \ldots, q\).

It is noted that \(\psi \in \mathbb{R}\) is an adapted cadlag non-negative, increasing process such that \(\psi(0^-) = 0\). Both controls \(v\) and \(\psi\) are associated with proportional as well as fixed transaction costs. The proportional and fixed transaction cost parameters are \(\delta \in (0, 1)\) and \(c \geq 0\), respectively. The proportional transaction costs may be thought of as taxation to be paid by shareholders and the fixed costs could possibly be salaries for staff responsible for payment of dividends. In this work, we shall refer to a control \(w := (\psi, v)\) as a combined singular and impulse control.

Consider that \(\{X^{(w)}(t)\}_{t \geq 0}\) denotes the controlled reserve process of the firm. Let \(\mathcal{W}\) be the set of all combined singular and impulse controls \(w = (\psi, v)\) such that \(X^{(w)}(t) \geq 0\). The family \(\mathcal{W}\) is called the set of admissible controls. Assume that the evolution of \(X^{(w)}(t)\) is governed by equations (2.2)-(2.3) given below.

\[
\begin{align*}
\text{d}X^{(w)}(t) &= \mu \text{d}t + \sigma \text{d}B(t) + \beta \int_{\mathbb{R}} z \tilde{N}(dt, dz) \\
&\quad - (1 + \delta) \text{d}\psi(t) - c H(\text{d}\psi(t)) \quad \text{if } \tau_{j-1} \leq t < \tau_j; \quad j = 1, 2, \ldots
\end{align*}
\]

(2.2)

\[
X^{(w)}(\tau_j) = \hat{X}^{(w)}(\tau_j^-) - (1 + \delta) \xi_j - c; \quad j = 1, 2, \ldots
\]

(2.3)

where \(H(y) := \chi_{(0, \infty)}(y)\) is the indicator function of the set \((0, \infty)\) and \(\hat{X}^{(w)}(\tau_j^-) = X^{(w)}(\tau_j^-) + \Delta_N X^{(w)}(\tau_j)\) represents the jump in \(X^{(w)}(\tau_j)\) which stems from the Poisson random measure \(N(., .)\).

For each combined control \(w = (\psi(t), v)\) define the performance functional, \(J^w(s, x)\), by

\[
J^w(s, x) := E_s^x \left[ \int_s^{\tau^w} e^{-\rho(t)} (\hat{X}^{(w)})^\alpha(t) d\psi(t) \right]
\]

(2.4)
where \( \tau^w_x = \inf\{t : X^{(w)}(t) \leq 0, \text{ and } X^{(w)}(s) = x \} \) (time to exhaustion of resources), \( \rho > 0 \) is a discount factor, \( 0 < \alpha \leq 1 \), \( E^{x,x} \) denotes the mathematical expectation with respect to probability law \( P \) given that the controlled process \( \{X^{(w)}(t)\}_{t \geq 0} \) has value \( x \) at time \( t = s \).

The problem is to find the optimal return function \( \Phi(s, x) \) and the optimal admissible control \( w^* := (\psi^*, v^*) \) such that

\[
\Phi(s, x) = \sup_{w \in W} J^{(w)}(s, x) = J^{(w^*)}(s, x) \tag{2.5}
\]

In order to examine this problem, we now develop a general combined singular and impulse control theory for Lévy processes.

### 2.3 Combined Singular and Impulse Control Theory

Let \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P) \) be a filtered complete probability space satisfying the usual conditions.

It is assumed that in the absence of interventions the state, \( Y(t) \in \mathbb{R}^k \), of a given system evolves according to the following equations

\[
dY(t) = b(Y(t))dt + \sigma(Y(t))dB(t) + \int_{\mathbb{R}^k} \gamma(Y(t^-), z)\tilde{N}(dt,dz); \tag{2.6}
\]

\[
Y(0^-) = y \in \mathbb{R}^k, \tag{2.7}
\]

where \( b : \mathbb{R}^k \to \mathbb{R}^k \), \( \sigma : \mathbb{R}^k \to \mathbb{R}^{k \times m} \) and \( \gamma : \mathbb{R}^k \times \mathbb{R}^d \to \mathbb{R}^{k \times d} \) are given functions satisfying the conditions for the existence and uniqueness of a strong solution, \( Y(t) \). For details concerning such conditions see, for example, Theorem 1.19 in [42]. Here, \( B(t) \) is \( m \)-dimensional Brownian motion with respect to \( \{\mathcal{F}_t\} \) and \( \tilde{N}_r(\cdot, \cdot) \) is a compensated Poisson random measure given by

\[
\tilde{N}_r(dt,dz) = N_r(dt,dz) - dt\nu_r(dz); \quad r = 1, 2, \ldots, d
\]

where \( \nu_r(.) \) is a Lévy measure associated with the Poisson random measure \( N_r(\cdot, \cdot) \). For a more extensive treatment of random measures and stochastic
differential equations with a jump component see, for example [5], [6], [34], [42] and [44].

Consider an increasing sequence of stopping times \((\tau_1, \tau_2, \ldots, \tau_j, \ldots)\), and a sequence of \(p\)-dimensional random variables \((\xi_1, \xi_2, \ldots, \xi_j, \ldots)\) such that \(\xi_j \in \mathbb{R}^p\) is \(\mathcal{F}_{\tau_j}\)-measurable for all \(j\). Suppose that at any given instant \(\tau_j\), the decision maker is free to give the system an impulse, \(\xi_j \in \mathcal{Z} \subset \mathbb{R}^p\).

We now state the following definition of an impulse control.

**Definition 2.3.1** An impulse control is a possibly finite double sequence, denoted by \(v\), and given by

\[
v := (\tau_1, \tau_2, \ldots; \xi_1, \xi_2, \ldots)_{j \leq M} \quad \text{for} \quad M \leq \infty. \tag{2.8}
\]

Let \(\mathcal{Z}\) be the set of all admissible impulses. We now consider that as a result of applying an admissible impulse control \(v = (\tau_1, \tau_2, \ldots; \xi_1, \xi_2, \ldots)\), the corresponding controlled state process, \(Y^{(v)}(t)\), evolves according to equations (2.9)-(2.11) stated below

\[
Y^{(v)}(0^-) = y \quad \text{and} \quad Y^{(v)}(t) = Y(t); \quad 0 < t < \tau_1 \tag{2.9}
\]

\[
Y^{(v)}(\tau_j) = \Gamma(Y^{(v)}(\tau_j^-), \xi_j); \quad j = 1, 2, \ldots \tag{2.10}
\]

\[
dY^{(v)}(t) = b(Y^{(v)}(t))dt + \sigma(Y^{(v)}(t))dB(t) + \int_{\mathbb{R}^d} \gamma(Y^{(v)}(t^-), z)\tilde{N}(dt, dz)
\]

\[
\quad \text{for} \quad \tau_j < t < \tau_{j+1} \tag{2.11}
\]

where

\[
\dot{Y}^{(v)}(\tau_j^-) = Y^{(v)}(\tau_j^-) + \Delta_N Y(\tau_j), \tag{2.12}
\]

35
\( \Delta_N Y(\tau_j) \) represents the jump in \( Y^{(v)}(\tau_j) \) which stems from \( N(.,.) \) and

\[ \Gamma : \mathbb{R}^k \times \mathcal{Z} \rightarrow \mathbb{R}^k \]

is a given function.

Let \( S \subset \mathbb{R}^k \) be a fixed Borel set in which we seek solutions to the problem such that \( S \subset \bar{S}^0 \).

Suppose we are given continuous functions \( f : S \rightarrow \mathbb{R}, \ g : \mathbb{R}^k \rightarrow \mathbb{R}, \kappa = [\kappa_{ie}] \in \mathbb{R}^{k \times p} \) and \( \theta = [\theta_i] \). Additionally, let the profit of making an intervention with impulse \( \xi \in \mathcal{Z} \) when the state is \( y \) be \( K(y, \xi) \), where \( K : S \times \mathcal{Z} \rightarrow \mathbb{R} \). Let \( \mathcal{V} \) be the set of admissible impulse controls, \( v \), and also assume that \( f, g \) and \( K \) satisfy appropriate integrability requirements.

Let

\[ \tau_S := \inf\{ t \geq 0 ; Y^{(v)}(t) \notin S \}. \]

The notion of intervention operator plays a crucial role in the rest of this work, so we define it below.

**Definition 2.3.2** Let \( \mathcal{H} \) be the space of all measurable functions \( h : S \rightarrow \mathbb{R} \). The intervention operator \( \mathcal{M} : \mathcal{H} \rightarrow \mathcal{H} \) is defined by

\[ \mathcal{M}h(y) = \sup\{ h(\Gamma(y, \xi)) + K(y, \xi) ; \xi \in \mathcal{Z} \}. \]  

(2.13)

We let

\[ \mathcal{T} = \{ \tau ; \tau \text{ stopping times, } 0 \leq \tau \leq \tau_S \}. \]

Suppose that at times \( t_n \in [\tau_j; \tau_{j+1}] \) one is allowed to intervene and apply, whenever it is profitable to do so, the singular control \( \psi \) for \( n = 1, 2, ..., q \).

We note that \( \psi \in \mathbb{R}^p \) is an adapted cadlag process with non-negative, increasing components such that \( \psi(0^-) = 0 \).

A control process of the form \( w := (\psi, v) \) shall be referred to as a combined singular and impulse control. Let \( \{X^{(w)}(t)\}_{t \geq 0} \) denote the controlled process obtained by applying the combined control \( w \) on \( X(t) \). Let \( \mathcal{W} \) be the set of
all admissible combined singular and impulse controls \(w\). It is worthwhile to note that the controlled process \(\{X^{(w)}(t)\}_{t \geq 0}\) has two sources of jumps, namely, \(N(dt, dz)\) and the control \(w\).

Consider that we are given generalised functions of the type \(\kappa = [\kappa_{ie}] : \mathbb{R}^k \rightarrow \mathbb{R}^{k \times p}\) and \(\theta = [\theta_i] : \mathbb{R} \rightarrow \mathbb{R}^p\). Assume that \(\kappa\) and \(\theta\) are continuous with respect to the usual Euclidean metric.

We shall denote by \(\triangle NY(t)\) the jump of the process \(\{X^{(w)}(t)\}_{t \geq 0}\) stemming directly from the Poisson random measure \(N(dt, dz)\), and we represent by \(\triangle \psi Y(t)\) the jump of \(\{X^{(w)}(t)\}_{t \geq 0}\) caused by the combined control \(w\). The jumps \(\triangle NY(t)\) and \(\triangle \psi Y(t)\) are defined by

\[
\triangle NY(t) := \int_{\mathbb{R}^d} \gamma(Y^w(t^-), z) \tilde{N}(dt, dz)
\]

and

\[
\triangle \psi Y(t) := \kappa(Y^w(t^-))\triangle \psi.
\]

Suppose that as a consequence of applying the combined singular and impulse control, \(w = (\psi, v)\), the state process \(Y^{(w)}\) satisfies equations (2.14)-(2.16) given as follows

\[
Y^{(w)}(0^-) = y \in \mathbb{R}^k \text{ and } Y^{(w)}(t) = Y(t); \quad 0 < t < \tau_1
\]  
(2.14)

\[
Y^{(w)}(\tau_j) = \Gamma(Y^w(\tau_j^-), \xi_j); \quad j = 1, 2, \ldots
\]  
(2.15)

\[
dY^{(w)}(t) = b(Y^{(w)}(t))dt + \sigma(Y^{(w)}(t))dB(t)
\]

\[
+ \int_{\mathbb{R}^d} \gamma(Y^{(w)}(t^-), z) \tilde{N}(dt, dz) + \kappa(Y^{(w)}(t))d\psi
\]

for \(\tau_j < t < \tau_{j+1} < \tau^*\)

(2.16)

where

\[
\tau^* = \tau^*(\omega) = \lim_{R \to \infty} (\inf \{t > 0; |Y^{(w)}(t)| \geq R\}) \leq \infty.
\]  
(2.17)

Define a performance functional, \(J^{(w)}\), for the controlled process \(Y^{(w)}\), by

\[
J^{(w)}(y) = \mathbb{E}^w \left[ \int_0^{\tau_S} f(Y^{(w)}(t))dt + g(Y^{(w)}(\tau_S^-))\chi_{\{\tau_S < \infty\}} + \int_0^{\tau_S} \theta^T(Y(t))d\psi(t) + \sum_{\tau_j \leq \tau_S} K(Y^{(w)}(\tau_j^-), \xi_j) \right].
\]

37
The combined singular and impulse control problem for jump diffusions is to find \( \Phi(y) \) and \( w^* \in W \) such that
\[
\Phi(y) = \sup \{ J(w)(y); w \in W \} = J^{(w^*)}(y).
\]  
(2.18)

Let \( \phi \in C^2(S^o) \cap C(\bar{S}) \). If \( t \in [\tau_{j-1}, \tau_j] \) for \( j = 1, 2, ... \) and \( dw = 0 \) then the generator of \( Y^{(w)}(t) \) coincides with the second order integro-differential operator \( L := L^w \), given by
\[
L\phi(y) = \sum_{i=1}^k b_i(y) \frac{\partial \phi}{\partial y_i} + \frac{1}{2} \sum_{i,s=1}^k (\sigma \sigma^T)_{is}(y) \frac{\partial^2 \phi}{\partial y_i \partial y_s}
\]
\[
+ \int_{\mathbb{R}^k} \sum_{r=1}^d \{(\phi(y + \gamma^{(r)}(y,z)) - \phi(y) - \nabla \phi(y)^T \gamma^{(r)}(y,z)) \nu_r(dz_r).
\]  
(2.19)

If \( t_i \in [\tau_{j-1}, \tau_j] \) are the jumping times of \( w \), for \( i = 1, 2, ... n ; j = 1, 2, .... \), then the increase in the function \( \phi \), at time \( t_i \), caused by the jump in \( w \) is denoted by \( \Delta_w \phi(Y(t_i)) \) and defined as
\[
\Delta_w \phi(Y(t_i)) = \phi(Y(t_i)) - \phi(Y(t_i^-)) + \Delta_N \phi(Y(t_i)).
\]  
(2.20)

In the next section we state and prove a verification theorem for the combined singular and impulse control problem of jump diffusions. The theorem that we present below constitutes the main result of this chapter.

### 2.4 Main Result

**Theorem 2.1 (Verification theorem)**

1. Suppose that we can find \( \phi : \bar{S} \rightarrow \mathbb{R} \) such that
   
   (i) \( \phi \in C^1(S^o) \cap C(\bar{S}) \),
   
   (ii) \( \phi \geq M \phi \) on \( S^o \),
   
   (iii) \( \sum_{i=1}^k \kappa_{ie}(y) \frac{\partial \phi}{\partial y_i}(y) + \theta_e(y) \leq 0 \) for all \( y \in S \), \( e = 1, 2, ..., p \),
Define
\[ D = \{ y \in S; \max_e \{ M\phi(y) - \phi(y), \sum_{i=1}^{k} \kappa_{ie}(y) \frac{\partial \phi}{\partial y_i}(y) + \theta_e(y) \} \leq 0 \} \]
(2.21)

Assume that \( Y^{(w)}(t) \) spends 0 time on \( \partial D \) a.s., that is,
(iv) \( E^y \left[ \int_0^{\tau_S} \chi_{\partial D}(Y^{(w)}(t)) dt \right] = 0 \) for all \( y \in S, w \in W \),
and suppose that
(v) \( \partial D \) is the graph of a Lipschitz function (i.e. \( \partial D \) is a Lipschitz surface),
(vi) \( \phi \in C^2(S \setminus \partial D) \) with locally bounded derivatives near \( \partial D \),
(vii) \( Y^{(w)}(\tau_S) \in \partial S \) a.s. on \( \{ \tau_S < \infty \} \) and
\[ \phi(Y^{(w)}(t)) \to g(Y^{(w)}(\tau_S)) \cdot \chi_{\{\tau_S < \infty\}} \text{ as } t \to \tau_S^- \text{ a.s., for all} \]
\( y \in S, w \in W \),
(ix) \( \{ \phi^-(Y^{(w)}(\tau)); \tau \in T \} \) is uniformly integrable, for all \( y \in S, w \in W \).
Then
\[ \phi(y) \geq \Phi(y) \quad y \in S. \]

2. Suppose that, in addition to conditions 1(i) – 1(vi),
(i) there exists a function \( \tilde{w} = (\tilde{\psi}, \tilde{v}) \in W \) such that
\[ \mathcal{L}\phi(y) + f(y, \tilde{w}(y)) = 0 \text{ for all } y \in D \]
(ii) \( Y^{\tilde{w}}(t) \in \tilde{D} \)
(iii) \( \sum_{e=1}^{p} \left\{ \sum_{i=1}^{k} \kappa_{ie}(y) \frac{\partial \phi}{\partial y_i}(Y^{(w)}(t^-)) + \theta_e \right\} d\tilde{\psi}^{(c)}_e(t) = 0 \) for all \( 1 \leq p \)
where \( \tilde{\psi}^{(c)}_e(t) \) is the continuous part of \( \tilde{\psi}^{(c)}_e \).
(iv) \( \triangle \tilde{w}_e \phi(Y^{(w)}(t_n)) + \sum_{e=1}^{p} \theta_e (Y^{(w)}(t_n^-)) \triangle \tilde{w}_e(t_n) = 0 \) for all jumping times \( t_n \) of \( \tilde{w} \) and
\[ \lim_{R \to \infty} E^y \left[ \phi(Y^{\tilde{w}}(T_R)) \right] = E^y \left[ g(Y^{\tilde{w}}(\tau_S)) \cdot \chi_{\{\tau_S < \infty\}} \right] \]
where

39
\[ T_R = \min(\tau_S, R) \text{ for } R < \infty. \]

and

\[(v) \hat{\xi}(y) \in \text{Argmax}\{\phi(\Gamma(y, .)) + K(y, .)\} \in \mathbb{Z} \text{ exists for all } y \in S. \]

Then

\[ \phi(y) = \Phi(y) \text{ for all } y \in S \]

and

\[ \hat{w} \in \mathcal{W} \text{ is an optimal combined singular impulse control}. \]

Proof

Proof of part 1 of the theorem

On the basis of assumptions 1(iv) – 1(vi) of the theorem we can use an approximation argument (see for example Theorem 2.1 in [42], Theorem 10.4.1 in [40]) to observe that \( \phi \in C^2(S) \cap C(\bar{S}) \).

Consider an arbitrarily chosen impulse control \( v = (\tau_1, \tau_2, ..., \tau_j, ..., \xi_1, \xi_2, ..., \xi_j, ...) \in \mathcal{V} \)

and let \( \tau_0 = 0 \). Since \( \dot{\phi} \) is twice continuously differentiable, see condition 1(vi) of the above theorem, we can apply Itô’s generalized formula for semimartingales, see for example [44](page 74 Theorem 33), between the stopping times \( \tau_j \) and \( \tau_{j+1} \) with \( y \in S \), to obtain

\[ \phi(\hat{Y}^{(w)}(\tau_{j+1}^-)) - \phi(Y^{(w)}(\tau_j)) = \int_{\tau_j}^{\tau_{j+1}} \mathcal{L}\phi(Y^{(w)}(t))dt + \]

\[ + \int_{\tau_j}^{\tau_{j+1}} \sum_{i=1}^{k} \left[ \frac{\partial \phi}{\partial y_i}(Y^{(w)}(t^-)) \right] \sum_{e=1}^{p} \kappa_{i e}(Y^{(w)}(t^-))d\psi^e_c(t) + \sum_{\tau_j < t_n < \tau_j + 1} \Delta \psi \phi(Y^{(w)}(t_n)) \]

(2.22)

where \( \hat{Y}^{(w)}(\tau_{j+1}^-) = Y^{(w)}(\tau_{j+1}^-) + \Delta N Y^{(w)}(\tau_{j+1}) \) and \( \psi^e_c(t) \) denotes the continuous part of \( \psi^e(t) \).

Taking expectations in (2.22) we get

\[ E^y \left[ \phi(\hat{Y}^{(w)}(\tau_{j+1}^-)) \right] - E^y \left[ \phi(Y^{(w)}(\tau_j)) \right] = E^y \left[ \int_{\tau_j}^{\tau_{j+1}} \mathcal{L}\phi(Y^{(w)}(t))dt \right] + \]

\[ + \int_{\tau_j}^{\tau_{j+1}} \sum_{i=1}^{k} \left[ \frac{\partial \phi}{\partial y_i}(Y^{(w)}(t^-)) \right] \sum_{e=1}^{p} \kappa_{i e}(Y^{(w)}(t^-))d\psi^e_c(t) + \sum_{\tau_j < t_n < \tau_j + 1} \Delta \psi \phi(Y^{(w)}(t_n)) \].
This last equation is equivalent to

\[ E_y \left[ \phi(Y^{(w)}(\tau_j)) \right] - E_y \left[ \phi(Y^{(w)}(\tau_j^{-})) \right] = -E_y \left[ \int_{\tau_j}^{\tau_j+1} \mathcal{L} \phi(Y^{(w)}(t)) dt + \int_{\tau_j}^{\tau_j+1} k \sum_{i=1}^p \frac{\partial \phi}{\partial y_i} (Y(t^-)) \sum_{e=1}^p \kappa_{ie} (Y^{(w)}(t^-)) d\psi^e_c(t) + \sum_{0<\tau_n<\tau_{j+1}} \Delta \phi \psi(Y^{(w)}(t_n)) \right]. \]

(2.24)

Focussing our attention on the first \( m \) stopping times, we can sum up from \( j = 0 \) to \( j = m \) to get

\[
\phi(y) + \sum_{j=1}^m E_y \left[ \phi(Y^{(w)}(\tau_j)) - \phi(Y^{(w)}(\tau_j^-)) \right] - E_y \left[ \phi(Y^{(w)}(\tau_{m+1})) \right] = -E_y \left[ \int_0^{\tau_{m+1}} \mathcal{L} \phi(Y^{(w)}(t)) dt + \int_0^{\tau_{m+1}} k \sum_{i=1}^p \frac{\partial \phi}{\partial y_i} (Y(t^-)) \sum_{e=1}^p \kappa_{ie} (Y^{(w)}(t^-)) d\psi^e_c(t) + \sum_{0<\tau_n<\tau_{m+1}} \Delta \phi \psi(Y^{(w)}(t_n)) \right].
\]

(2.25)

Using the fact that \( \dot{Y}^{(w)}(\tau_j^-) = Y^{(w)}(\tau_j^-) + \Delta_N Y(\tau_j) \) and combining this with the definition of the intervention operator we can get

\[
\phi(Y^{(w)}(\tau_j)) = \phi(\Gamma(\dot{Y}^{(w)}(\tau_j^-), \xi_j)) \leq \mathcal{M} \phi(Y^{(w)}(\tau_j^-)) - K(\dot{Y}^{(w)}(\tau_j^-), \xi_j) \text{ if } \tau_j < \tau_S
\]

(2.26)

and by assumption 1(viii) of the theorem we have

\[
\phi(Y^{(w)}(\tau_j)) = \phi(Y^{(w)}(\tau_S))
\]

From (2.26) we get

\[
\mathcal{M} \phi(\dot{Y}(\tau_j^-)) - \phi(Y(\tau_j^-)) \geq \phi(Y(\tau_j^-)) - \phi(\dot{Y}(\tau_j^-)) + K(\dot{Y}(\tau_j^-), \xi_j).
\]

(2.27)
Applying the mean value theorem we obtain

\[
\Delta \psi \phi(Y^{(n)}(t_n)) = \nabla \phi(Y^{(n)}(t_n))^T \Delta \psi(Y^{(w)}(t_n)) = \sum_{i=1}^{k} \sum_{l=1}^{p} \frac{\partial \phi}{\partial y_i}(\tilde{Y}^{(n)}((t_n)))[\kappa_l(Y^{(n)}(t_n^-))(\Delta \psi)(t_n)]
\]

(2.28)

where \(\tilde{Y}^{(n)}\) is some point on the straight line between \(Y(t_n)\) and \(Y(t_n^-)+\Delta N Y(t_n)\).

Using assumption 1(vii) of the theorem combined with (2.27) and (2.28) we get

\[
\phi(y) + \sum_{j=1}^{m} E^y \left[ \{ M \phi(\tilde{Y}^{(w)}(\tau_j^-)) - \phi(\tilde{Y}^{(w)}(\tau_j^-)) \} \chi(\tau_j < \tau_S) \right]
\]

\[
\geq E^y \left[ \phi(\tilde{Y}^{(w)}(\tau_{m+1}^-)) - \int_0^{\tau_{m+1}} \mathcal{L} \phi(Y^{(w)}(t))dt 
- \int_0^{\tau_{m+1}} \sum_{i=1}^{k} \frac{\partial \phi}{\partial y_i}(Y^{(w)}(t^-)) \sum_{e=1}^{p} \kappa_{ie}(Y^{(w)}(t^-))d\psi_{e}(t) - \sum_{0 < t_n < \tau_{j+1}} \Delta \psi \phi(Y^{(w)}(t_n)) + 
+ \sum_{i=1}^{k} K(Y^{(w)}(\tau_j^-), \xi_j) \right] \geq E^y \left[ \int_0^{\tau_{m+1}} f(Y^{(w)}(t), u(t))dt + \phi(\tilde{Y}^{(w)}(\tau_{m+1}^-)) + 
+ \sum_{e=1}^{p} \int_0^{\tau_{m+1}} \theta_{e}(Y^{(w)}(t))d\psi_{e}(t) + \sum_{i=1}^{k} K(Y^{(w)}(\tau_j^-), \xi_j) \right]
\]

(2.29)

Considering \(M\) as defined in (2.8) and letting \(m \rightarrow M\), we have

\[
\phi(y) \geq E^y \left[ \int_0^{\tau_S} f(Y^{(w)}(t), u(t))dt + g(Y^{(w)}(\tau_S))\chi(\tau_S < \infty) + 
+ \int_0^{\tau_S} \theta(Y^{(w)}(t))d\psi_{1}(t) + \sum_{i=1}^{k} K(Y^{(w)}(\tau_j^-), \xi_j) \right] = J^w(y) \text{ for all } y \in S
\]

(2.30)

If we assume that conditions 2(i) – (vi) hold, and apply the above reasoning to
\[ w = (v, \xi), \] then we get the following equalities, from (2.29) and (2.30), respectively,

\[
\phi(y) + \sum_{j=1}^{m} E_y \left[ \{ M \phi(\check{Y}^w(\tau_j^-)) - \phi(\check{Y}^w(\tau_j^-)) \} \chi_{\{\tau_j < \tau_S\}} \right] = E_y \left[ \phi(\check{Y}(\tau_{m+1}^-)) \right] \\
- \int_0^{\tau_{m+1}} L\phi(Y^w(t)) dt - \int_0^{\tau_{m+1}} \sum_{i=1}^{k} \frac{\partial \phi}{\partial y_i}(\check{Y}(t^-)) \sum_{i=1}^{p} \kappa_i \psi(\check{Y}(t^-)) d\psi_e(t) \\
- \sum_{0 < t_n < \tau_{m+1}} \Delta_t \phi(Y(t_n)) + \sum_{i=1}^{k} K(\check{Y}^w(\tau_j^-), \xi_j) \\
= E_y \left[ \int_0^{\tau_{m+1}} f(Y^w(t), \check{u}(t)) dt + \phi(\check{Y}^w(\tau_{m+1}^-)) \right] \\
+ \sum_{e=1}^{p} \int_0^{\tau_{m+1}} \theta_e(Y^w(t)) d\psi_e(t) + \sum_{i=1}^{k} K(\check{Y}^w(\tau_j^-), \xi_j) \right]
\]

(2.31)

\[
\phi(y) = E_y \left[ \int_0^{\tau_S} f(Y^{\hat{w}}(t), \check{u}(t)) dt + g(Y^{\hat{w}}(\tau_S)) \chi_{\{\tau_S < \infty\}} \right] + \\
+ \int_0^{\tau_S} \theta(Y^{\hat{w}}(t)) d\psi_e(t) + \sum_{i=1}^{k} K(\check{Y}^{\hat{w}}(\tau_j^-), \xi_j) \right] = J^{\hat{w}}(y) \quad \text{for all } y \in \mathcal{S}.
\]

(2.32)

Consequently, we obtain

\[
\phi(y) = \Phi(y) = \sup \{ J^w(y); w \in W \} = J^{\hat{w}}(y).
\]

(2.33)

This completes the proof of the theorem.

2.5 Application to Research Problem

We apply Theorem 2.1 to solve the problem presented in Section 2.2. Here we separate cases.
Case 1: $0 < \alpha \leq 1$

This case is a jump diffusion extension of Example 3.1 in [1]. However, it is worthwhile to note that in [1] the diffusion version of the problem is formulated and solved using singular control theory only whereas in this work we use the more general combined singular and impulse control for Lévy processes.

It can easily be observed that in light of Theorem 2.1 we have

$$K = u = g = f = 0, \quad \theta = e^{-\rho s} x^\alpha, \quad \kappa(s, x) = -(1 + \delta),$$

$$\Gamma(s, x, \xi) = x - (1 + \delta) \xi - c, \quad \mathcal{S} = \{y = (y_1, y_2) := (s, x) \in [0, \infty) \times \mathbb{R}; \ x > 0\}, \ b(.) = \mu, \ \sigma(.) = \sigma \text{ and } \gamma(.) = \beta z.$$

If there are no interventions, the generator of the controlled process

$$Y(t) = \begin{bmatrix} s \\ X(t) \end{bmatrix}; \quad Y(0) = y = \begin{bmatrix} 0 \\ x \end{bmatrix}$$

coincides with the second order integro-partial differential operator, $\mathcal{L}$, given by

$$\mathcal{L}\phi(s, x) = \frac{\partial \phi}{\partial s} + \mu \frac{\partial \phi}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 \phi}{\partial x^2} + \int_{\mathbb{R}} \{\phi(s, x + \beta z) - \phi(s, x) - \beta z \frac{\partial \phi}{\partial x}\} \nu(dz).$$

We suggest a solution of the form

$$\phi(s, x) = e^{-\rho s} \varphi(x).$$

With this solution candidate we have

$$\mathcal{L}\phi(s, x) = e^{-\rho s} \mathcal{L}' \varphi(x),$$

where

$$\mathcal{L}' \varphi(x) = -\rho \varphi(x) + \mu \varphi'(x) + \frac{1}{2} \sigma^2 \varphi''(x) + \int_{\mathbb{R}} \{\varphi(x + \beta z) - \varphi(x) - \beta z \varphi'(x)\} \nu(dz).$$

In this example the intervention operator, $\mathcal{M}$, is given by

$$\mathcal{M} \varphi(x) = \sup_{\xi} \left\{ \varphi(x - (1 + \delta) \xi - c); \ \xi \in \mathcal{Z} \right\}$$

$$= \sup_{\xi} \left\{ \varphi(x - (1 + \delta) \xi - c); \ 0 \leq \xi \leq \frac{x - c}{1 + \delta} \right\}$$
and the continuation region is described as follows

\[
D = \left\{ y := (y_1, y_2) := (s, x) \in \mathcal{S}; \right\}
\max\{\mathcal{M}\phi(s, x) - \phi(s, x), \max_e\left\{ \sum_{i=1}^{2} \kappa_i(y) \frac{\partial \phi}{\partial y_i}(y) + \theta_e(y) \right\} \right\} \leq 0
\]

\[
= \left\{ (s, x) \in \mathcal{S}; \sup_{\xi} \left\{ \varphi(x - (1 + \delta)\xi - c); 0 \leq \xi \leq \frac{x - c}{1 + \delta} \right\} - \varphi(x) \leq 0 \right\}
\cap \left\{ (s, x) \in \mathcal{S}; -(1 + \delta)\varphi'(x) + x^\alpha < 0 \right\}.
\]

We propose that \( D \) be given by

\[
D = \left\{ (s, x) \in \mathcal{S}; 0 < x < x^* \right\}
\]

for some \( x^* > 0 \).

For all \( x, x + \beta z \in D \), condition 2(i) of Theorem 2.1 yields

\[-\rho\varphi(x) + \mu \varphi'(x) + \frac{1}{2} \sigma^2 \varphi''(x) + \int_{\mathbb{R}} \left\{ \varphi(x + \beta z) - \varphi(x) - \beta z \varphi'(x) \right\} \nu(dz) = 0.
\]

To solve this last equation we try

\[
\varphi(x) = e^{rx}
\]

for some \( r \in \mathbb{R} \).

Then \( r \) must solve

\[
h(r) := -\rho + \mu r + \frac{1}{2} \sigma^2 r^2 + \int_{\mathbb{R}} \left\{ e^{r\beta z} - 1 - r\beta z \right\} \nu(dz) = 0.
\]

Since \( h(0) = -\rho < 0 \) and \( \lim_{|r| \to \infty} h(r) = \infty \) we see that there exist two solutions \( r_1, r_2 \) of \( h(r) = 0 \) such that \( r_2 < 0 < r_1 \). Moreover, since \( e^{r\beta z} - 1 - r\beta z \geq 0 \) for all \( r, z \) we have \(|r_2| > r_1 \). With such a choice of \( r_1 \) and \( r_2 \) we try

\[
\varphi(x) = A_1 e^{r_1 x} + A_2 e^{r_2 x}
\]

where \( A_i \) (\( i = 1, 2 \)) is a constant.

We recall that \( \phi(s, x) = e^{-\rho^* \varphi(x)} \) is a value function and as such \( \varphi(0) = 0 \). This yields

\[
A_1 = A = -A_2 > 0.
\]

Thus

\[
\varphi(x) = A(e^{r_1 x} - e^{r_2 x}); \quad 0 < x < x^*.
\]

(2.34)
Outside $D$ we require that

$$-(1 + \delta)\varphi'(x) + x^\alpha = 0.$$  

From this last equation we get

$$\varphi(x) = \frac{x^{1+\alpha}}{(1 + \alpha)(1 + \delta)} + A_3, \quad \text{for } x \geq x^*$$  

(2.35)

where $A_3$ is an arbitrary constant. Combining (2.34) and (2.35) we get

$$\varphi(x) = \begin{cases} A(e^{r_1 x} - e^{r_2 x}); & 0 < x < x^* \\ \frac{x^{1+\alpha}}{(1 + \alpha)(1 + \delta)} + A_3, & \text{for } x \geq x^*. \end{cases}$$

To determine $A$, $A_3$ and $x^*$ we need three equations. Using the fact that $\varphi$ is continuous at $x^*$ we obtain

$$A(e^{r_1 x^*} - e^{r_2 x^*}) = \frac{(x^*)^{1+\alpha}}{(1 + \alpha)(1 + \delta)} + A_3.$$  

(2.36)

Also, since $\varphi \in C^1$ at $x = x^*$ then

$$A(r_1 e^{r_1 x^*} - r_2 e^{r_2 x^*}) = \frac{(x^*)^\alpha}{1 + \delta}.$$  

(2.37)

From $\varphi \in C^2$ at $x = x^*$ we infer that

$$A(r_1^2 e^{r_1 x^*} - r_2^2 e^{r_2 x^*}) = \frac{\alpha(x^*)^{\alpha-1}}{1 + \delta}.$$  

(2.38)

Using (2.37) and (2.38) we get

$$e^{(r_1-r_2)x^*} = \frac{r_2(r_2 x^* - \alpha)}{r_1(r_1 x^* - \alpha)}.$$  

(2.39)

Now, $x^*$ is determined by solving the equation

$$x^* = \frac{\alpha r_1 - \alpha r_2 e^{(r_2-r_1)x^*}}{r_1^2 - r_2^2 e^{(r_2-r_1)x^*}}.$$  

(2.40)

The value of $x^*$ obtained from this last equation is used to find $A$ and $A_3$ from equations (2.38) and (2.36) by substitution.

We now examine $\mathcal{M}\varphi$.

\begin{align*}
\mathcal{M}\varphi(x) &= \sup_{\xi} \left\{ \varphi(x - (1 + \delta)\xi - c); \ 0 \leq \xi \leq \frac{x-c}{1+\delta} \right\} \\
&= \sup_{\xi} \left\{ A(e^{r_1[x-(1+\delta)\xi-c]} - e^{r_2[x-(1+\delta)\xi-c]}); \ 0 \leq \xi \leq \frac{x-c}{1+\delta} \right\} \\
&\leq \sup_{\xi} \left\{ A(e^{r_1 x} - e^{r_2 x}.e^{-(1+\delta)\xi}.e^{-r_2 c}); \ 0 \leq \xi \leq \frac{x-c}{1+\delta} \right\} \\
&\leq A(e^{r_1 x} - e^{r_2 x}) \\
&= \varphi(x).
\end{align*}
This shows that the point of maximum, \( \hat{\xi}(x) \), is given by

\[ \hat{\xi}(x) = 0. \]

The above results are summarised in the following theorem

**Theorem 2.5.1** Let \( X(t) \) be given by (2.2)-(2.3) and \( J^{(w)}(s,x) \) be defined by (2.4). Assume that \( \rho \geq 0 \) and \( 0 < \alpha \leq 1 \). Then

\[
\Phi(s,x) := \sup_{w \in W} J^{(w)}(s,x) = \begin{cases} 
A e^{-\rho s} (e^{r_1 x} - e^{r_2 x}); & \text{if } 0 < x < x^* \\
A e^{-\rho s} \left( \frac{x^{1+\alpha}}{(1+\delta)(1+\alpha)} + A_3 \right), & \text{if } x \geq x^*.
\end{cases}
\]

where \( x^* \), \( A \) and \( A_3 \) are determined by solving equations (2.40), (2.37) and (2.36) simultaneously. The corresponding optimal combined singular and impulse optimal control \( \hat{w} = (\hat{\psi}, \hat{v}) \) is the following

- If \( x \leq x^* \) it is optimal to do nothing.
- If \( x > x^* \) it is optimal to take out an amount \( \xi = \frac{(x - x^* - c)^+}{1 + \delta} \).

**Proof**

In this proof we verify that the function \( \phi(s,x) \) given by

\[
\phi(s,x) := \sup_{w \in W} J^{(w)}(s,x) = \begin{cases} 
A e^{-\rho s} (e^{r_1 x} - e^{r_2 x}); & \text{if } 0 < x < x^* \\
A e^{-\rho s} \left( \frac{x^{1+\alpha}}{(1+\alpha)(1+\delta)} + A_3 \right), & \text{if } x \geq x^*.
\end{cases}
\]

where \( x^* \), \( A \) and \( A_3 \) are determined by (2.40), (2.37) and (2.36) , satisfies all the requirements of Theorem 2.1.

It is not difficult to observe that \( \phi(s,x) \) is continuous on \( \bar{S} \) and also differentiable on \( S \). Thus, condition 1(i) is satisfied.

We now show that \( \phi \geq \mathcal{M} \phi \) on \( S \). But for \( x \leq x^* \) we have
\[ \mathcal{M}_\varphi(x) = \sup_{\xi} \{ \varphi(x - (1 + \delta)\xi - c); \ 0 \leq \xi \leq \frac{x - c}{1 + \delta} \} \]

\[ = \sup_{\xi} \{ A(e^{r_1[x-(1+\delta)\xi-c]} - e^{r_2[x-(1+\delta)\xi-c]}); \ 0 \leq \xi \leq \frac{x - c}{1 + \delta} \} \]

\[ \leq \sup_{\xi} \{ A(e^{r_1x} - e^{r_2x}e^{-r_2(1+\delta)\xi}e^{-r_2c}); \ 0 \leq \xi \leq \frac{x - c}{1 + \delta} \} \]

\[ \leq A(e^{r_1x} - e^{r_2x}). \]

\[ = \varphi(x). \]

For \( x \geq x^* \) we note that

\[ \mathcal{M}_\varphi(x) = \sup_{\xi} \{ \varphi(x - (1 + \delta)\xi - c); \ 0 \leq \xi \leq \frac{x - c}{1 + \delta} \} \]

\[ = \sup_{\xi} \{ \frac{x - (1 + \delta)\xi - c}{(1 + \alpha)(1 + \delta)} + A_3 \ 0 \leq \xi \leq \frac{x - c}{1 + \delta} \} \]

\[ \leq \frac{x^{1+\alpha}}{(1 + \alpha)(1 + \delta)} + A_3 \]

\[ = \varphi(x). \]

Hence, \( \phi \geq \mathcal{M}_\phi \). Thus, condition 1(ii) is satisfied. To verify 1(iii) we just have to show that \( -(1 + \delta)\varphi'(x) + x^\alpha \leq 0 \) since \( e^{-\rho s} > 0 \) for all \( s \geq 0 \). For \( x \geq x^* \) this condition holds by construction of \( \varphi \).

Now, for \( x \in D \) we refer to Example 3.2 in [36].

This proves that condition 1(iii) is satisfied.

The process \( Y^w(t) \) spends no time on the boundary of \( D \), that is to say \( \chi_{\partial D}(Y^w(t)) = 0 \ a.e. \). Consequently,

\[ E_y \left[ \int_0^{T^S} \chi_{\partial D}(Y^w(t))dt \right] = 0 \quad \text{for all } y \in S, \ w \in W \]

Thus, 1(iv) is satisfied.

We note that \( \partial D = \{ x^* \} \). So \( \partial D = \{ x^* \} \) is the graph of the Lipschitz function \( p(x) := x^* \). This verifies condition 1(v).

To prove condition 1(vi), let us consider the intervals

\[ I_1 := (0, \delta), \ I_2 := (x^* - \delta, x^*) \text{ and } I_3 := (x^*, x^* + \delta) \]
where $\delta$ is an arbitrarily small positive number.

For $x \in I_1$ we have

$$0 \leq \varphi'(x) = A(r_1 e^{r_1 x} - r_2 e^{r_2 x}) < (r_1 e^{\delta r_1} - r_2),$$

and

$$-A(r_2^2 e^{\delta r_1} + r_2^2) \leq \varphi''(x) = A(r_2^2 e^{r_1 x} - r_2^2 e^{r_2 x}) < Ar_1^2 e^{\delta r_1}.$$

If $x \in I_2$ we have

$$0 \leq \varphi'(x) = A(r_1 e^{r_1 x} - r_2 e^{r_2 x}) < A(r_1 e^{r_1 x^*} - r_2),$$

whereas

$$-A(r_2^2 e^{r_2 x^*} + r_2^2) \leq \varphi''(x) = A((r_1^2 e^{r_1 x} - r_2^2 e^{r_2 x}) < Ar_1^2 e^{r_1 x^*}.$$

Finally, taking $x \in I_3$ then

$$0 \leq \varphi'(x) = A(r_1 e^{r_1 x} - r_2 e^{r_2 x}) < (r_1 e^{r_1 (x^* + \delta)} - r_2),$$

and

$$-A(r_2^2 e^{r_2 (x^* + \delta)} + r_2^2) \leq \varphi''(x) = (r_1^2 e^{r_1 x} - r_2^2 e^{r_2 x}) < Ar_1^2 e^{r_1 (x^* + \delta)}.$$

Using these results we can conclude that $\varphi$ has locally bounded first and second order derivatives near $\partial D$, and so condition 1(vi) is verified. The rest of the conditions of Theorem 2.1 hold by construction of $\varphi$.■

**Conclusion**

Since $\phi(s, x) = e^{-\rho s} \varphi(x)$ satisfies all the conditions of Theorem 2.1 we conclude that

$$\phi(s, x) = \Phi(s, x) = J^{w^*}(s, x).$$

We note that

$$X(\tau_j) = \dot{X}(\tau_j^-) - (1 + \delta)[\dot{X}(\tau_j^-) - x^*] - c$$

$$= \dot{X}(\tau_j^-) - (1 + \delta)\dot{X}(\tau_j^-) + (1 + \delta)x^* - c$$

$$= \dot{X}(\tau_j^-) - \dot{X}(\tau_j^-) + (1 + \delta)x^* - c$$

$$= -\delta \dot{X}(\tau_j^-) + (1 + \delta)x^* - c.$$
The requirement that \( X(\tau_j) \geq 0 \) yields
\[
x^* \geq \frac{c + \delta X(\tau^-_j)}{1 + \delta}.
\]
The optimal impulse control, \( \hat{v} \), is then given by:
\[
\begin{align*}
\hat{\tau}_0 & : = 0 \\
\hat{\tau}_{j+1} & : = \inf \left\{ \hat{\tau}_j : x^* \geq c + \delta X(\tau^-_j) \right\} \wedge \tau_S; \ j = 0, 1, 2, ...
\end{align*}
\]
and the optimal strategy is to wait until the time, \( \tau_j \), that the resources reach or exceed \( x^* \) and then take out an amount \( \xi_{\tau_j} \) given by
\[
\xi_{\tau_j} = \max\{x(\tau_j) - x^*, 0\}
\]

**Case 2:** \(-1 \leq \alpha < 0\)

Without loss of generality we consider \( \alpha = -\frac{1}{2} \). In this case the performance functional, \( J^w(s, x) \), is given by
\[
J^w(s, x) := E^s_x \left[ \int_s^T e^{-\rho t} (X(t^-))^{-\frac{1}{2}} d\psi(t) \right]. \tag{2.41}
\]
Just like in the previous case we observe that \( K = u = g = f = 0 \), \( \theta = e^{-\rho s - \frac{\alpha}{2}} \), \( \kappa(s, x) = -(1 + \delta) \), \( \Gamma(s, x, \xi_j) = x - (1 + \delta) \xi_j - c \)
\( S = \{(s, x); x > 0\} \), \( b(.) = \mu \), \( \sigma(.) = \sigma \) and \( \gamma(.) = \beta z \). It is worthwhile to note that \( \theta : \mathbb{R} \rightarrow \mathbb{R} \) is a non-increasing function and for that reason we apply some of the arguments in [1]. In our case the discussion takes transaction costs into account. Additionally, here we examine the problem as a combined singular and impulse control for jump diffusions, whereas in [1] it is treated as a purely singular control problem for the diffusion case.

Let us consider the "take-the-money-carefully-and-run" strategy, \( \dot{w} \). Such a strategy is described by \( \dot{\psi}(s^-) = 0 \) and
\[
\dot{\psi}(s) = \dot{\xi} = (1 - \delta) \dot{X}^w(s^-) - c. \tag{2.42}
\]
If \( \dot{X}^w(s^-) = x \), then
\[
\dot{X}^w(s) = x - (1 + \delta) \dot{\xi} - c = x - (1 + \delta)[(1 - \delta)x - c] = \delta^2 x + (1 + \delta)c. \tag{2.43}
\]
Using (2.41) we note that the performance functional for the "take-the-money-carefully-and-run" strategy is given by

$$J^\dagger(s, x) = E_s^x [e^{-\rho s} (\dot{X}^\dagger(s^-))^{-\frac{1}{2}} \dot{\xi}] = e^{-\rho s} x^{-\frac{1}{2}} [(1 - \delta) x - c]$$  \hspace{1cm} (2.44)

According to this strategy we also have $\dot{w}(\tau^-) = 0$ and

$$\dot{w}(\tau) := \dot{\xi} := \frac{1}{1 + \delta} \left( \dot{X}^\dagger(\tau^-) - c \right)$$  \hspace{1cm} (2.45)

for the stopping time

$$\tau' := \inf \{ t \geq s : \dot{X}^\dagger(t^-) \geq c \}$$

It follows that

$$J^\dagger(s, x) = E_s^x [e^{-\rho \tau'} (\dot{X}^\dagger(\tau^-))^{-\frac{1}{2}} \dot{\xi} \chi(\tau \leq \tau')]$$  \hspace{1cm} (2.46)

and on the set $\{ \dot{X}^\dagger(s^-) \geq c \}$ we have

$$J^\dagger(s, x) = e^{-\rho s} x^{-\frac{1}{2}} \frac{x - c}{1 + \delta}.$$  \hspace{1cm} (2.47)

Apparently, this strategy is not optimal simply because it does not take into account the fact that the marginal utility /profit increases as the financial reserves diminish. Consequently, we seek a kind of "chattering strategy", denoted by $\tilde{w}^{(m, \eta)} = \tilde{\psi}^{(m, \eta)}$ where $m$ is a fixed positive integer and $\eta > 0$. Under this strategy, we denote by $\tilde{\xi}_j$ the value of withdrawals taken out at stopping time $j$.

At stopping times $\tau_j$ given by

$$\tau_j = \left( s + \frac{j}{m} \eta \right) \wedge \tau : \ j = 1, 2, ..., m$$  \hspace{1cm} (2.48)

an amount of resources $\Delta \tilde{\psi}(\tau_j)$ given by

$$\Delta \tilde{\psi}(\tau_j) := \tilde{\xi}_j = \frac{1}{m} x$$

is taken out. Considering $m$ withdrawals of this nature the expected total value of harvested resources is given by

$$J_0^{\tilde{w}^{(m, \eta)}}(s, x) := E_s^x \left[ \sum_{j=1}^{m} e^{-\rho \tau_j} \left[ (\dot{X}^\dagger(\tau_j^-))^+ \right]^{-\frac{1}{2}} \tilde{\xi}_j \right]$$  \hspace{1cm} (2.49)
Taking transaction costs into consideration we may present this as

\[
J(\tilde{\omega}) (s, x) = E^s, x \left[ \sum_{j=1}^{m} e^{-\rho \tau_j} [(x - (1 + \delta) \xi_j - c)^+]^{-\frac{1}{2}} \xi_j \right]
\]  

Letting \( \eta \to 0 \) we realise that \( \tau_j \to s \) for \( j = 1, 2, ..., m \) and we get

\[
J(\tilde{\omega}(m, 0)) (s, x) := \lim_{\eta \to 0} J(\tilde{\omega}(m, \eta)) (s, x)
\]

\[
= \lim_{\eta \to 0} E^s, x \left[ \sum_{j=1}^{m} e^{-\rho \tau_j} [(x - \frac{j}{m} (1 + \delta) x - c)^+]^{-\frac{1}{2}} \frac{x}{m} \right]
\]

\[
= e^{-\rho s} \sum_{j=1}^{m} h(x_j) \Delta x_j.
\]

where \( h(y) = [(x - (1 + \delta) y - c)^+]^{-\frac{1}{2}} \), \( x_j = \frac{jx}{m} \) and \( \Delta x_j = x_{j+1} - x_j = \frac{x}{m} \).

Given \( \epsilon > 0 \) there exists a positive integer \( m \) such that

\[
e^{-\rho s} \left| \int_0^x [(x - (1 + \delta) y - c)^+]^{-\frac{1}{2}} dy - \sum_{j=1}^{m} h(x_j) \Delta x_j \right| < \epsilon.
\]  

(2.51)

By making an appropriate choice of \( m \) and \( \eta \) we obtain the following

\[
| J(\tilde{\omega}(m, \eta)) (s, x) - e^{-\rho s} \int_0^x [(x - (1 + \delta) y - c)^+]^{-\frac{1}{2}} dy | < \epsilon.
\]  

(2.52)

We conclude that

\[
\lim_{\eta \to 0, m \to \infty} J(\tilde{\omega}) (s, x) = e^{-\rho s} \int_0^x [(x - (1 + \delta) y - c)^+]^{-\frac{1}{2}} dy \frac{2 e^{-\rho s}}{1 + \delta} \sqrt{x - c}.
\]  

(2.53)

We call this "chattering policy" of applying \( \tilde{\omega}(m, \eta) \) in the limit as \( \eta \to 0 \) and \( m \to \infty \) the policy of immediate chattering down to 0. Let us now investigate whether the function

\[
\phi(s, x) := \frac{2 e^{-\rho s}}{1 + \delta} \sqrt{x - c}.
\]

satisfies the conditions of Theorem 2.1.

Condition 1(i) holds since the function

\[
\phi(s, x) := \frac{2 e^{-\rho s}}{1 + \delta} \sqrt{x - c}
\]  

52
is differentiable on \( S \) and continuous on the closure of \( S \) whenever \( x - c > 0 \). To investigate condition 1(ii) we observe that

\[
M \phi = \sup_{\xi} \left\{ \phi(\Gamma(s, x, \xi)) : 0 \leq \xi \leq \frac{x - c}{1 + \delta} \right\}
\]

\[
= \frac{2e^{-\rho s}}{1 + \delta} \sup_{\xi} \left\{ \sqrt{x - (1 + \delta)\xi - c} : 0 \leq \xi \leq \frac{x - c}{1 + \delta} \right\}
\]

\[
\leq \frac{2e^{-\rho s}}{1 + \delta} \sqrt{x - c}
\]

\[
= \phi(s, x,).\]

Hence, \( \phi(s, x, \) satisfies condition 1(ii).

To find out whether \( \phi(s, x, \) satisfies condition 1(iii) we proceed as follows

\[
\sum_{i=1}^{k} \kappa_{i}(y) \frac{\partial \phi}{\partial y_i}(y) + \theta_{i}(y) = -(1 + \delta) \frac{2e^{-\rho s}}{1 + \delta} \cdot \frac{d}{dx} \left[ (x - c)^{\frac{1}{2}} \right] + e^{-\rho s}x^{-\frac{1}{2}}
\]

\[
= -e^{-\rho s} \left[ -(x - c)^{-\frac{1}{2}} + x^{-\frac{1}{2}} \right]
\]

\[
\leq e^{-\rho s} \left[ -\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x}} \right]
\]

\[
= 0.
\]

This proves that \( \phi(s, x, \) satisfies condition 1(iii).

Using the second-order integro-partial-differential operator

\[
L \phi(s, x) = \frac{\partial \phi}{\partial s} + \mu \frac{\partial \phi}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 \phi}{\partial x^2}
\]

\[
+ \int_{\mathbb{R}} \{ \phi(s, x + \beta z) - \phi(s, x) - \beta z \frac{\partial \phi}{\partial x} \} \nu(dz),
\]

we obtain

\[
L \phi(s, x) = e^{-\rho s} \frac{1}{1 + \delta} \left[ -2\rho(x - c)^{\frac{1}{2}} + \mu(x - c)^{-\frac{1}{2}} - \frac{1}{4} \sigma^2(x - c)^{-\frac{3}{2}}
\]

\[
+ \int_{\mathbb{R}} \{ 2\sqrt{x + \beta z - c} - 2(x - c)^{\frac{1}{2}} - \beta z(x - c)^{-\frac{1}{2}} \} \nu(dz) \right]
\]

\[
\leq e^{-\rho s} \frac{1}{1 + \delta} \left[ -2\rho(x - c)^{\frac{1}{2}} + \mu(x - c)^{-\frac{1}{2}} - \frac{1}{4} \sigma^2(x - c)^{-\frac{3}{2}}
\]

\[
+ \int_{\mathbb{R}} \{ 2\sqrt{(x - c)} - 2(x - c)^{\frac{1}{2}} - \beta z(x - c)^{-\frac{1}{2}} \} \nu(dz) \right]
\]

\[
= e^{-\rho s} \frac{1}{1 + \delta} \left[ -2\rho(x - c)^{\frac{1}{2}} + \mu(x - c)^{-\frac{1}{2}} - \frac{1}{4} \sigma^2(x - c)^{-\frac{3}{2}}
\]

\[
- \int_{\mathbb{R}} \beta z(x - c)^{-\frac{1}{2}} \nu(dz) \right].
\]
We have applied the fact that $\beta z \leq 0$. Thus

$$L \phi(s, x) \leq \frac{-2\rho e^{-\rho s}}{1 + \delta} (x - c)^{-\frac{3}{2}} [(x - c)^2 - \frac{\mu}{2\rho} (x - c) + \frac{\sigma^2}{8\rho} + (x - c) \int_{\mathbb{R}} \beta z \nu(dz)]$$

$$= \frac{-2\rho e^{-\rho s}}{1 + \delta} (x - c)^{-\frac{3}{2}} [(x - c)^2 + \left( \int_{\mathbb{R}} \beta z \nu(dz) - \frac{\mu}{2\rho} \right) (x - c) + \frac{\sigma^2}{8\rho}] .$$

So, condition 1(vii) holds if $x \geq c$ and

$$\left( \int_{\mathbb{R}} \beta z \nu(dz) - \mu \right)^2 \leq 2 \rho \sigma^2 .$$

We now state the following result:

**Theorem 2.2** Let $X^{(w)}(t)$ be given by (2.2)-(2.3).

1. Assume that $x \geq c$ and

$$\left( \int_{\mathbb{R}} \beta z \nu(dz) - \mu \right)^2 \leq 2 \rho \sigma^2 . \quad (2.54)$$

Then

$$\Phi(s, x) := \frac{2e^{-\rho s}}{1 + \delta} \sqrt{x - c} . \quad (2.55)$$

where $\sigma$ and $\rho$ are defined as before. This value is achieved in the limit if we apply the strategy $\tilde{w}^{(m, \eta)}$ described above with $\eta \to 0$ and $m \to \infty$, that is, by applying the policy of immediate chattering to 0.

2. If

$$\left( \int_{\mathbb{R}} \beta z \nu(dz) - \mu \right)^2 > 2 \rho \sigma^2 . \quad (2.56)$$

then the value function has the form

$$\Phi(s, x) = \begin{cases} 
\frac{e^{-\rho s} A(e^{r_1 x} - e^{r_2 x})}{1 + \delta} & \text{for} \ 0 \leq x < x^* \\
\frac{e^{-\rho s} (2 \sqrt{x - c} - \frac{2}{1+\delta} \sqrt{x^* - c} + B)}{1+\delta} & \text{for} \ x^* \leq x 
\end{cases} \quad (2.57)$$

for some constants $A > 0$, $B > 0$ and $x^* > 0$ where $r_1$ and $r_2$ are the solutions of the equation

54
\[-\rho + \mu r + \frac{1}{2} \sigma^2 r^2 + \int_{\mathbb{R}} \{e^{r\beta z} - 1 - r\beta z\} \nu(dz) = 0. \quad (2.58)\]

with \( r_2 < 0 < r_1 \) and \( |r_2| > r_1 \).

In both cases 1. and 2. the corresponding optimal policy is the following:

- If \( x > x^* \) it is optimal to apply immediate chattering from \( x \) down to \( x^* \). The policy of immediate chattering is as defined at the beginning of the discussion of this Case 2.

- If \( 0 < x < x^* \) it is optimal to apply the harvesting equal to the local time of the downward reflected process \( \bar{X}(t) \) at \( x^* \).

Proof

We need to show that the proposed value function satisfies all the conditions of Theorem 2.1. Let us first examine the case

\[
\left( \int_{\mathbb{R}} \beta z \nu(dz) - \frac{\mu}{2\rho} \right)^2 \leq \frac{\sigma^2}{2\rho}.
\]

In this case we have

\[
\phi(s, x) := \frac{2e^{-\rho s}}{1 + \delta} \sqrt{x - c}.
\]

From the construction of \( \phi(s, x) \) we can state that conditions 1(i) – (iii) and 1(vii) are satisfied.

Since \( X(t) \) spends no time on \( \partial D \), then \( \chi_{\partial D} X(t) = 0 \) a.e and this leads to

\[
E^y \left[ \int_0^T \chi_{\partial D}(X(t)) dt \right] = 0 \quad \text{for all} \ y \in \mathcal{S}, \ v \in \mathcal{V}.
\]

So, condition 1(iv) is satisfied.

In this example the boundary, \( \partial D \), of the non-intervention region, \( D \), is given by

\[
\partial D = \partial D_1 \cup \partial D_2
\]

where \( \partial D_1 = \{0\} \) and \( \partial D_2 = \{x^*\} \). But \( \partial D_1 \) and \( \partial D_2 \) are both Lipschitz surfaces since each of them is a singleton which consists of a constant. Hence, \( \partial D \) is also a Lipschitz surface. Thus \( \phi(s, x) \) satisfies condition 1(v).
For $x > c$ it can easily be verified that the function

$$\phi(s, x) := \frac{2e^{-\rho s}}{1 + \delta} \sqrt{x - c}.$$ 

is twice continuously differentiable on $S \setminus \partial D$ and none of its derivatives explodes near $\partial D$. This establishes the requirements of 1(vi).

Condition 1(vii) is satisfied since

$$\lim_{t \to \tau} \phi(t, x) = \lim_{t \to \tau} \frac{2e^{-\rho t}}{1 + \delta} \sqrt{x - c} = 0 \text{ as } \tau \to \infty.$$ 

This proves that 1(vii) is satisfied.

The remaining conditions in part 1. of Theorem 2.1 can also be verified without much difficulty.

Up to this point we have proved that

$$\phi(s, x) \geq \Phi(s, x).$$

Now, by construction of $\phi(s, x)$ we observe that

$$\mathcal{L} \hat{\phi}(y) + f(y, \hat{\phi}(y)) = 0 \quad \text{for all } y \in D.$$ 

That is to say, 2(i) is satisfied.

Conditions 2(ii) – 2(vii) can easily be verified from the construction of the function.

So, in this case

$$\phi(s, x) := \frac{2e^{-\rho s}}{1 + \delta} \sqrt{x - c}.$$ 

satisfies all the requirements of Theorem 2.1. Hence it is a value function for the given problem.

We now treat the case

$$\left( \int_{\mathbb{R}} \beta z\nu(dz) - \frac{\mu}{2\rho} \right)^2 > \frac{\sigma^2}{2\rho}.$$ 

56
For this case we need to show that the function \( \phi(s,x) \) given by

\[
\phi(s,x) = \begin{cases} 
  e^{-\rho s} A(e^{r_1 x} - e^{r_2 x}); & \text{for } 0 \leq x < x^* \\
  e^{-\rho s} \left( \frac{2}{1+\delta} \sqrt{x-c} - \frac{2}{1+\delta} \sqrt{x^*-c} + B \right) & \text{for } x^* \leq x
\end{cases}
\]

(2.59)

also satisfies conditions of Theorem 2.1 where \( A, B, x^*, r_1, r_2 \) are as specified in Theorem 2.2.

Here we follow closely arguments used to prove part (b) of Theorem 3.2 in [1], where we effect the necessary extension arguments to cater for the jump component as well as transaction costs. First, we observe that if we apply the policy of immediate chattering from \( x \) to \( x^* \) where \( 0 < x^* < x \), then the value of the dividends paid out is given by

\[
e^{-\rho s} \int_{0}^{x-x^*} [(x - (1 + \delta)y - c)^+]^{-\frac{1}{2}} dy = \frac{2e^{-\rho s}}{1+\delta} \left[ \sqrt{x-c} - \sqrt{(1+\delta)x^*-\delta x - c} \right].
\]

This follows by the argument (2.44)-(2.51) presented above. To verify the conclusions of part 2 of Theorem 2.2 we observe that \( r_1 \) and \( r_2 \) are the roots of the equation

\[-\rho + \mu r + \frac{1}{2} \sigma^2 r^2 + \int_{\mathbb{R}} \{e^{r\beta z} - 1 - r\beta z\} \nu(dz) = 0.\]

Hence, by defining \( \phi(s,x) \) as in (65) it is relatively easy to show that for \( x < x^* \)

\[\mathcal{L}\phi(s,x) = 0\]

(2.60)

and

\[\phi(s,0) = 0.\]

(2.61)

Combining the smooth contact principle and the requirement that \( \phi(s,x) \) be \( C^2 \) at \( x = x^* \), we obtain the following three equations

\[A(e^{r_1 x^*} - e^{r_2 x^*}) = B\]

(2.62)

\[A(r_1 e^{r_1 x^*} - r_2 e^{r_2 x^*}) = (x^*)^{-\frac{1}{2}}\]

(2.63)

\[A(r_1^2 e^{r_1 x^*} - r_2^2 e^{r_2 x^*}) = -\frac{1}{2} (x^*)^{-\frac{3}{2}}\]

(2.64)
Dividing (2.63) by (2.64) yields
\[
\frac{r_1e^{r_1x^*} - r_2e^{r_2x^*}}{r_1^2e^{r_1x^*} - r_2^2e^{r_2x^*}} = -2x^*
\tag{2.65}
\]
Now, observing that
\[
\lim_{x^* \to 0} \frac{r_1e^{r_1x^*} - r_2e^{r_2x^*}}{r_1^2e^{r_1x^*} - r_2^2e^{r_2x^*}} = \frac{1}{r_1 + r_2} < 0
\tag{2.66}
\]
and
\[
\lim_{x^* \to \infty} \frac{r_1e^{r_1x^*} - r_2e^{r_2x^*}}{r_1^2e^{r_1x^*} - r_2^2e^{r_2x^*}} = \frac{1}{r_1} > 0
\tag{2.67}
\]
the intermediate value theorem guarantees the existence of \(x^*\) satisfying equation (2.65). With this value of \(x^*\) we define \(A\) by (2.63) and \(B\) by (2.62) We have proved the existence of a solution of the system (2.62)-(2.64) where \(A > 0,\ B > 0,\ x^* > 0\). With this choice of \(A > 0,\ B > 0,\ x^* > 0\) the function \(\phi(s, x)\) becomes a \(C^2\) and we can easily verify that \(\phi\) satisfies conditions \(1(i) - (x)\) of Theorem 2.1. Hence,
\[
\phi(s, x) \geq \Phi(s, x) \quad \text{for all } s, x.
\tag{2.68}
\]
Moreover, the non-intervention region \(D\) is identified to be
\[
D = \{(s, x) : 0 < x < x^*\}.
\tag{2.69}
\]
Consequently, by (2.59) we know that condition 2(i) of Theorem 2.1 holds.

Additionally, it is an established fact that the local time \(\hat{\psi}\) of the downward reflected process \(\bar{X}(t)\) at \(x^*\) satisfies conditions 2(ii) - 2(vii) (see, for example [1] and [42], and references therein).

By Theorem 2.1 we conclude that if \(x \leq x^*\) then
\[
\psi^* := \hat{\psi}
\]
is optimal and
\[
\phi(s, x) = \Phi(s, x).
\]
Finally, if \(x > x^*\) then it follows from previous arguments that immediate chattering from \(x\) to \(x^*\) gives the value
\[
\Phi(s, x) \geq e^{-\rho s}[\sqrt{x}(1 - \delta) - cx^{-\frac{1}{2}}] + \Phi(s, x^*) \quad \text{for all } x > x^*
\tag{2.70}
\]
58
Combined with (2.68) this proves that
\[ \phi(s, x) = \Phi(s, x) \quad \text{for all } s, x \]  
(2.71)

and the proof of part 2 of Theorem 2.2 is complete.
Chapter 3

Combined Regular-Singular Control

3.1 Introduction

A more complex challenge for a typical insurance company is that of determining an optimal dividend policy in the presence of friction and simultaneously controlling the level of risk exposure. In such a case the firm is confronted with a mixed stochastic control problem of determining the optimal combined dividend and proportional reinsurance policy. Here, we investigate the problem using the combined regular and singular stochastic control theory.

The jump-diffusion case of the generalised combined regular and singular control problem is considered. As far as we know this problem has not been examined before.

Risk control action consists in reinsuring a proportion of the incoming claims the insurance firm is contractually obliged to pay. Such a practice requires the insurance company to divert a certain percentage of the premiums to the reinsurance company. For earlier work on combined risk and dividend control the reader is referred to [51] and [52]. Combined impulse and regular control is treated in [7]. The major purposes of reinsurance are:

(a) To protect against catastrophic events.
(b) To allow ceding company to assume individual risks greater than its size, and to protect the cedant against big losses.

(c) To help in making an insurance company’s results more predictable by absorbing larger losses and hence ensuring a smooth income.

(d) To reduce the amount of net liability, and increase surplus for the insurance

In this chapter the notion of control variables is extended to include the coefficients of the drift, volatility and jump components of a Levy process.

We now present the mathematical theoretical framework.

### 3.2 Problem Formulation

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) be a stochastic basis and \(B(t)\) be 1-dimensional standard Brownian motion with respect to \(\{\mathcal{F}_t\}\).

Assume that if there are no interventions a stochastic process \(\{X_t\}\) evolves according to the following equation

\[
dX_t = \eta(t, X_t) dt + \beta(t, X_t) dB_t + \int_{\mathbb{R}} h(t, X_{t-}, z) \tilde{N}(dt, dz); \quad X_0 = x > 0, \quad (3.1)
\]

where \(\eta, \beta\) and \(h\) are Borel measurable real valued functions. Denote by \(\mathcal{H}\) the set of all real valued bounded functions such that (3.1) has a unique strong solution. \(\tilde{N}(.,.)\) is a compensated Poisson random measure given by

\[
\tilde{N}(dt, dz) = N(dt, dz) - dt \nu(dz);
\]

where \(\nu(.)\) is a Levy measure associated with the Poisson random measure \(N(.,.)\).

Now, suppose that we consider harvesting from the system described by (3.1). Then the process \(\{X_t\}_{t \geq 0}\) has dynamics governed by

\[
dX_t = \eta(t, X_t) dt + \beta(t, X_t) dB_t + \int_{\mathbb{R}} h(t, X_{t-}, z) \tilde{N}(dt, dz) - (1 + \delta)dL_t; \quad X_0 = x > 0(3.2)
\]

where \(L_t\) represents the total amount of resources harvested from the system up to time \(t\) and \(0 \leq \delta \leq 1\). \(L_t\) is right continuous, nonnegative, \(\mathcal{F}_t\) adapted.
If $L = 0$, then the time-state process $(t, X(t))$ is a jump diffusion whose generator on $C^2_0(\mathbb{R}^2)$ coincides with the integro-differential operator $\mathcal{L}$ given by

\[
\mathcal{L} \phi(s, x) = \frac{\partial \phi}{\partial s}(s, x) + \eta(s, x) \frac{\partial \phi}{\partial x}(s, x) + \frac{1}{2} \beta^2(s, x) \frac{\partial^2 \phi}{\partial x^2} + \int_{\mathbb{R}} \{\phi(s, x + h(s, x, z)) - \phi(s, x) - h(s, x, z) \frac{\partial \phi}{\partial x}(s, x)\} \nu(dz).
\]

(3.3)

Define the terminal time $T$ by

\[
T = \inf\{t > s : (t, X_t) \notin S\} \leq \infty
\]

(3.4)

where $S$ is an open and connected subset of $\mathbb{R}^2$ and $T$ is the bankruptcy time.

Let $\mathcal{A}$ be the set of all mixed regular-singular controls of the form $(\eta, \beta, h, L)$ such that (3.2) has a unique strong solution. We call $\mathcal{A}$ the set of admissible controls. Define the performance functional, $J^{(\eta, \beta, h, L)}(s, x)$, by

\[
J^{(\eta, \beta, h, L)}(s, x) = \mathbb{E}^{s,x}[\int_s^T u(s + t, X(t)) dt + \int_s^T \pi(s + t, X(t)) dL_t]
\]

(3.5)

where $\mathbb{E}^{s,x}$ is the expectation with respect to the probability law $P$, given that $X(s) = x$.

In this case we impose the requirement that the utility rate $u : \mathbb{R}^2 \to \mathbb{R}$ be continuous, non-decreasing and concave.

It is also assumed that $u(x)$ satisfies the following integrability condition

\[
\mathbb{E}^{s,x}[\int_s^T |u(s + t, X(t))| dt] < \infty, \text{ for all } s, x \in S.
\]

(3.6)

The problem is to find the function $\Phi$ and the optimal mixed control $(\eta^*, \beta^*, h^*, L^*) \in \mathcal{A}$ such that

\[
\Phi(s, x) = \sup_{(\eta, \beta, h, L) \in \mathcal{A}} J^{(\eta, \beta, h, L)}(s, x) = J^{(\eta^*, \beta^*, h^*, L^*)}(s, x).
\]

(3.7)
3.3 Integro-Variational Inequalities

**Theorem 3.1** Assume that $\pi(t, \xi)$ is decreasing with respect to $\xi$ and for all $t$.

1. Suppose there exists a non-negative function $\phi \in C_0^2(S) \cap C(\bar{S})$ such that
   (i) $\pi(t, x) - (1 + \delta) \frac{\partial \phi}{\partial x} \leq 0$ for all $(t, x) \in S$.
   (ii) $L\phi(t, x) \leq 0$ for all $(t, x) \in S$ and for all controls $\eta, \beta, h \in \mathcal{H}$.
   (iii) $E_s,x \left[ \int_0^T \left( |\beta(t, X(t)) \nabla \phi(t, X(t))|^2 + \int_\mathbb{R} |\phi(t, x + h) - \phi(t, x)|^2 \nu(dz) \right) dt \right] < \infty$
   (iv) $\{\phi(t, X(t))\}_{t \leq T}$ is uniformly integrable for all $(t, x) \in S$.

Then
\[ \phi(s, x) \geq \Phi(s, x) \text{ for all } (s, x) \in S. \quad (3.8) \]

2. Define the non-intervention region $D$ by
\[ D = \left\{ (t, x) \in S : \max \{ \pi(t, x) - (1 + \delta) \frac{\partial \phi}{\partial x}, L\phi(t, x) \} \leq 0 \right\}. \quad (3.9) \]

Suppose also that for all $(t, x) \in \bar{D}$
   (v) $L\phi(t, x) = 0$
   Moreover, suppose there exists a mixed control $\tilde{v} := (\tilde{\eta}, \tilde{\beta}, \tilde{h}, \tilde{L}) \in \mathcal{A}$ such that
   (vi) $(t, X^{\tilde{v}}(t)) \in \bar{D}$ for all $t > s$
   (vii) $\left( \frac{\partial \phi}{\partial x}(t, X^{\tilde{v}}(t)) - \pi(t, x) \right) d\tilde{L}^{(\cdot)}(t) = 0$ where $i = 1, 2, \ldots, n$
   (viii) $\triangle \phi(t_j, X^{\tilde{v}}(t_j)) := \phi(t_j, X^{\tilde{v}}(t_j)) - \phi(t_j, X^{\tilde{v}}(t_j^-))$ at all jumping times $t_j \geq s$; and
   (ix) $\lim_{R \to \infty} E^{s,x}[\phi(T_R, X^{\tilde{v}}(T_R^-))] = 0$

where
\[ T_R = T \wedge R \wedge \inf \{ t > s : |X^v(t)| \geq R \} \text{ for } R > 0. \]
Then
\[\phi(s, x) = \Phi(s, x) \quad \text{for all} \quad (s, x) \in S\] (3.10)

and
\[v^* := \hat{v} \text{ is an optimal combined regular-singular control.}\]

**Proof**

Using the generalised Itô’s formula for semimartingales and (3.2) the following is obtained

\[
\begin{align*}
\phi(s + T_R, X(T_R)) - \phi(s, X(0)) &= \int_0^{T_R} \left( \frac{\partial\phi}{\partial s}(s, x) + \eta(s, x) \frac{\partial\phi}{\partial x}(s, x) + \frac{1}{2} \beta^2(s, x) \frac{\partial^2\phi}{\partial x^2} \right. \\
&\left. + \int_\mathbb{R} \left\{ \phi(s, x + h(s, x, z)) - \phi(s, x) - h(s, x, z) \frac{\partial\phi}{\partial x}(s, x) \right\} v(dz) \right) dt \\
&\quad + \int_0^{T_R} \beta(s, x) \frac{\partial\phi}{\partial x} dB(t) - (1 + \delta) \int_0^{T_R} \frac{\partial\phi}{\partial x} dL(t) \\
&\quad + \int_0^{T_R} \int_\mathbb{R} \left\{ \phi(s, x + h(s, x, z)) - \phi(s, x) \right\} \tilde{N}(ds, dz) \\
&\quad + \sum_{0 < t_j < T_R} \left\{ \phi(t_j, X_t) - \phi(t_j, X_t^-) - \Delta X_{t_j}(s+, x, z) \frac{\partial\phi}{\partial x}(s, x) \right\}
\end{align*}
\] (3.11)

where the sum is taken over all the jumping times \( t_j \in [0, T_R] \) and

\[\Delta X_{t_j} = X_t - X_{t_j^-}\]

represents jumps at \( t_j \), and similarly for \( \Delta \phi \). Applying Dynkin’s formula to (3.11)

\[
\begin{align*}
E^{s,x}[\phi(s + T_R, X(T_R))] &= \phi(s, x) + E^{s,x}\left[ \int_0^{T_R} \mathcal{L}\phi(t, x) dt - \int_0^{T_R} u dt - \int_0^{T_R} \frac{\partial\phi}{\partial x} dL(t) \right] \\
&\quad + E^{s,x}\left[ \sum_{0 < t_j < T_R} \left\{ \phi(t_j, X_{t_j}) - \phi(t_j, X_{t_j}^-) - \Delta X_{t_j}(s+, x, z) \frac{\partial\phi}{\partial x}(s, x) \right\} \right]
\end{align*}
\] (3.12)

From condition 1(ii) we obtain

\[
\begin{align*}
E^{s,x}[\phi(s + T_R, X(T_R))] &\leq \phi(s, x) - E^{s,x}\left[ \int_0^{T_R} u dt + \int_0^{T_R} \frac{\partial\phi}{\partial x} dL(t) \right] \\
&\quad + E^{s,x}\left[ \sum_{0 < t_j < T_R} \left\{ \phi(t_j, X_{t_j}) - \phi(t_j, X_{t_j}^-) - \Delta L(t_j) \frac{\partial\phi}{\partial x}(s, x) \right\} \right]
\end{align*}
\] (3.13)
where $\triangle L(t_j) = L(t_j) - L(t_{j-})$. Consequently,

$$E^{s,x}[\phi(s + T_R, X(T_R))] \leq \phi(s, x) - E^{s,x}\left[\int_0^{T_R} u dt + \int_0^{T_R} \frac{\partial \phi}{\partial x} dL(t)\right] + E^{s,x}\left[\sum_{0 < t_j < T_R} \{\phi(t_j, X_{t_j}) - \phi(t_j, X_{t_j-})\}\right]$$

(3.15)

where $L^c(t)$ is the continuous part of $L(t)$ defined by

$$L^c(t) = L(t) - \sum_{0 < t_j < t} \triangle L(t_j)$$

(3.16)

Applying the mean value theorem on the two terms under the summation in (3.15) we obtain

$$\triangle \phi(t_j, X_{t_j}) = -\frac{\partial \phi}{\partial x}(s + t_j, \hat{X}_{t_j}). \triangle L(t_j)$$

(3.17)

for some point $\hat{X}_{t_j}$ joining the two points $X_{t_{j-}}$ and $X_{t_j}$. From (3.17) and condition (i) of Theorem (3.1) the following result is obtained

$$\phi(s, x) \geq E^{s,x}\left[\int_0^{T_R} u dt + \int_0^{T_R} \pi(s + t) dL(t) + \phi(s + T_R, X^L(T_R))\right]$$

(3.18)

Letting $R \to \infty$ in (3.18) we obtain

$$\phi(s, x) \geq \lim_{R \to \infty} \sup E^{s,x}\left[\int_0^{T_R} u dt + \int_0^{T_R} \pi(s + t) dL(t)\right] \geq J^L(s, x)$$

(3.19)

This proves the requirement (3.8) for an arbitrarily chosen $L(t)$.

Now, using conditions (iv), (v) and (vi) and replacing $L(t)$ by $L^*(t)$ we get equality in (3.15), (3.18) and (3.19). It follows that

$$\phi(s, x) = \lim_{R \to \infty} E^{s,x}\left[\int_0^{T_R} u dt + \int_0^{T_R} \pi(s + t) dL(t)\right]$$

(3.20)

Requirement (3.10) is proved by combining (3.8) and (3.20). Thus

$v^*$ is an optimal combined regular singular control.

This completes the proof of Theorem 3.1.
3.4 Application

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a filtered probability space satisfying the usual conditions. \(\{B_t\}\) is a 1-dimensional standard Brownian motion with respect to \(\{\mathcal{F}_t\}\).

Assume that the management of an insurance company are allowed to control the risk exposure of the firm by applying proportional reinsurance policy. Let \(a(t)\) be the retention level, which is the fraction of all incoming claims that the firm will insure by itself. In addition, the company also applies a dividend policy. Suppose that if a dividend of magnitude \(\xi\) is paid out to share holders, a transaction cost \(\delta \xi\) is incurred by the company as they process the dividends. Here \(\delta \in (0, 1)\) is a constant of proportionality. So, if the total dividend paid out upto time \(t\) is \(L(t)\), it means the total transaction cost incurred is \(\delta L(t)\).

Now suppose that as a result of applying the two dimensional policy, \(c(t)\), given by

\[
c(t) = (a(t), L(t)),
\]

the liquid reserves, \(X(t)\) at time \(t\), of the insurance company evolve according to the following stochastic differential equation

\[
dX(t) = \mu a(t)dt + \sigma a(t)dB(t) + \gamma a(t) \int_{\mathbb{R}} z \tilde{N}(dt, dz) - (1 + \delta) dL(t), \quad X(0) = x > 0;
\]

\[(3.21)\]

where \(L(t)\) is the cumulative dividend paid out up to time \(t\), \(0 < \delta < 1\) is as explained above and \(\mu\) and \(\sigma\) are positive constants and \(\tilde{N}(., .) = N(dt, dz) - \nu(dz)dt\) is a compensated Poisson random measure with Lévy measure \(\nu\). It is also assumed that \(\gamma z < 0\).

Define the performance functional, \(J\), by

\[
J^c(x) := E\left[\int_0^T e^{-\rho t} X^\alpha(t) dL(t)\right];
\]

\[(3.22)\]

where \(\{X(t)\}\) is a solution of \((3.21)\), \(\tau\) is a stopping time given by

\[
\tau := T = \inf\{t : X(t) \leq 0\} \quad \text{(time to bankruptcy)},
\]
$\rho > 0$ is a discount factor, $-1 < \alpha \leq 0$ and $E[.]$ denotes expectation with respect to probability law $P$.

**Problem 3.4.1** The problem is to find the optimal combined control $c^* := (a^*(t), L^*)$ and the function $V(x)$ such that

$$V(x) = \sup_{c \in \mathcal{A}} J^c(x) = J^{c^*}(x)$$

(3.23)

where $\mathcal{A}$ is the space of all admissible policies $c(t)$.

**Solution**

We apply Theorem 3.1 to solve the problem.

In view of the problem represented by (3.7) we observe that

$$\eta(t, X_t) = \mu a(t), \quad \beta(t, X_t) = \sigma a(t), \quad h(t, X_{t-}, z) = \gamma a(t) z$$

where $\mu, \sigma, \gamma$ are given constants and $0 \leq a(t) \leq 1$.

We state the following result without proof.

**Lemma 3.4.1** The function $V$ defined by (3.23) is non-negative and concave.

A similar lemma is proved in [29].

Our next task is to find the function $V$ and the optimal control $c^*$.

**Lemma 3.4.2** If $dL(t)=0$ the function $V$ satisfies the following Hamilton-Jacobi-Bellman equation

$$\max_{a \in [0,1]} \left[ \frac{\partial V}{\partial s}(s, x) + \mu a (s, x) + \frac{1}{2} \sigma^2 a^2 \frac{\partial^2 V}{\partial x^2}(s, x) \right] + \int_{\mathbb{R}} \{ V(s, x + \gamma a z) - V(s, x) - \gamma a z \frac{\partial V}{\partial x}(s, x) \} \nu(d z) = 0$$

68
with initial condition
\[ V(0) = 0 \] (3.25)

For proof of a similar result see [29] We propose a candidate value function of the form
\[ V(s, x) = e^{-\rho s} f(x). \]

Then, (3.24) can be expressed as
\[
\max_{a \in [0,1]} \left[ -\rho f(x) + \mu a f'(x) + \frac{1}{2} \sigma^2 a^2 f''(x) + \int_{\mathbb{R}} \{ f(x + \gamma az) - f(x) - \gamma az f'(x) \} \nu(dz) \right] = 0
\]
(3.26)

Using (3.26) we observe that the first order condition for the optimal control \( a := \hat{a} \) is given by
\[
\mu f'(x) + \sigma^2 \hat{a} f''(x) + \int_{\mathbb{R}} \{ f'(x + \gamma \hat{a} z) - f'(x) \} \gamma z \nu(dz) = 0
\]
(3.27)

Now let us choose \( f(x) = e^{rx} \) for some constant \( r \in \mathbb{R} \). Putting \( a = \hat{a} \) and \( V(s, x) = e^{-\rho s} e^{rx} \) into (3.24) we get
\[
g(r) := -\rho + \mu \hat{a} r + \frac{1}{2} \sigma^2 \hat{a}^2 r^2 + \int_{\mathbb{R}} \{ e^{\gamma \hat{a} z} - 1 - r \gamma \hat{a} z \} \nu(dz) = 0.
\]
(3.28)

Since \( g(0) = -\rho < 0 \) and
\[
\lim_{r \to -\infty} g(r) = \lim_{r \to -\infty} g(r) = \infty,
\]
we observe that the equation \( g(r) = 0 \) has two distinct real solutions \( r_1 = r_1(a), r_2 = r_2(a) \) such that
\[
r_2(a) < 0 < r_1(a)
\]
The solution to (3.24)-(3.25) is
\[ C(e^{r_{1}x} - e^{r_{2}x}). \] (3.29)
where \( C \) is an arbitrary constant. Now, setting
\[ f(x) = C(e^{r_{1}(a)x} - e^{r_{2}(a)x}) \] (3.30)
in (3.27) we conclude that the optimal proportional reinsurance policy \( a := \hat{a} \) is the solution of equation (3.31)
\[
\begin{align*}
&\mu[r_{1}(\hat{a})e^{r_{1}(\hat{a})x} - r_{2}(\hat{a})e^{r_{2}(\hat{a})x}] + \sigma^{2}\hat{a}[r_{1}^{2}(\hat{a})e^{r_{1}(\hat{a})x} - r_{2}^{2}(\hat{a})e^{r_{2}(\hat{a})x}] \\
&+ \int_{\mathbb{R}} \{r_{1}(\hat{a})e^{r_{1}(\hat{a})(x+\gamma z)} - r_{2}(\hat{a})e^{r_{2}(\hat{a})(x+\gamma z)} \} \gamma z \nu(dz) = 0.
\end{align*}
\] (3.31)
We now turn to the dividend control. Here we proceed in a similar fashion as in Chapter 2.
Considering \( \alpha = -\frac{1}{2} \) the performance functional is given by
\[
J^{(u)}(s,x) := E\left[ \int_{s}^{T} e^{-\rho t}(X^{(u)}(t))^{-\frac{1}{2}} dL(t) \right].
\]
Again we note that if we apply the "take-the-money-carefully-and-run" strategy, \( \dot{L} \), then all the resources are taken out immediately, and no reinsurance policy is applied, that is \( a^{*}(t) = 0 \). Such a strategy is described by
\[
\dot{L}(x) = \frac{x}{1 + \delta}.
\] (3.32)
The value function obtained from this strategy is
\[
\Phi(s,x) = e^{-\rho s}x^{-\frac{1}{2}}\frac{x}{1 + \delta} = e^{-\rho s} \frac{\sqrt{x}}{1 + \delta}; \quad x > 0.
\] (3.33)
This strategy is not likely to be optimal since it does not take into account the increase in utility as the financial reserves diminish. The strategy also does not consider the benefits that accrue from reinsuring a proportion of the incoming claims.
Proceeding as before, we seek a kind of "chattering strategy", similar to the one considered in Chapter 2 and we obtain

$$\phi(s, x) := \frac{2e^{-\rho s}}{1 + \delta} \sqrt{x}. \quad (3.34)$$

as a candidate value function.

It is not very difficult to prove that the function given by (3.34) satisfies the conditions of Theorem 3.1.

Using the second-order integro-partial-differential operator

$$L \phi(s, x) = \frac{\partial \phi}{\partial s} + \mu a \frac{\partial \phi}{\partial x} + \frac{1}{2} \sigma^2 a^2 \frac{\partial^2 \phi}{\partial x^2} + \int_{\mathbb{R}} \left\{ \phi(s, x + \gamma az) - \phi(s, x) - \gamma az \frac{\partial \phi}{\partial x} \right\} \nu(dz),$$

we obtain

$$L \phi(s, x) = \frac{e^{-\rho s}}{1 + \delta} \left[ -2\rho x^3 + \mu ax^{-\frac{1}{2}} - \frac{\sigma^2 a^2}{4} x^{-\frac{3}{2}} \right] + \int_{\mathbb{R}} \left\{ \frac{2}{\sqrt{x}} + \gamma az - 2x^{\frac{1}{2}} - \gamma az x^{-\frac{1}{2}} \right\} \nu(dz)$$

$$\leq \frac{e^{-\rho s}}{1 + \delta} \left[ -2\rho x^3 + \mu ax^{-\frac{1}{2}} - \frac{\sigma^2 a^2}{4} x^{-\frac{3}{2}} \right] + \int_{\mathbb{R}} \left\{ 2\sqrt{x} - 2x^{\frac{1}{2}} - \gamma ax x^{-\frac{1}{2}} \right\} \nu(dz)$$

$$= \frac{e^{-\rho s}}{1 + \delta} \left[ -2\rho x^3 + \mu ax^{-\frac{1}{2}} - \frac{\sigma^2 a^2}{4} x^{-\frac{3}{2}} \right] + \int_{\mathbb{R}} \gamma a x^{-\frac{1}{2}} \nu(dz).$$

We have applied the fact that $\gamma z \leq 0$ and $a \in [0, 1]$. Thus

$$L \phi(s, x) \leq \frac{-2\rho e^{-\rho s}}{1 + \delta} x^{-\frac{3}{2}} \left[ x^2 - \frac{\mu a}{2\rho} x + \frac{\sigma^2}{8\rho} + x \int_{\mathbb{R}} \gamma az \nu(dz) \right]$$

$$= \frac{-2\rho e^{-\rho s}}{1 + \delta} x^{-\frac{3}{2}} \left[ x^2 + \left( \int_{\mathbb{R}} \gamma az \nu(dz) - \frac{\mu a}{2\rho} \right) x + \frac{\sigma^2 a^2}{8\rho} \right].$$

This shows that if

$$\left( \int_{\mathbb{R}} \gamma z \nu(dz) - \mu \right)^2 \leq 2\rho \sigma^2.$$

then $\phi(x) = \Phi(x)$. 71
Putting
\[ f(x) = \frac{2\sqrt{x}}{1 + \delta} \]  
(3.35)
in (3.27) gives
\[
\frac{\mu}{\sqrt{x}} - \frac{\sigma^2 \hat{a}}{2 \sqrt{x^3}} + \int_{\mathbb{R}} \left\{ \frac{1}{\sqrt{x + \gamma az}} - \frac{1}{\sqrt{x}} \right\} \gamma z \nu \, (dz) = 0.
\]
(3.36)

We now state the following result:

**Theorem 3.2** Let \( X^{(a)}(t) \) be given by (3.21).

1. Assume that
\[
\left( \int_{\mathbb{R}} \gamma z \nu \, (dz) - \mu \right)^2 \leq 2 \rho \sigma^2.
\]
(3.37)

Then
\[
\Phi(s, x) := \frac{2e^{-\rho s}}{1 + \delta} \sqrt{x}.
\]
(3.38)
where \( \sigma \) and \( \rho \) are defined as before. This value is achieved in the limit if we apply the strategy \( \tilde{u}^{(m, \eta)} \) described above with \( \eta \to 0 \) and \( m \to \infty \), that is, by applying the policy of immediate chattering to 0.

2. If
\[
\left( \int_{\mathbb{R}} \gamma z \nu \, (dz) - \mu \right)^2 > 2 \rho \sigma^2.
\]
(3.39)
then the value function has the form
\[
\Phi(s, x) = \begin{cases} 
  e^{-\rho s} A(e^{r_1 x} - e^{r_2 x}); & \text{for } 0 \leq x < x^* \\
  e^{-\rho s}\left(\frac{2}{1 + \delta} \sqrt{x} - \frac{2}{1 + \delta} \sqrt{x^*} + B\right) & \text{for } x^* \leq x 
\end{cases}
\]
(3.40)
for some constants $A > 0$, $B > 0$ and $x^* > 0$ where $r_1$ and $r_2$ are the solutions of the equation

$$-\rho + \mu x + \frac{1}{2} \sigma^2 a^2 r^2 + \int_{\mathbb{R}} \{e^{r \gamma az} - 1 - r \gamma az\} \nu(dz) = 0. \quad (3.41)$$

with $r_2 < 0 < r_1$ and $|r_2| > r_1$.

In both cases 1. and 2. the corresponding optimal divided policy is the following:

(a) If $x > x^*$ it is optimal to apply immediate chattering from $x$ down to $x^*$.

(b) If $0 < x < x^*$ it is optimal to apply the harvesting equal to the local time of the downward reflected process $\bar{X}(t)$ at $x^*$.

Constants $A, B$ and $x^*$ can be determined using the principle of smooth fit at $x^*$.

The optimal reinsurance policy $a^* := \hat{a}$ is determined by the solution of

(c) Equation (3.36) if $x > x^*$.

(d)

$$\mu[r_1(\hat{a})e^{r_1(\hat{a})x} - r_2(\hat{a})e^{r_2(\hat{a})x}] + \sigma^2 [r_1^2(\hat{a})e^{r_1(\hat{a})x} - r_2^2(\hat{a})e^{r_2(\hat{a})x}]$$

$$+ \int_{\mathbb{R}} \{r_1(\hat{a})e^{r_1(\hat{a})(x+\gamma az)} - r_2(\hat{a})e^{r_2(\hat{a})(x+\gamma az)} - r_1(\hat{a})e^{r_1(\hat{a})x} - r_2(\hat{a})e^{r_2(\hat{a})x} \} \gamma z \nu(dz) = 0$$

if $0 < x < x^*$. 

73
Chapter 4

Optimal Differential Game Strategies in Insurance

4.1 Introduction

In Chapter 2 we derived optimal dividend policy for a company that insures all its claims. However, the results of Chapter 3 showed how a typical insurance firm can optimally control the level of risk exposure using proportional reinsurance. One of the reasons for reinsuring a proportion of the company’s claims is to generate regular and smooth income so as to achieve the goal of paying guaranteed interests and dividends. This demonstrates that those responsible for an insurance company’s capital investments are confronted with the task of minimising the risk of terminal wealth of the firm. It should be noted that the firm’s management are involved in economic game wars, on a day to day basis. We therefore deem it necessary to extend our study to stochastic differential games.

This chapter considers two-player zero-sum stochastic differential games for jump diffusions. We extend the work in [37] and our major contribution is to allow singular control in stochastic differential game theory for jump diffusions. The chapter also explores the problem of minimizing convex risk of terminal wealth for an insurance firm in the presence of proportional transaction costs. It is considered that management are allowed to invest the company’s capital on a generalised Black-Scholes market.
4.2 Preliminary Results and Motivation.

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) be a filtered complete probability space satisfying the usual conditions. A financial position is described by a map \(X: \Omega \to \mathbb{R}\), in which case \(X(\omega)\) is the discounted net worth of the position at the end of the trading period if the scenario \(\omega \in \Omega\) is realised.

Suppose that the reserves of an insurance company are invested in a market that consists of the following three assets

(i) a **safe investment** (e.g. a bank account) whose unit price at time \(t\), denoted by \(S_0(t)\), satisfies the following equation

\[
dS_0(t) = r(t)S_0(t)dt; \quad S_0(0) = 1; \tag{4.1}
\]

(ii) a **risky investment** (e.g. a stock) with unit price \(S_1(t)\) satisfying the equation

\[
dS_1(t) = S_1(t^-)[\alpha(t)dt + \beta(t)dB_1(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}_1(dt,dz)]; \quad S_1(t) = s_1 > 0; \tag{4.2}
\]

where \(\alpha(t), \beta(t), \gamma(t)\) and \(r(t)\) are \(\mathcal{F}_t\)-adapted processes satisfying the conditions for the existence and uniqueness of a solution for the system of equations (4.1)- (4.2), \(B_1(t)\) is 1-dimensional standard Brownian motion and \(\tilde{N}_1(.,.)\) is a compensated Poisson random measure.

(iii) **insurance position/ amount retained by the insurance firm** where an amount \(S_2(t)\) satisfies the equation

\[
dS_2(t) = \mu dt + \sigma dB_2(t) + \int_{\mathbb{R}} \eta z\tilde{N}_2(dt,dz); \quad S_2(t) = s_2 > 0; \tag{4.3}
\]

where \(\mu, \sigma\) and \(\eta\) are constants, \(B_2(t)\) is 1-dimensional standard Brownian motion and \(\tilde{N}_2(.,.)\) is a compensated Poisson random measure. The processes \(B_1(t), B_2(t), \tilde{N}_1(.,.)\) and \(\tilde{N}_2(.,.)\) are considered to be pairwise independent.

At each point in time \(t\), the insurance firm with current wealth \(V(t)\) will invest an amount \(\pi_1(t)\) into the risky asset, and an amount \(\pi_2(t)\) is retained in the insurance position while the remainder, that is, \(V(t) - \pi_1(t) - \pi_2(t)\) is invested in
the safe asset. The consolidated position of the insurance firm’s wealth has the following dynamics

\[
dV^{(\pi)}(t) = r^*(t)\{V^*(t) - \pi_2(t)\}dt + 2\pi_1(t^-)\left\{\{(\alpha(t) - r^*(t))dt + \beta(t)dB_1(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}_1(dt,dz)\right\} \\
+ \mu^*dt + \sigma dB_2(t) + \eta \int_{\mathbb{R}} z\tilde{N}_2(dt,dz); \quad V^{(\pi)}(0) > 0
\]

where \( \pi := \pi(t) := (\pi_1(t); \pi_2(t)) \) is an admissible investment strategy.

We impose the following integrability condition

\[
\int_0^T \left[ \left| (V(t) - \pi_1(t) - \pi_2(t))r^*(t) \right| + \left| \pi_1(t)\alpha(t) \right| + \pi_1^2(t)\beta^2(t) \\
+ \pi_1^2(t^-)\int_{\mathbb{R}} \gamma^2(t,z)\tilde{N}_1(dt,dz) \right] dt < \infty;
\]

where \( T \) is a fixed finite time horizon. At time \( T \) the terminal wealth is associated with a proportional transaction cost denoted by \( \delta V(T) \) where \( 0 \leq \delta \leq 1 \). This cost may be interpreted as the friction associated with liquidation of a portfolio.

The problem is:

To find the portfolio \( \pi(t) := (\pi_1(t); \pi_2(t)) \) that minimizes the convex risk of the terminal net wealth \( (1 - \delta)V(T) \)

For a more detailed treatment of convex risk measures we refer to [3], [20], [25], [37] and [43], and references therein.

Let \( \mathcal{X} \) be the vector space of bounded financial positions, \( \mathcal{X} \), say.

**Definition 4.2.1** The map \( \varrho : \mathcal{X} \rightarrow \mathbb{R} \) is called a convex measure of risk if it satisfies the following conditions for all \( X, Y \in \mathcal{X} \)

(i) convexity: \( \varrho(\lambda X + (1 - \lambda)Y) \leq \lambda \varrho(X) + (1 - \lambda)\varrho(Y) \), where \( \lambda \in (0; 1) \);

(ii) monotonicity: If \( X \leq Y \), then \( \varrho(X) \geq \varrho(Y) \);
(iii) translation invariance: If \( m \in \mathbb{R} \), then \( \varrho(X + m) = \varrho(X) - m \).

In financial circles \( \varrho(X) \) may be interpreted as the minimal amount of financial resources which should be added to the position \( X \) in order to make it acceptable according to a system of criteria laid down by a supervising agent or authority.

Taking into account the general representation formula for convex risk measures (see, for example [20], [25] and [37]) and also considering the presence of proportional transaction costs, we adopt the following generalised representation of the risk measure \( \varrho \)

\[
\varrho((1 - \delta)X) = \sup_{Q \in \mathcal{M}_a} \left\{ E_Q[-(1 - \delta)X] - \zeta(Q) \right\}
\]

where \( E_Q \) is the expectation with respect to probability measure \( Q \) and \( \mathcal{M}_a \) is a given family of measures that are absolutely continuous with respect to probability law \( P \). \( \zeta : \mathcal{M} \to \mathbb{R} \) is some given convex penalty function defined on \( \mathcal{M} \). For example, \( \zeta \) could be the entropy penalty, defined by

\[
\zeta(Q) = \mathbb{E} \left[ \frac{dQ}{dP} \log \frac{dQ}{dP} \right].
\]

We can now rephrase our problem in terms of the convex risk measure \( \varrho \) as follows:

Find the portfolio \( \pi(t) \) which minimises

\[
\varrho((1 - \delta)V^{(\pi)}(T)) = \sup_{Q \in \mathcal{M}_a} \left\{ E_Q[-(1 - \delta)V^{(\pi)}(T)] - \zeta(Q) \right\}
\] (4.5)

Closely related to the notion of a convex risk measure is that of a monetary utility function defined below.

**Definition 4.2.2** The map \( U : \mathcal{X} \to \mathbb{R} \) is called a monetary utility function if it satisfies the following axioms.

For all \( X, Y \in \mathcal{X} \)

(i) concavity: \( U(\lambda X + (1 - \lambda)Y) \geq \lambda U(X) + (1 - \lambda)U(Y) \), where \( \lambda \in (0; 1) \);
(ii) monotonicity: If $X \leq Y$, then $U(X) \geq U(Y)$;

(iii) translation invariance: If $m \in \mathbb{R}$, then $\varrho(X + m) = \varrho(X) + m$.

It is observable that if $\varrho$ is a convex risk measure, then $U(X) := -\varrho(X)$ is a monetary utility function and the converse also holds.

In view of the preceding discussion we can also adopt the following general representation of a monetary utility function

$$U(X) = \inf_{Q \in \mathcal{M}} \left\{ E_Q[X] + \zeta(Q) \right\} \quad (4.6)$$

If we apply (4.6) to the risk minimising problem stated in (4.5) we obtain the following utility maximizing problem

**Problem 4.2.1** Find $\Psi$, $\pi^*$ and $Q^* \in \mathcal{M}$ such that

$$\Psi = \sup_{\pi} \left( \inf_{Q \in \mathcal{M}} \left\{ E_Q[(1 - \delta) V^\pi(t)] + \zeta(Q) \right\} \right)$$

$$= E_{Q^*}[(1 - \delta) V^{\pi^*}(t)] + \zeta(Q^*)$$

We note that Problem 4.2.1 is an extension of Problem 2.4 in [37] to markets with friction in the form of proportional transaction costs.

Problem 4.2.1 is a two-player game, where the first player is the market and the second player is the trading agent. The probability measure $Q$ is the control for player number 1 (the market/nature) and the trading strategy $\pi(t)$ is the control for player number 2 (the agent).

As far as we know this is the first time transaction costs are studied in differential game problems of this type. We apply singular control theory for jump diffusions to solve our problem.

Let us now construct the general theoretical framework for the singular control two-player zero-sum stochastic differential game problem for jump diffusions.
4.3 Formulation of the General Problem and Passage to the First Main Result

4.3.1 Formulation of the Problem

The presentation in this subsection is motivated by Chapter 5 in [42]. We extend the theory to a game theoretic context.

Let $\kappa = [\kappa_{ie}] \in \mathbb{R}^{k \times p}$, $f$, $g$, $S$, $\tau_S$ and $\vartheta = [\vartheta_i] : \mathbb{R}^k \to \mathbb{R}^p$ be defined as in Chapter 2. Suppose that the state $Y(t) = Y^{u_0(t),u_1(t),\psi(\cdot)}(t) \in \mathbb{R}^k$ satisfies the equations

$$
\begin{align*}
\frac{dY(t)}{dt} &= b(Y(t),u_0(t))dt + \sigma(Y(t),u_0(t))dB(t) \\
&\quad + \int_{\mathbb{R}^l} \gamma(Y(t^-),u_1(t^-,z),z)N(dt,dz) + \kappa(Y(t^-))d\psi; \\
Y(0^-) &= y \in \mathbb{R}^k,
\end{align*}
$$

(4.7)

(4.8)

where $b : \mathbb{R}^k \times G \to \mathbb{R}^k$, $\sigma : \mathbb{R}^k \times G \to \mathbb{R}^{k \times d}$, $\gamma : \mathbb{R}^k \times G \times \mathbb{R}^l \to \mathbb{R}^{k \times l}$ are continuous functions with respect to $y, u_1, u_0$ and Borel measurable with respect to $z$. In this case $u_0(t) = u_0(t,\omega) \in G$ and $u_1(t,z) = u_1(t,z,\omega) \in G$ are the control processes assumed to be cadlag and adapted to $\mathcal{F}_t$, where $G \subset \mathbb{R}^p$ is compact with respect to the usual metric in $\mathbb{R}^k$, $B(t)$ is $d$-dimensional Brownian motion and $\hat{N}_j(dt,dz) = N_j(dt,dz) - dt\nu_j(dz)$; $j = 1, 2, \ldots, l$ is a compensated Poisson random measure where $\nu_j(.)$ is a Levy measure associated to the Poisson measure $N_j(.,.)$. The controls $u_0(t)$ and $u_1(t,z)$ are given by

$$
u_0(t) = (\theta_0(t),\nu_0(t)); \quad u_1(t,z) = (\theta_1(t,z),\nu_1(t,z),z))$$

where $t \geq 0; z \in \mathbb{R}^l$. It is also noted that $\psi \in \mathbb{R}^p$ is an adapted cadlag finite variation process whose components are nondecreasing and $\psi(0^-) = 0$. We call $\psi$ the singular control for player 2, and $\theta = (\theta_0,\theta_1) \in \Theta$ is treated as the control for player 1 and, similarly, $\nu = (\nu_0,\nu_1) \in \mathcal{V}$ is regarded as the absolutely continuous control for player 2 where $\Theta$, $\mathcal{V}$ are families of admissible controls $\theta$, $\nu$, respectively.
Setting \( u(t) = (u_0(t), u_1(t)) \) and considering \((u(t), \psi(t)) \in \mathcal{A}\), define the performance functional \( J^{u,\psi}(y) \) by

\[
J^{u,\psi}(y) = E\left[ \int_0^{\tau_S} f(Y(t), u_0(t)) dt + g(Y(\tau_S)) \chi_{\tau_S < \infty} + \int_0^{\tau_S} \vartheta^T(Y(t)) d\psi(t) \right]
\]

where \( g(Y(\tau_S)) \) is interpreted to be 0 if \( \tau_S = \infty \), \( \mathcal{A} \) is the set of all admissible controls \((u_0(t), u_1(t), \psi(t))\) and \( E[\cdot] \) denotes the expectation with respect to the probability \( P \).

If we apply a Markov control \( u(t) = (u_0(t), u_1(t)) \in \mathcal{U} \subset \mathcal{G} \times \mathcal{G} \) and assume that \( d\psi = 0 \), then the generator of \( Y(t) = Y^{u,0} \) coincides with the second-order integro-partial differential operator, \( \mathcal{L} \), defined as in Chapter 2 of this thesis.

The general singular control two-player zero-sum stochastic differential game problem is the following:

**Problem 4.3.1** Find \( \Phi(y) \) and \((\theta^*, v^*, \psi^*) \in \Theta \times \mathcal{V} \times \Psi \) such that

\[
\Phi(y) = \sup_{(v,\psi) \in \mathcal{V} \times \Psi} \left( \inf_{\theta \in \Theta} J^{\theta,v,\psi}(y) \right) = J^{\theta^*, v^*, \psi^*}(y).
\]

We now state the variational inequalities for the singular control zero-sum stochastic differential game problem presented above.

### 4.3.2 Passage to the First Main Result

The theorem stated below constitutes the first main result of this work.

**Theorem 4.1** 1. Suppose that we can find \( \phi : \tilde{S} \to \mathbb{R} \) and a Markov control \((\hat{\theta}, \hat{\psi}) \in \Theta \times \mathcal{V} \) such that

\begin{enumerate}
  \item \( \phi \in C^2(\mathcal{S}^\circ) \cap C(\tilde{S}) \),
  \item \( \mathcal{L}_{\hat{\theta},\hat{\psi}}(y) \phi(y) + f(y, \hat{\theta}(y), \hat{\psi}(y)) \geq 0 \), for all \( \theta \in \Theta \), \( y \in \mathcal{S} \)
  \item \( \mathcal{L}_{\hat{\theta}}(y,v) \phi(y) + f(y, \hat{\theta}(y), v) \leq 0 \), for all \( v \in \mathcal{V} \), \( y \in \mathcal{S} \)
  \item \( \mathcal{L}_{\hat{\theta}}(y,\hat{\psi})(y) + f(y, \hat{\theta}(y), \hat{\psi}(y)) = 0 \), for all \( y \in \mathcal{S} \)
\end{enumerate}
\( (v) \) \( \sum_{i=1}^{k} \kappa_{ie}(y) \frac{\partial \phi}{\partial y_i}(y) + \vartheta_e(y) \leq 0, \) for all \( y \in S, \ e = 1, \ldots, p \)

\( (vi) \) \( E^y \left[ \int_0^{\tau_S} \left\{ \sigma^T(Y(t), \hat{\theta}(y), \hat{v}(y)) \nabla \phi(Y(t)) \right\}^2 \right. \\
+ \sum_{m=1}^{l} \int_{\mathbb{R}^k} | \phi(Y(t) + \gamma^{(m)}) - \phi(Y(t)) \|^2 \nu_m(dz) \} \, dt \right] < \infty, \) for all \( y \in S. \)

\( (vii) \) the family \( \{ \phi^-(Y^{\theta,v}({\tau})) ; \ \tau \in T \} \) is uniformly integrable, for all \( (\theta, v) \in \Theta \times \mathcal{V}, \ y \in S. \)

Define the non-intervention region, \( D, \) by

\[
D = \left\{ y \in S; \ \max_{e} \left\{ \sum_{i=1}^{k} \kappa_{ie}(y) \frac{\partial \phi}{\partial y_i}(y) + \vartheta_e(y) \right\} \leq 0 \right\} \quad (4.10)
\]

2. Assume, in addition to \( 1(i) - (vii), \) that

\( (i) \) \( E^y \left[ \int_0^{\tau_S} \chi_{\partial D}(Y^{\theta,v,\hat{\psi}}(t)) \, dt \right] = 0, \) for all \( y \in S, \ (\theta, v, \psi) \in \Theta \times \mathcal{V} \times \Psi \)

\( (ii) \) \( \partial D \) is a Lipschitz surface,

\( (iii) \phi \in C^2(S \setminus \partial D) \) with locally bounded derivatives near \( \partial D, \)

Moreover, suppose there exists \( \hat{\psi} \in \Psi \) such that \( (\hat{\theta}, \hat{v}, \hat{\psi}) \in \Theta \times \mathcal{V} \times \Psi \) and

\( (iv) \) \( \triangle_{\hat{\psi}} \phi(Y(t_n)) + \sum_{e=1}^{p} \vartheta_e(Y(t_n^-)) \triangle_{\hat{\psi}}(t_n) = 0 \) for all jumping times \( t_n \) of \( \hat{\psi}(t) \) and

\( (v) \) \( \lim_{R \to \infty} E^y \left[ \phi(Y^{\hat{\theta},v,\hat{\psi}}(T_R)) \right] = E^y \left[ g(Y^{\hat{\theta},v,\hat{\psi}}(T)) \cdot \chi_{\{T \leq \infty\}} \right] \)

where

\[ T_R = \min(\tau_S, R) \text{ for } R < \infty. \]

Then

\[
\phi(y) = \Phi(y) = \sup_{(v, \psi) \in \mathcal{V} \times \Psi} \left( \inf_{\theta \in \Theta} J^{\theta,v,\psi}(y) \right) = \inf_{\theta \in \Theta} \left( \sup_{(v, \psi) \in \mathcal{V} \times \Psi} J^{\theta,v,\psi}(y) \right) \\
= \sup_{(v, \psi) \in \mathcal{V} \times \Psi} J^{\hat{\theta},v,\hat{\psi}}(y) = \inf_{\theta \in \Theta} J^{\hat{\theta},v,\hat{\psi}}(y) = J^{\hat{\theta},v,\hat{\psi}}(y) \text{ for all } y \in S
\]

and

\( (\hat{\theta}, \hat{v}, \hat{\psi}) = (\hat{\theta}(y), \hat{v}(y), \hat{\psi}(y)) \in \Theta \times \mathcal{V} \times \Psi \) is an optimal control.
Proof
Considering an arbitrarily chosen control \((\theta, v) \in \Theta \times V\) and applying Itō’s generalized formula for semimartingales, we obtain

\[
E^y \left[ \phi(Y(T_R)) \right] = \phi(y) + E^y \left[ \int_0^{T_R} \mathcal{L}^{\theta,v} \phi(Y(t)) dt \right]
\]

\[
+ \int_0^{T_R} \sum_{i=1}^k \sum_{e=1}^p \frac{\partial \phi}{\partial y_i} (Y(t^-)) \kappa_{ie}(Y(t^-)) d\psi^e(t) + \sum_{0 < t_n < T_R} \Delta \phi(Y(t_n))
\]

(4.11)

where \(\psi^e(t)\) denotes the continuous part of \(\dot{\psi}(t)\), and \(\Delta \phi(Y(t))\) denotes the change in value of \(\phi\) due to the jump in the singular control \(\psi\).

If we apply (4.11) to \((\theta, \dot{v}, \dot{\psi})\) and use 1(ii) for all \(y = Y(t)\) we get

\[
E^y \left[ \phi(Y(T_R)) \right] \geq \phi(y) - E^y \left[ \int_0^{T_R} f(Y(t), \theta(Y(t)), \dot{v}(Y(t))) dt \right]
\]

\[
- \int_0^{T_R} \sum_{i=1}^k \sum_{e=1}^p \frac{\partial \phi}{\partial y_i} (Y(t^-)) \kappa_{ie}(Y(t^-)) d\psi^e(t) - \sum_{0 < t_n < T_R} \Delta \phi(Y(t_n))
\]

(4.12)

or

\[
\phi(y) \leq E^y \left[ \phi(Y(T_R)) \right] + E^y \left[ \int_0^{T_R} f(Y(t), \theta(Y(t)), \dot{v}(Y(t))) dt \right]
\]

\[
- \int_0^{T_R} \sum_{i=1}^k \sum_{e=1}^p \frac{\partial \phi}{\partial y_i} (Y(t^-)) \kappa_{ie}(Y(t^-)) d\psi^e(t) - \sum_{0 < t_n < T_R} \Delta \phi(Y(t_n))
\]

(4.13)

Applying the mean value theorem we obtain

\[
\Delta \phi(Y(t_n)) = (\nabla \phi'(Y^{(n)})) T \Delta \psi(t_n)
\]

\[
= \sum_{i=1}^k \sum_{e=1}^p \frac{\partial \phi}{\partial y_i} (Y^{(n)}) \kappa_{ie}(Y(t_n^-)) \Delta \psi^e(t_n)
\]
where $\tilde{Y}(n)$ is some point on the line segment which joins $Y(t_n)$ and $Y(t_n^-) + \Delta_N Y(t_n)$, where $\Delta_N Y(.)$ denotes the change in value of $Y(.)$ caused by the Poisson random jumps at $t = t_n$.

By (vi); (4.13) and (4.14)

$$
\phi(y) \leq E^y \left[ \int_0^{T_R} f(Y(t), \theta(Y(t)), \hat{v}(Y(t))) dt + \phi(Y(T_R)) \right. \\
- \sum_{e=1}^{p} \sum_{i=1}^{k} \left\{ \int_0^{T_R} \frac{\partial \phi}{\partial y_i}(Y(t^-)) \kappa_{ie}(Y(t^-)) d\hat{\psi}_e^c(t) \\
+ \sum_{0 < t_n < T_R} \frac{\partial \phi}{\partial y_i}(\tilde{Y}(n)) \kappa_{ie}(Y(t^-)) \Delta \hat{\psi}_e(t_n) \right\} \\
\left. \leq E^y \left[ \int_0^{T_R} f(Y(t), \theta(Y(t)), \hat{v}(Y(t))) dt + \phi(Y(T_R)) \right. \\
+ \sum_{e=1}^{p} \int_0^{T_R} \vartheta_e(Y(t^-)) d\hat{\psi}_e^c(t) \right].
$$

(4.15)

Letting $R \to \infty$ and applying (vii); (viii) and (viii) we have

$$
\phi(y) \leq E^y \left[ \int_0^{T} f(Y(t), \theta(Y(t)), \hat{v}(Y(t))) dt \\
+ g(Y(T)) \chi_{\{T < \infty\}} + \int_0^{T} \vartheta(Y(t)) d\hat{\psi}_e^c(t) \right] \\
= J^{\theta, \hat{v}, \hat{\psi}}(y) \text{ for all } y \in \mathcal{S}.
$$

That is

$$
\phi(y) \leq J^{\theta, \hat{v}, \hat{\psi}}(y). \quad (4.16)
$$

Since this holds for all $\theta \in \Theta$ we deduce that

$$
\phi(y) \leq \inf_{\theta \in \Theta} J^{\theta, \hat{v}, \hat{\psi}}(y). \quad (4.17)
$$
Consequently,
\[
\phi(y) \leq \sup_{(v,\psi) \in V \times \Psi} \left( \inf_{\theta \in \Theta} J^{\theta,v,\psi}(y) \right) = \Phi(y). \tag{4.18}
\]

If we apply (4.11) to \((\hat{\theta}, v, \psi)\) and use 1(iii) for all \(y = Y(t)\) we get
\[
\begin{align*}
E^y \left[ \phi(Y(T_R)) \right] &\leq \phi(y) - E^y \left[ \int_0^{T_R} f(Y(t), \hat{\theta}(Y(t), v(Y(t)))\phi(Y(t)))dt \right. \\
&\quad - \int_0^{T_R} \sum_{i=1}^k \frac{\partial \phi}{\partial y_i}(Y(t^-)) \sum_{e=1}^p \kappa_{ie}(Y(t^-))d\psi^e_c(t) - \sum_{0 < t_n < T_R} \Delta_\psi \phi(Y(t_n)) \bigg] \\
&= \Phi(y). \quad \text{(4.19)}
\end{align*}
\]

or
\[
\begin{align*}
\phi(y) &\geq E^y \left[ \phi(Y(T_R)) \right] + \int_0^{T_R} f(Y(t), \hat{\theta}(Y(t), v(Y(t)))\phi(Y(t)))dt \\
&\quad - \int_0^{T_R} \sum_{i=1}^k \frac{\partial \phi}{\partial y_i}(Y(t^-)) \sum_{e=1}^p \kappa_{ie}(Y(t^-))d\psi^e_c(t) - \sum_{0 < t_n < T_R} \Delta_\psi \phi(Y(t_n)) \bigg] \\
&= \Phi(y). \quad \text{(4.20)}
\end{align*}
\]

Letting \(R \to \infty\) and using 1(vii) and 1(viii) we have
\[
\phi(y) \geq J^{\hat{\theta},\hat{v},\hat{\psi}}(y) \geq \inf_{\theta \in \Theta} J^{\theta,v,\psi}(y). \tag{4.21}
\]

Since this holds for all \(v \in V\) we deduce that
\[
\phi(y) \geq \sup_{(v,\psi) \in V \times \Psi} \left( \inf_{\theta \in \Theta} J^{\theta,v,\psi}(y) \right) = \Phi(y). \tag{4.22}
\]

Finally, applying (4.11) to \((\hat{\theta}, \hat{v}, \hat{\psi})\) and proceeding as before we get
\[
\phi(y) = J^{\hat{\theta},\hat{v},\hat{\psi}}(y) = \Phi(y). \tag{4.23}
\]

Combining (4.22), (4.23) and (4.18) we conclude that
\[
\Phi(y) \leq \phi(y) = J^{\hat{\theta},\hat{v},\hat{\psi}}(y) \leq \Phi(y). \tag{4.24}
\]

Combining (4.21) and (4.16) we get
\[
\begin{align*}
\inf_{\theta \in \Theta} \left( \sup_{(v,\psi) \in V \times \Psi} J^{\theta,v,\psi}(y) \right) & \leq \sup_{(v,\psi) \in V \times \Psi} J^{\hat{\theta},v,\hat{\psi}}(y) \leq \phi(y) \leq \inf_{\theta \in \Theta} J^{\theta,v,\psi} \\
& \leq \sup_{(v,\psi) \in V \times \Psi} \left( \inf_{\theta \in \Theta} J^{\theta,v,\psi} \right) = \Phi(y).
\end{align*}
\]

But the following relationship always holds
\[
\sup_{(v,\psi) \in V \times \Psi} \left( \inf_{\theta \in \Theta} J^{\theta,v,\psi} \right) \leq \inf_{\theta \in \Theta} \left( \sup_{(v,\psi) \in V \times \Psi} J^{\theta,v,\psi}(y) \right).
\]

From (4.25), (4.26) and (4.23) we obtain
\[
\phi(y) = \Phi(y) = \sup_{(v,\psi) \in V \times \Psi} \left( \inf_{\theta \in \Theta} J^{\theta,v,\psi}(y) \right) = \inf_{\theta \in \Theta} \left( \sup_{(v,\psi) \in V \times \Psi} J^{\theta,v,\psi}(y) \right) = \sup_{(v,\psi) \in V \times \Psi} J^{\hat{\theta},v,\hat{\psi}}(y) = \inf_{\theta \in \Theta} J^{\hat{\theta},\hat{v},\hat{\psi}}(y) = J^{\hat{\theta},\hat{v},\hat{\psi}}(y)
\]
for all \( y \in S \)

\[
(\hat{\theta}, \hat{v}, \hat{\psi}) = (\hat{\theta}(y), \hat{v}(y), \hat{\psi}(y)) \in \Theta \times V \times \Psi \text{ is an optimal control.}
\]

We apply the theorem to solve Problem 4.2.1.

**Example 1**

We now revisit the market described by equations (4.1)-(4.3). Let \( \Pi \) be the set of all admissible two dimensional trading strategies \( \pi(t) := (\pi_1(t), \pi_2(t)) \)
It is assumed that the mean rate of return, $\alpha(t)$, of the stock is determined by the choice of the two-dimensional trading strategy $\pi(.) = (\pi_1(t), \pi_2(t)) \in \Pi$. The firm’s aim is to maximise the expected utility of terminal wealth in the presence of friction, by judiciously applying the strategy $\pi$, while the market tries to minimise this maximum expected utility by choosing $\alpha(t) \in U$ accordingly. Such a situation can be described as a mini-max problem as follows

$$\inf_{\alpha(t) \in U} \left( \sup_{\pi \in \Pi} \{ E[U_0((1 - \delta)Y_{1\alpha,\pi}^T(T))] \} \right)$$

(4.28)

where $U_0$ is a given monetary utility and $U$, $\Pi$ are given families of admissible controls $\alpha(t)$ and $\pi(t)$. It is assumed that $\gamma(t,z) = z > -1$ a.s. $\nu$ and $\beta$ is a constant.

Setting $Y_1(t) := V(\pi)(t)$, we get

$$dY_1(t) = r(t)[Y_1(t) - \pi_2(t)]dt$$

$$+ \pi_1(t^-)[\{\alpha(t) - r(t)\}dt + \beta dB_1(t) + \eta(t^-)\int_{\mathbb{R}} z \tilde{N}_1(dt, dz)]$$

$$+ \mu dt + \sigma dB_2(t) + \gamma \int_{\mathbb{R}} z \tilde{N}_2(dt, dz); \quad Y_1(0) = y_1 = x > 0.$$  

(4.29)

Problem (4.28) can now be represented as

Find $(\alpha^*, \pi^*) \in U \times \Pi$ and $\Phi(y) = \Phi(s, x)$ such that

$$\Phi(s, x) = \inf_{\alpha \in U} \left( \sup_{\pi \in \Pi} \{ E^{s,x}[U_0((1 - \delta)Y_{1\alpha,\pi}^T(T))] \} \right) = E^{s,x}[U_0((1 - \delta)Y_{1\alpha^*,\pi^*}^T(T))]$$

(4.30)

The generator coincides with the following integro-differential operator

$$\mathcal{L}^{\alpha,\pi} \phi(s, x) = \frac{\partial \phi}{\partial s} + [\mu + r(s)(x - \pi_2(s)) + \pi_1(s)\{\alpha(s) - r(s)\}]\frac{\partial \phi}{\partial x}$$

$$+ \frac{1}{2}\beta^2 \pi_1^2(s) \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{2}\sigma^2 \frac{\partial \phi}{\partial x^2}$$

$$+ \int_{\mathbb{R}} \{ \phi(s, x + \pi_1 \eta z) - \phi(s, x) - \pi_1(s)\eta z \frac{\partial \phi}{\partial x} \} \nu_1(dz)$$

$$+ \int_{\mathbb{R}} \{ \phi(s, x + \gamma z) - \phi(s, x) - \gamma z \frac{\partial \phi}{\partial x} \} \nu_2(dz).$$
and the HJB for the problem may be given as

$$\inf_{\alpha \in U} \left( \sup_{\pi \in \Pi} \left\{ L^{\alpha, \pi} \phi(s, x) \right\} \right) = 0; \ s < T$$  \hspace{1cm} (4.32)

$$\phi(T, x) = U_0((1 - \delta)x)$$  \hspace{1cm} (4.33)

We propose a function of the form

$$\phi(s, x) = U_0[(1 - \delta)x \exp(\int_s^T r(t) dt)].$$

Applying the operator $L^{\alpha, \pi}$ on $\phi(s, x)$ we obtain

$$L^{\alpha, \pi} \phi(s, x) = -(1 - \delta)xU'_0[(1 - \delta)x \exp(\int_s^T r(t) dt)] \exp(\int_s^T r(t) dt) r(s)$$

$$+ (1 - \delta)[r x - r \pi_2 + \mu + \pi_1(\alpha - r)]U'_0[(1 - \delta)x \exp(\int_s^T r(t) dt)] \exp(\int_s^T r(t) dt)$$

$$+ \frac{1}{2} \beta^2 \pi_1^2 (1 - \delta)^2 U''_0[(1 - \delta)x \exp(\int_s^T r(t) dt)] \exp(2 \int_s^T r(t) dt)$$

$$+ \frac{1}{2} \sigma^2 (1 - \delta)^2 U''_0[(1 - \delta)x \exp(\int_s^T r(t) dt)] \exp(2 \int_s^T r(t) dt)$$

$$+ \int_R \left\{ U_0[(1 - \delta)(x + \pi \eta z) \exp(\int_s^T r(t) dt)] - U_0[(1 - \delta)x \exp(\int_s^T r(t) dt)] \right\} \nu_1(dz)$$

$$- \pi_1(1 - \delta) \eta z U'_0[(1 - \delta)x \exp(\int_s^T r(t) dt)] \exp(\int_s^T r(t) dt)] \nu_1(dz)$$

$$+ \int_R \left\{ U_0[(1 - \delta)(x + \gamma z) \exp(\int_s^T r(t) dt)] - U_0[(1 - \delta)x \exp(\int_s^T r(t) dt)] \right\} \nu_2(dz)$$

$$- (1 - \delta)\gamma z U'_0[(1 - \delta)x \exp(\int_s^T r(t) dt)] \exp(\int_s^T r(t) dt)] \nu_2(dz)$$  \hspace{1cm} (4.34)
Define $h(\pi_1, \pi_2)$ by

$$h(\pi_1, \pi_2) :=$$

$$- (1 - \delta) U_0' \left[ (1 - \delta) x \exp \left( \int_s^T r(t) dt \right) \right] \exp \left( \int_s^T r(t) dt \right) r(s)$$

$$+ (1 - \delta) \left[ r x - r \pi_2 + \mu + \pi_1 (\alpha - r) \right] U_0' \left[ (1 - \delta) x \exp \left( \int_s^T r(t) dt \right) \right] \exp \left( \int_s^T r(t) dt \right)$$

$$+ \frac{1}{2} \beta^2 \pi_1^2 (1 - \delta)^2 U_0'' \left[ (1 - \delta) x \exp \left( \int_s^T r(t) dt \right) \right] \exp \left( \int_s^T r(t) dt \right)$$

$$+ \frac{1}{2} \sigma^2 (1 - \delta)^2 U_0'' \left[ (1 - \delta) x \exp \left( \int_s^T r(t) dt \right) \right] \exp \left( \int_s^T r(t) dt \right)$$

$$+ \int \left\{ U_0 \left[ (1 - \delta) \left( x + \pi_1 \eta z \right) \exp \left( \int_s^T r(t) dt \right) \right] - U_0 \left[ (1 - \delta) x \exp \left( \int_s^T r(t) dt \right) \right] \right\} \nu_1 (dz)$$

$$- \pi_1 (1 - \delta) \eta z U_0' \left[ (1 - \delta) x \exp \left( \int_s^T r(t) dt \right) \right] \exp \left( \int_s^T r(t) dt \right) \} \nu_1 (dz)$$

$$+ \int \left\{ U_0 \left[ (1 - \delta) \left( x + \gamma z \right) \exp \left( \int_s^T r(t) dt \right) \right] - U_0 \left[ (1 - \delta) x \exp \left( \int_s^T r(t) dt \right) \right] \right\} \nu_2 (dz)$$

$$- (1 - \delta) \gamma z U_0' \left[ (1 - \delta) x \exp \left( \int_s^T r(t) dt \right) \right] \exp \left( \int_s^T r(t) dt \right) \} \nu_2 (dz)$$

(4.35)

Now, we would like to maximise $h$ over the point $\pi = (\pi_1, \pi_2)$. It can easily be observed that if $U_0 > 0$ then $h$ is linear and decreasing in $\pi_2$. The maximum value of $h$ is attained when $\pi_2 = 0$.

Setting $\pi_2 = 0$ in (4.35) and maximizing the resultant expression over $\pi_1$ we get the following first order condition for the maximum point $\hat{\pi}_1 = \hat{\pi}_1(\alpha)$

$$(1 - \delta) [\alpha - r(s)] U_0' \left[ (1 - \delta) x \exp \left( \int_s^T r(t) dt \right) \right] \exp \left( \int_s^T r(t) dt \right)$$

$$+ \beta^2 \hat{\pi}_1 (1 - \delta)^2 U_0'' \left[ (1 - \delta) x \exp \left( \int_s^T r(t) dt \right) \right] \exp \left( \int_s^T r(t) dt \right)$$

$$+ \int \left\{ U_0' \left[ (1 - \delta) \left( x + \hat{\pi}_1 \eta z \right) \exp \left( \int_s^T r(t) dt \right) \right] \exp \left( \int_s^T r(t) dt \right) \right\} \nu_1 (dz)$$

$$- U_0' \left[ (1 - \delta) x \exp \left( \int_s^T r(t) dt \right) \right] \exp \left( \int_s^T r(t) dt \right)$$

$$\{(1 - \delta) \eta z \nu_1 (dz) = 0.$$

(4.36)
Equation (4.36) simplifies to

\[ [\alpha - r(s)]U'_0 [(1 - \delta)x \exp \left( \int_s^T r(t) dt \right)] \]

\[ + \beta^2 \hat{\pi}_1 (1 - \delta)U''_0 [(1 - \delta)x \exp \left( \int_s^T r(t) dt \right) \exp \left( \int_s^T r(t) dt \right)] \]

\[ + \int_R \left\{ U'_0 [(1 - \delta)(x + \hat{\pi}_1 \eta z) \exp \left( \int_s^T r(t) dt \right)] \right\} \eta \hat{\pi}_1 \left( \alpha \right) \eta z \nu (dz) = 0. \]

\[ (4.37) \]

Now substituting \( \pi_1 = \hat{\pi}_1 (\alpha) \) and \( \pi_2 = 0 \) into (4.32) and minimizing with respect to \( \alpha \) we get

\[ (1 - \delta)(-r(s)\hat{\pi}'_1 (\alpha) + \hat{\pi}_1 (\alpha) + \alpha \hat{\pi}'_1 (\alpha)]U'_0 [(1 - \delta)x \exp \left( \int_s^T r(t) dt \right)] \]

\[ + \beta^2 \hat{\pi}_1 (\alpha)\hat{\pi}_1 (\alpha)(1 - \delta)^3 xU''_0 [(1 - \delta)x \exp \left( \int_s^T r(t) dt \right) \exp \left( \int_s^T r(t) dt \right)] \]

\[ + \int_R \left\{ U'_0 [(1 - \delta)(x + \hat{\pi}_1 x z) \exp \left( \int_s^T r(t) dt \right)] \right\} \hat{\pi}_1 (\alpha)z \nu (dz) = 0. \]

\[ (4.38) \]

Equation (4.38) can be written as

\[ (1 - \delta)(\alpha - r(s))\hat{\pi}'_1 (\alpha)U'_0 [(1 - \delta)x \exp \left( \int_s^T r(t) dt \right)] \]

\[ + (1 - \delta)\hat{\pi}_1 (\alpha)U'_0 [(1 - \delta)x \exp \left( \int_s^T r(t) dt \right)] \]

\[ + \beta^2 \hat{\pi}_1 (\alpha)\hat{\pi}_1 (\alpha)(1 - \delta)^3 xU''_0 [(1 - \delta)x \exp \left( \int_s^T r(t) dt \right) \exp \left( \int_s^T r(t) dt \right)] \]

\[ + \int_R \left\{ U'_0 [(1 - \delta)(x + \hat{\pi}_1 (\alpha) \eta z) \exp \left( \int_s^T r(t) dt \right)] \right\} \hat{\pi}_1 (\alpha)z \nu (dz) = 0 \]

\[ 90 \]
Comparing (4.37) and (4.39) we obtain

$$\hat{\pi}_1(\alpha) = 0.$$  \hfill (4.40)

Putting $\alpha = \hat{\alpha}$ and $\hat{\pi}_1(\alpha) = 0$ in (4.37) yields

$$\hat{\alpha} = r(s)$$  \hfill (4.41)

Putting these values of $\hat{\alpha}, \hat{\pi}_1$ into (4.34) we obtain

$$L^{\hat{\alpha}, \hat{\pi}_1} \phi(s, x) = - (1 - \delta) x U'_0 \left[ (1 - \delta) x \exp \left( \int_s^T r(t) dt \right) \exp \left( \int_s^T r(t) dt \right) r(s) \right] + \left( 1 - \delta \right) \left[ r(1 - \delta) x + \mu + \pi(\alpha - r) \right] U'_0 \left[ (1 - \delta) x \exp \left( \int_s^T r(t) dt \right) \exp \left( \int_s^T r(t) dt \right) \right] + \frac{1}{2} \sigma^2 (1 - \delta)^2 U''_0 \left[ (1 - \delta) x \exp \left( \int_s^T r(t) dt \right) \exp \left( \int_s^T r(t) dt \right) \right] + \int_{\mathbb{R}} \left\{ U_0 \left[ (1 - \delta) \left( x + \gamma z \right) \exp \left( \int_s^T r(t) dt \right) \right] - U_0 \left[ (1 - \delta) x \exp \left( \int_s^T r(t) dt \right) \right] \right\} \nu_2(dz)$$  \hfill (4.42)

Our results are summarised in the following theorem

**Theorem 4.2** Suppose $L_0 := L^{\hat{\alpha}, \hat{\pi}_1} \leq 0$ where $\hat{\pi}_1$ and $\hat{\alpha}$ are given by (4.40) and (4.41), respectively, and $\pi_2 = 0$. Then the value function is given by

$$\Phi(s, x) = \phi(s, x) = U_0 \left[ (1 - \delta) x \exp \left( \int_s^T r(t) dt \right) \right].$$  \hfill (4.43)

### 4.4 Passage to the Second Main Result: An HJBIIVI for Nash equilibria

Assume that $\kappa^{(q)} = [\kappa^{(q)}_{ic}] \in \mathbb{R}^{k \times p}$ and $\vartheta^{(q)} = [\vartheta^{(q)}_{ic}] : \mathbb{R}^k \to \mathbb{R}^p$ are given continuous functions, where $q = 1, 2$.  

91
Let \( \pi^{(q)} \in \mathbb{R}^p \) be the singular control applied by player \( q \) with \( \pi^{(q)}(0^-) = 0 \), and assume that \( v^{(q)} = (v_0^{(q)}, v_1^{(q)}) \in \mathcal{V}^{(q)} \) is the absolutely continuous (with respect to the Lebesgue measure \( dt \)) control for player \( q \) where \( q = 1, 2 \).

Consider \( Q^{(1)} = \mathcal{V}^{(1)} \times \Pi^{(1)} \) and \( Q^{(2)} = \mathcal{V}^{(2)} \times \Pi^{(2)} \) to be the families of admissible controls \( \psi^{(1)} = (v^{(1)}, \pi^{(1)}) \) and \( \psi^{(2)} = (v^{(2)}, \pi^{(2)}) \), respectively.

Now, suppose that as a result of applying the double singular control \( \psi = (\psi^{(1)}, \psi^{(2)}) \in Q = Q^{(1)} \times Q^{(2)} \) the doubly controlled stochastic process \( Y(t) = Y^{\psi}(t) \) satisfies the following equations

\[
\begin{align*}
\text{d}Y(t) & = b\left(Y(t), v_0^{(1)}(t), v_0^{(2)}(t)\right)\text{d}t + \sigma\left(Y(t), v_0^{(1)}(t), v_0^{(2)}(t)\right)\text{d}B(t) \\
 & + \int_{\mathbb{R}^l} \gamma\left(Y(t^-), v_1^{(1)}(t^-, z), v_1^{(2)}(t^-, z), z\right) \tilde{N}(dt, dz) \\
 & + \kappa^{(1)}(Y(t^-))d\pi^{(1)}(t) + \kappa^{(2)}(Y(t^-))d\pi^{(2)}(t)
\end{align*}
\]

\[ (4.44) \]

\[
Y(0^-) = y \in \mathbb{R}^k.
\]

Also suppose that \( f_q : \mathcal{S} \times \mathcal{V}^{(q)} \to \mathbb{R} \), \( g_q : \mathbb{R}^k \to \mathbb{R} \), and \( \vartheta^{(q)} : \mathbb{R}^k \to \mathbb{R}^{k \times p} \) are given continuous functions such that

\[
\mathbb{E}^y\left[ \int_0^{\tau_S} \left| f_q\left(Y(t), v_0^{(1)}(t), v_0^{(2)}(t)\right) \right| \text{d}t + \left| g_q(Y(\tau_S)) \right| \chi_{\{\tau_S < \infty\}} + \int_0^{\tau_S} \sum_{e=1}^p \left| \vartheta_e^{(q)}(Y(t^-)) \right| d\pi_e^{(q)}(t) \right] < \infty
\]

for all \( y \in \mathcal{S} \).

If we apply a Markov control \( (v^{(1)}, v^{(2)}) \) and assume that \( d\pi^{(q)} = 0 \), then the generator of \( Y(t) = Y^{v^{(1)}, v^{(2)}, 0, 0}(t) \) coincides with the second-order integro-partial...
differential operator, \( L \), given by

\[
L^{v(1),v(2)}(y)(\phi) = \sum_{i=1}^{k} b_i \left( y, v_0^{(1)}(y), v_0^{(2)}(y) \right) \frac{\partial \phi}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^{k} (\sigma \sigma^T)_{ij} \left( y, v_0^{(1)}(y), v_0^{(2)}(y) \right) \frac{\partial^2 \phi}{\partial y_i \partial y_j} + \sum_{j=1}^{l} \int_{\mathbb{R}^k} \{ \phi(y + \gamma^{(j)}(y, v_1^{(1)}(y, z), v_1^{(2)}(y, z), z)) - \phi(y) \} \nu_j(dz_j)
\]

where \( \phi \in C^2_0(\mathbb{R}^k) \).

Now, consider that there are two performance functionals, \( J_q^{\psi}(y) \), given by

\[
J_q^{\psi}(y) = E^y \left[ \int_0^{\tau_S} f_q(Y(t), v_0^{(1)}(t), v_0^{(2)}(t)) dt + g_q(Y(\tau_S))\chi_{\{\tau_S < \infty\}} + \int_0^{\tau_S} \vartheta^{(q)}(Y(t))d\pi^{(q)}(t) \right];
\]

(4.46)

where \( q = 1, 2 \), \( S \) and \( \tau_S \) are defined as before. Here, \( J_q^{\psi}(y) \) is regarded as the payoff to player number \( q \) \( (q = 1, 2) \), if the players 1 and 2 use the controls \( (v_1^{(1)}, \pi_1^{(1)}) \) and \( (v_1^{(2)}, \pi_1^{(2)}) \), respectively.

**Definition 4.4.1** A pair \( (\psi^{(1)*}, \psi^{(2)*}) \in Q^{(1)} \times Q^{(2)} \) is called a Nash equilibrium for the singular control stochastic differential game (4.44)-(4.45), (4.46) if the following holds:

\[
J_1^{\psi^{(1)*},v^{(2)*}}(y) \leq J_1^{\psi^{(1)*},\psi^{(2)*}}(y) \text{ for all } \psi^{(1)} \in Q^{(1)}, \ y \in S \quad (4.47)
\]

\[
J_2^{\psi^{(2)*},\psi^{(2)*}}(y) \leq J_2^{\psi^{(1)*},\psi^{(2)*}}(y) \text{ for all } \psi^{(2)} \in Q^{(2)}, \ y \in S. \quad (4.48)
\]

Condition (4.47) states that if player 2 uses the singular control \( \psi^{(2)*} = (v_2^{(2)*}, \pi_2^{(2)*}) \) then it is optimal for player 1 to use the singular control \( \psi^{(1)*} = (v_1^{(1)*}, \pi_1^{(1)*}) \). Similarly, condition (4.48) states that if player 1 uses the control \( \psi^{(1)*} = (v_1^{(1)*}, \pi_1^{(1)*}) \) then it is optimal for player 2 to use the singular control \( \psi^{(2)*} = (v_2^{(2)*}, \pi_2^{(2)*}) \).
In other words, \((\psi^{(1)*}, \psi^{(2)*})\) is an equilibrium point. In this work we restrict ourselves to Markov controls.

The following Theorem is a generalization of Theorem 4.1 and it constitutes the main result of this section.

**Theorem 4.3 (HJBIVI for Nash Equilibria)**

Suppose that we can find functions \(\phi_q \in C^2(S) \cap C(S)\); \(q = 1, 2\) and a Markov double singular control

\[
(\hat{\psi}^{(1)}, \hat{\psi}^{(2)}) = \left(\hat{\psi}^{(1)}, \hat{\nu}^{(1)}, \hat{\nu}^{(2)}, \hat{\nu}^{(2)}\right) \in Q^{(1)} \times Q^{(2)}
\]
such that

(i) \(L^{(1)} \phi_1(y) + f_1(y, \psi^{(1)}(y), \psi^{(2)}(y)) \leq L^{(1)} \phi_1(y) + f_1(y, \hat{\psi}^{(1)}(y), \hat{\psi}^{(2)}(y)) = 0\), for all \(\psi^{(1)} \in V^{(1)}\), \(y \in S\)

(ii) \(L^{(1)} \phi_2(y) + f_2(y, \hat{\psi}^{(1)}(y), \psi^{(2)}(y)) \leq L^{(1)} \phi_2(y) + f_2(y, \hat{\psi}^{(1)}(y), \hat{\psi}^{(2)}(y)) = 0\), for all \(\psi^{(2)} \in V^{(2)}\), \(y \in S\)

(iii) \(\sum_{i=1}^{k} \kappa_{ie}^{(1)} (Y^{(1)} \hat{\psi}^{(2)} (t)) \frac{\partial \phi_1}{\partial y_i} (Y^{(1)} \hat{\psi}^{(2)} (t)) + \frac{\partial \psi^{(1)}}{\partial y_i} (Y^{(1)} \hat{\psi}^{(2)} (t)) = 0\); for all \(\psi^{(1)} \in Q^{(1)}\), \(e = 1, \ldots, p\)

(iv) \(\sum_{i=1}^{k} \kappa_{ie}^{(2)} (Y^{(1)} \hat{\psi}^{(2)} (t)) \frac{\partial \phi_2}{\partial y_i} (Y^{(1)} \hat{\psi}^{(2)} (t)) + \frac{\partial \psi^{(2)}}{\partial y_i} (Y^{(1)} \hat{\psi}^{(2)} (t)) = 0\); for all \(\psi^{(2)} \in Q^{(2)}\), \(e = 1, \ldots, p\)

(v) \(E^\psi \left[ \int_0^{\tau_S} \sigma^T (Y^{(1)} \psi (t)) \nabla \phi_q (Y^{(1)} \psi (t)) \right] < \infty\), for all \(\psi \in Q^{(1)} \times Q^{(2)}\), \(\psi \in S\); \(q = 1, 2\)

(vi) \(Y^{(1)} (\tau_S) \in \partial S\) a.s. on \(\{\tau_S < \infty\}\) for all \(\psi \in Q^{(1)} \times Q^{(2)}\); \(q = 1, 2\) and \(\phi_q (Y^{(1)} \psi (t)) \rightarrow g_q (Y^{(1)} \hat{\psi}^{(2)} (\tau_S)) \chi_{\{\tau_S < \infty\}}\) as \(t \rightarrow \tau_S^-\) a.s., for all \(y \in S\)

(vii) the family \(\{\phi_q (Y^{(1)} \psi (\tau)); \tau \in \mathcal{T}\}\) is uniformly integrable.

Define the non-intervention region, \(D_q\), for player \(q\), by

\[
D_q = \left\{ y \in S; \max_e \left\{ \sum_{i=1}^{k} \kappa_{ie}^{(q)} (y) \frac{\partial \phi_q}{\partial y_i} (y) + \frac{\partial \phi_q}{\partial y_i} (y) \right\} \leq 0 \right\}; q = 1, 2.
\]
Assume, in addition to (i) – (vii), that

(viii) \( E^y \left[ \int_0^{\tau_S} \chi_{0D_q}(Y^\psi(t)) \, dt \right] = 0 \), for all \( y \in S, \ \psi \in Q^{(1)} \times Q^{(2)} \)

(ix) \( \partial D_q \) is a Lipschitz surface,

(x) \( \phi_q \in C^2(S \setminus \partial D_q) \) with locally bounded derivatives near \( \partial D_q \), \( q = 1, 2 \)

(xi) \( Y^\hat{\psi}(t) \in \bar{D}_q \) for all \( t \)

(xii) \( \sum_{e=1}^p \left\{ \sum_{i=1}^k \kappa^{(q)}_{ic}(y) \frac{\partial \phi}{\partial y_i}(Y(t^-)) + \hat{\psi}^{(q)}_e \right\} d\hat{\pi}^{(q)(c)}_e(t) = 0 \) where \( \hat{\pi}^{(q)(c)}_e(t) \) is the continuous part of \( \pi^{(q)}_e \); \( q = 1, 2 \).

(xiii) \( \Delta \hat{\psi}^{(q)}(Y(t_n)) + \sum_{e=1}^p \hat{\psi}^{(q)}_e(Y(t^-_n)) \Delta \hat{\pi}^{(q)}_e(t_n) = 0 \) for all jumping times \( t_n \) of \( \hat{\pi}^{(q)} \)

and

\[ \lim_{R \to \infty} E^y \left[ \phi_q(Y^\hat{\psi}(T_R)) \right] = E^y \left[ g_q(Y^\hat{\psi}(T)) \chi_{\{T<\infty\}} \right] \]

where

\[ T_R = \min(\tau_S, R) \text{ for } R < \infty \text{ and } q = 1, 2. \]

Then

\( (\hat{\psi}^{(1)}, \hat{\psi}^{(2)}) \in Q^{(1)} \times Q^{(2)} \) is a Nash equilibrium for the game (4.44)-(4.45), (4.46).

and

\[ \phi_1(y) = \sup_{\psi^{(1)} \in Q^{(1)}} J_1^{\psi^{(1)}, \psi^{(2)}}(y) = J_1^{\hat{\psi}^{(1)}, \hat{\psi}^{(2)}}(y) \] \hspace{1cm} (4.50)

\[ \phi_2(y) = \sup_{\psi^{(2)} \in Q^{(2)}} J_2^{\psi^{(1)}, \psi^{(2)}}(y) = J_2^{\hat{\psi}^{(1)}, \hat{\psi}^{(2)}}(y). \] \hspace{1cm} (4.51)

Proof

The proof of this verification theorem proceeds as that of Theorem 4.1.

95
Once more, on the basis of approximation arguments we assume that \( \phi_1 \in C^2(S) \cap C(\bar{S}) \). Considering \( Y(t) = Y^{\psi(1),\hat{\psi}(2)}(t) \) and applying Itô’s generalized formula for semimartingales we obtain

\[
E^y \left[ \phi_1(T_R) \right] = E^y \left[ \phi_1(y) \right] + E^y \left[ \int_0^{T_R} \mathcal{L}^{\psi(1),\hat{\psi}(2)}(Y(t))dt \right]
+ \int_0^{T_R} \sum_{i=1}^k \frac{\partial \phi_1}{\partial y_i}(Y(t^-)) \sum_{e=1}^p \kappa_{ie}^{(1)}(Y(t^-))d\pi_e^{(1)}(t)
+ \sum_{0<t_n<T_R} \Delta_{\pi_e^{(1)}}(Y(t_n)) \tag{4.52}
\]

where \( \pi_e^{(1)}(t) \) denotes the continuous part of \( \pi_e^{(1)}(t) \).

Using this last equation and 1(i) of Theorem 4.2 we get

\[
E^y \left[ \phi_1(T_R) \right] \leq \phi(y) - E^y \left[ \int_0^{T_R} f_1(Y(t), v^{(1)}(Y(t)), \hat{v}^{(2)}(Y(t)))dt \right]
- \int_0^{T_R} \sum_{i=1}^k \frac{\partial \phi_1}{\partial y_i}(Y(t^-)) \sum_{e=1}^p \kappa_{ie}^{(1)}(Y(t^-))d\pi_e^{(1)}(t)
- \sum_{0<t_n<T_R} \Delta_{\pi_e^{(1)}}(Y(t_n)) \tag{4.53}
\]

or

\[
\phi_1(y) \geq E^y \left[ \int_0^{T_R} f_1(Y(t), v^{(1)}(Y(t)), \hat{v}^{(2)}(Y(t)))dt + \phi_1(T_R) \right]
- \int_0^{T_R} \sum_{i=1}^k \frac{\partial \phi_1}{\partial y_i}(Y(t^-)) \sum_{e=1}^p \kappa_{ie}^{(1)}(Y(t^-))d\pi_e^{(1)}(t)
- \sum_{0<t_n<T_R} \Delta_{\pi_e^{(1)}}(Y(t_n)) \tag{4.54}
\]

Applying the mean value theorem we obtain

\[
\Delta_{\pi_e^{(1)}}(Y(t_n)) = \sum_{i=1}^k \sum_{e=1}^p \frac{\partial \phi_1}{\partial y_i}(\bar{Y}(n))\kappa_{ie}^{(1)}(Y(t_n^-))\Delta\pi_e^{(1)}(t_n) \tag{4.55}
\]
where $\hat{Y}^{(n)}$ is some point on the line segment which joins $Y(t_n)$ and $Y(t_n^-) + \Delta_N Y(t_n)$, where $\Delta_N Y(.)$ denotes the change in value of $Y(.)$ caused by the Poisson random jumps at $t = t_n^-$. This yields

$$
\phi_1(y) \geq E_y\left[\int_0^{T_R} f_1(Y(t), v^1(Y(t)), \hat{v}^{(2)}(Y(t)))dt + \phi_1(Y(T_R))\right] - \sum_{e=1}^p \sum_{i=1}^k \left\{ \int_0^{T_R} \frac{\partial \phi_1}{\partial y_i}(Y(t^-)) \kappa_{ie}^{(1)}(Y(t^-)) d\pi_e^{(1)}(t) + \int_0^{T_R} \frac{\partial \phi}{\partial y_i}(\hat{Y}(n)) \kappa_{ie}^{(1)}(Y(t^-)) \Delta \pi_e^{(1)}(t_n) \right\}.
$$

(4.56)

Letting $R \to \infty$ we get

$$
\phi_1(y) \geq E_y\left[\int_0^{T_S} f_1(Y(t))dt + g_1(Y(\tau_S))\chi_{\{\tau_S < \infty\}}\right] + \int_0^{T_S} \phi^{(1)}_{e}(Y(t))d\pi_e^{(1)}(t) \right] = J_{\hat{\psi}^{(1)},\hat{\psi}^{(2)}}(y) \text{ for all } y \in S
$$

(4.57)

Since this holds for all $\psi^{(1)} \in Q^{(1)}$ we deduce that

$$
\phi_1(y) \geq \sup_{\psi^{(1)} \in Q^{(1)}} J_{\psi^{(1)},\hat{\psi}^{(2)}}(y).
$$

(4.58)

By applying similar reasoning to the control $(\hat{\psi}^{(1)}, \hat{\psi}^{(2)})$ we get

$$
\phi_1(y) = J_{\hat{\psi}^{(1)},\hat{\psi}^{(2)}}(y).
$$

(4.59)

Combining (4.58) and (4.59) we obtain

$$
\phi_1(y) \geq \sup_{\psi^{(1)} \in Q^{(1)}} J_{\psi^{(1)},\hat{\psi}^{(2)}}(y) = J_{\hat{\psi}^{(1)},\hat{\psi}^{(2)}}(y).
$$

(4.60)

as required in (4.50).

We can also prove statement (4.51) using analogous arguments.

Therefore, the point $(\hat{\psi}^{(1)}, \hat{\psi}^{(2)})$ is a Nash equilibrium and this completes the proof of Theorem 4.3.

The following example is an extension of Example 5.3 in [13] to the case with transaction costs.
Example 2

Assume that at any time $t$ the respective unit prices, $X_1(t), X_2(t)$, of two financial assets, are governed by the following equations

$$dX_1(t) = u_1(t)dt + \beta_{11}X_1(t)d\eta_1(t) + \beta_{12}X_1(t)d\eta_2(t); \quad X_1(0) = x_1 > 0 \quad (4.61)$$

$$dX_2(t) = u_2(t)dt + \beta_{21}X_2(t)d\eta_1(t) + \beta_{22}X_1(t)d\eta_2(t); \quad X_2(0) = x_2 > 0 \quad (4.62)$$

In this case $u_1(t), u_2(t)$ are trading strategies or controls applied by agents 1 and 2, respectively. We assume that $\beta_{ij}$ are constants, and $\eta_i(t) = \int_0^t \int_R z\tilde{N}_i(ds, dz)$ (i=1,2) are Levy martingale processes.

The performance functionals associated with assets 1 and 2, respectively, are defined as follows

$$J_1^{\psi(1),\psi(2)}(s, x_1, x_2) = E^{x_1, x_2} \left[ \int_s^T -\alpha_1 (1 - \delta)^2 u_1^2(t)X_2^2(t)dt + \gamma_1 (1 - \delta)^4 X_1^2(T)X_2^2(T) \right] \quad (4.63)$$

and

$$J_2^{\psi(1),\psi(1)}(s, x_1, x_2) = E^{x_1, x_2} \left[ \int_s^T -\alpha_2 (1 - \delta)^2 u_2^2(t)X_1^2(t)dt + \gamma_2 (1 - \delta)^4 X_1^2(T)X_2^2(T) \right]. \quad (4.64)$$

In this case $\alpha_i, \gamma_i > 0$, for $i = 1, 2$, and $\delta \in [0, 1]$. $\theta_i(t)$ represents investment rate in asset $i$, $(i = 1, 2)$.

For each financial asset the level of investment and activity of the other stimulates the general market in such a way that both the terminal payoff and the transaction cost is proportional to the size of the other. We now proceed to find the Nash equilibrium $(\hat{\psi}(1), \hat{\psi}(2))$. Since in this case $\pi^{(1)}(t) = \pi^{(2)}(t) = 0$ then the generator $\mathcal{L}$ of the controlled process $Y(t) = Y^{\psi}(t) = (s + t, X_1^{\psi(1),\psi(2)}(t), X_2^{\psi(1),\psi(2)}(t))$ with
\[ y = (s, x_1, x_2), \] is given by
\[
\mathcal{L}\phi(y) = \frac{\partial \phi}{\partial s} + u_1 \frac{\partial \phi}{\partial x_1} + u_2 \frac{\partial \phi}{\partial x_2} \\
+ \int_{\mathbb{R}} \left\{ \phi(s, x_1 + x_1 \beta_{11} z, x_2 + x_2 \beta_{21} z) - \phi(s, x_1, x_2) \right. \\
- x_1 \beta_{11} z \frac{\partial \phi}{\partial x_1} - x_2 \beta_{21} z \frac{\partial \phi}{\partial x_2} \} \nu_1(dz) \\
+ \int_{\mathbb{R}} \left\{ \phi(s, x_1 + x_1 \beta_{12} z, x_2 + x_2 \beta_{22} z) - \phi(s, x_1, x_2) \right. \\
- x_1 \beta_{12} z \frac{\partial \phi}{\partial x_1} - x_2 \beta_{22} z \frac{\partial \phi}{\partial x_2} \} \nu_2(dz)
\]

where \( \nu_i(dz) \) are the Levy measures of \( \eta_i(\cdot) \), respectively, \( i = 1, 2 \). In this example it is also not difficult to observe that
\[
\vartheta^{(1)} = 0, \quad f_1(s, x_1, x_2) = -\alpha_1 (1 - \delta)^2 u_2^2(t) x_2^2, \\
v^{(1)} = u_1, \quad g_1(s, x_1, x_2) = \gamma_1 (1 - \delta)^4 x_1^2 x_2^2,
\]
and
\[
\vartheta^{(2)} = 0, \quad f_2(s, x_1, x_2) = -\alpha_2 (1 - \delta)^2 u_2^2(t) x_1^2, \\
v^{(2)} = u_2, \quad g_2(s, x_1, x_2) = \gamma_2 (1 - \delta)^4 x_1^2 x_2^2.
\]

Now, fix \( u_2 \in \mathbb{R}, \ (s, x_1, x_2) \in \mathbb{R}^3 \) and maximize
\[
\mathcal{L}^{u_1,u_2}\phi_1(y) + f_1(y, u_1, u_2) \tag{4.65}
\]
with respect to \( u_1 \) for a given function \( \phi_1 \). This is equivalent to maximizing
\[
h_1(u_1) := u_1 \frac{\partial \phi_1}{\partial x_1} - \alpha_1 (1 - \delta)^2 u_1^2 x_2^2, \quad u_1 \in \mathbb{R}.
\]
We note that \( h_1(u_1) \) is quadratic and concave in \( u_1 \), so the maximum is attained at
\[
u_1 = \hat{u}_1 = \frac{1}{2} \alpha_1^{-1} (1 - \delta)^{-2} x_2^{-2} \frac{\partial \phi_1}{\partial x_1}, \tag{4.66}
\]
For this choice of \( u_1 \) and with \( u_2 = \hat{u}_2 \) we require that
\[
\mathcal{L}^{\hat{u}_1,\hat{u}_2}\phi_1(y) + f_1(y, \hat{u}_1, \hat{u}_2) = 0.
\]

99
Combining (4.67) and these last four equations we obtain

Thus

\[
\frac{\partial \phi_1}{\partial s} + \frac{1}{2} \alpha_1^{-1}(1-\delta)^{-2}x_2^{-2} \left( \frac{\partial \phi_1}{\partial x_1} \right)^2 + \hat{u}_2 \frac{\partial \phi_1}{\partial x_2} \\
+ \int_{\mathbb{R}} \left\{ \phi_1(s, x_1 + x_1 \beta_{11} z, x_2 + x_2 \beta_{21} z) - \phi_1(s, x_1, x_2) \right\} \nu_1(dz) \\
- x_1 \beta_{11} z \frac{\partial \phi_1}{\partial x_1} - x_2 \beta_{21} z \frac{\partial \phi_1}{\partial x_2} \right\} \nu_1(dz) \\
+ \int_{\mathbb{R}} \left\{ \phi_1(s, x_1 + x_1 \beta_{12} z, x_2 + x_2 \beta_{22} z) - \phi_1(s, x_1, x_2) \right\} \nu_2(dz) - \alpha_1(1-\delta)^2 \hat{u}_1^2(t) x_2^2 = 0.
\]

(4.67)

If we try functions \( \phi_i \) of the form

\[
\phi_i(s, x_1, x_2) = k_i(s)(1-\delta)^4 x_1^2 x_2^2; \quad i = 1, 2
\]

(4.68)

where \( k_i(s) \) are functions to be determined, we get

\[
\frac{\partial \phi_1}{\partial s} = k_1'(s)(1-\delta)^4 x_1^2 x_2^2, \quad \frac{\partial \phi_1}{\partial x_1} = 2k_1(s)(1-\delta)^4 x_1 x_2^2, \\
\frac{\partial \phi_1}{\partial x_2} = 2k_1(1-\delta)^4 x_1^2 x_2, \quad \hat{u}_1 = \alpha_1^{-1} k_1(s)(1-\delta)^2 x_1.
\]

Combining (4.67) and these last four equations we obtain

\[
k_1'(s)(1-\delta)^4 x_1^2 x_2^2 + 2\alpha_1^{-1} k_1^2(s)(1-\delta)^6 x_1^2 x_2^2 + 2k_1(s)(1-\delta)^4 \hat{u}_2 x_1^2 x_2 \\
+ \int_{\mathbb{R}} \left\{ k_1(s)(1-\delta)^4 (1 + \beta_{11} z)^2 (1 + \beta_{21} z)^2 x_1^2 x_2^2 - k_1(s)(1-\delta)^4 x_1^2 x_2^2 \right\} \nu_1(dz) \\
- 2k_1(s)(1-\delta)^4 \beta_{11} z x_1^2 x_2 - 2k_1(s)(1-\delta)^4 \beta_{21} z x_1^2 x_2 \nu_1(dz) \\
+ \int_{\mathbb{R}} \left\{ k_1(s)(1-\delta)^4 (1 + \beta_{12} z)^2 (1 + \beta_{22} z)^2 x_1^2 x_2^2 - k_1(s)(1-\delta)^4 x_1^2 x_2^2 \right\} \nu_2(dz) \\
- 2k_1(s)(1-\delta)^4 \beta_{12} z x_1^2 x_2 - 2k_1(s)(1-\delta)^4 \beta_{22} z x_1^2 x_2 \nu_2(dz) \\
- \alpha_1^{-1} k_1^2(s)(1-\delta)^6 x_1^2 x_2^2 = 0.
\]

(4.69)
Equation (4.69) simplifies to

\[ k_1'(s) + \alpha_1^{-1}k_1^2(s)(1 - \delta)^2 + 2k_1(s) \frac{\hat{u}_2}{x_2} + \int_R \{k_1(s)(1 + \beta_{11}z)^2(1 + \beta_{21}z)^2 - 1 - 2k_1(s)\beta_{11}z - 2k_1(s)\beta_{21}z\} \nu_1(dz) + \int_R \{k_1(s)(1 + \beta_{12}z)^2(1 + \beta_{22}z)^2 - 1 - 2k_1(s)\beta_{12}z - 2k_1(s)\beta_{22}z\} \nu_2(dz) = 0. \]  

(4.70)

Writing this last equation more compactly, we have

\[ k_1'(s) + \alpha_1^{-1}(1 - \delta)^2k_1^2(s) + 2k_1(s) \frac{\hat{u}_2}{x_2} + ak_1(s) = 0 \]  

(4.71)

where

\[ a = \sum_{j=1}^{2} \int_R z^2 \{4\beta_{1j}\beta_{2j} + \beta_{1j}^2 + \beta_{2j}^2 + z^2(2\beta_{1j}\beta_{2j} + 2\beta_{1j}^2) \nu_j(dz) \} \]  

(4.72)

(4.73)

By fixing \( u_1 \in \mathbb{R} \), and maximizing \( L^{u_1,u_2}\phi_2(y) + f_2(y, u_1, u_2) \) with respect to \( u_2 \) for a given function \( \phi_2 \) we find the maximum point, denoted by \( \hat{u}_2 \), is given by

\[ u_2 = \hat{u}_2 = \frac{1}{2} \alpha_1^{-1} x_1^{-2} \frac{\partial \phi_2}{\partial x_2} = \alpha_2^{-1} k_2(s) x_2. \]  

(4.74)

According to Theorem 4.3 we should have

\[ L^{\hat{u}_1,\hat{u}_2}\phi_2(y) + f_2(y, \hat{u}_1, \hat{u}_2) = 0. \]

This condition yields

\[ k_1'(s) + 2k_2(s) \frac{\hat{u}_1}{x_1} + \alpha_2^{-1}(1 - \delta)^2k_2(s) + ak_2(s) = 0 \]  

(4.75)

where \( a \) is once again defined by (4.73). Putting \( \alpha_1^{-1}k_1(s)(1 - \delta)^2x_1 \) for \( \hat{u}_1 \) in (4.75) yields

\[ k_2'(s) + 2\alpha_1^{-1}(1 - \delta)^2k_1(s)k_2(s) + \alpha_2^{-1}(1 - \delta)^2k_3(s) + ak_2(s) = 0 \]  

(4.76)
Similarly, substituting $\alpha_2^{-1}k_2(s)(1 - \delta)^2x_2$ for $\hat{u}_2$ in (4.71) yields

$$k_1'(s) + \alpha_1^{-1}(1 - \delta)^2k_1^2(s) + 2\alpha_2^{-1}(1 - \delta)^2k_1(s)k_2(s) + ak_1(s) = 0 \quad (4.77)$$

Equations (4.71) and (4.77) constitute a two dimensional system of Riccati equations. The system can be solved using standard methods for solving Riccati equations, provided the conditions for existence and uniqueness of solutions of Riccati equations are satisfied. For more details on Riccati equations the reader is referred to [46].
Chapter 5

Risk Minimization in Lévy Markets Using g-expectation

5.1 Introduction

In Chapter 4 we derived convex risk minimising portfolios for a firm that operates in Lévy markets characterised by friction. However, the question of how to choose an appropriate measure to quantify the risk was not addressed. In this chapter we investigate the problem of minimizing the convex risk using g-expectation in the context of Lévy markets. We extend and generalise the results in [25] and [43] to incomplete markets. Our main contribution is the five step scheme for solving forward backward stochastic differential equations (FBSDE’s) with a jump component. We also prove that whenever the five step scheme is realisable the solution obtained is unique.

Risk measures constitute axiomatic tools for a quantitative assessment of the riskiness of financial positions. Various classes of risk measures have been proposed and studied over the years, see for example, [3], [20], [25], [37], and [43]. [37] investigated the problem of risk minimization of terminal wealth and this lead to the notion of static risk measure. g-expectations are a relatively new mechanism for representing risk measures. For the diffusion case, g-expectations were introduced by [43] as particular ”nonlinear expectations” based on Backward Stochastic Differential Equations and depending on a functional g. Later on, interesting properties and applications of risk measures defined via g-expectations
for diffusions were studied in [25]. However, not much research has been done in
the area of g-expectation for jump diffusions.

5.2 Preliminary Results

This chapter is developed in the framework of a filtered complete probability
space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) satisfying the usual conditions. \(T > 0\) is a fixed time
horizon and \(\{B_t\}_{0 \leq t \leq T}\) is a d-dimensional standard Brownian motion with respect
to the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\).

Let \(J := \mathbb{R}^l \setminus \{0\}\) and denote by \(\mathcal{B} := \mathcal{B}(J)\) the Borel \(\sigma\)-algebra of subsets of \(J\).

Once again, we recall that \(\tilde{N}_j(dt, dz) = N_j(dt, dz) - dt \nu_j(dz); \ j = 1, 2, \ldots, l\) is a
compensated Poisson random measure where \(\nu_j(.)\) is a Levy measure associated
to the Poisson measure \(N_j(., .)\). We now define some spaces essential for this
chapter.

Define \(S^2(\mathcal{F}) = L^2(\Omega)\) to be the space of square integrable random variables \(Y_t\),
for \(t \in [0; T]\) such that
\[
\| Y \|_{S^2} := \sup_{0 \leq t \leq T} | Y_t | \|_{L^2(\Omega)} < \infty.
\]

Consider \(L^2(\mathcal{F})\) to be the set of \(\mathcal{F}\)-progressively measurable \(k \times d\)-dimensional
processes \(\{Z_t\}_{t \in [0; T]}\) such that
\[
\| Z \|_{L^2(\mathcal{F})} := \left( E \int_0^T | Z_t |^2 \, dt \right)^{1/2} < \infty.
\]

Let \(L^2(\tilde{N})\) be the set of mappings
\[
U : \Omega \times [0, T] \times \mathbb{R}^n \to \mathbb{R}^k
\]
which are \(\mathcal{P} \times \mathcal{B}(\mathbb{R}^n)\) -measurable such that
\[
\| U \|_{L^2(\tilde{N})} := \left( E \int_0^T \int_{\mathcal{B}(\mathbb{R}^n)} U_t^2(e) \lambda(de) \, dt \right)^{1/2} < \infty.
\]

Let \(\mathcal{M}\) be a family of probability measures \(Q = Q_\theta\) parametrized by processes
\(\theta = (\theta_0(t), \theta_1(t, z))\) such that
\[
dQ_\theta = Z_\theta(T)dP(\omega), \text{ on } \mathcal{F}_T
\]

104
where

$$Z_{\theta} = \exp\left[ -\int_{0}^{t} \theta_0(s)dB(s) - \frac{1}{2} \int_{\mathbb{R}} \theta_1^2(s)(ds) - \int_{0}^{t} \int_{\mathbb{R}} \log(1 - \theta_1(s,z))\tilde{N}(ds,dz) \right] + \int_{0}^{t} \int_{\mathbb{R}} \left\{ \log(1 - \theta_1(s,z)) + \theta_1(s,z) \right\} \nu(dz)ds ; 0 \leq t \leq T. \tag{5.1}$$

provided $\theta_1 < 1$ and

$$\int_{0}^{T} [\theta_0^2(t)dB(t) + \frac{1}{2} \int_{\mathbb{R}} \theta_1^2(s,z)\nu(dz)] < \infty; \ a.s.$$  

It is well known that if $E[Z_{\theta}(T)] = 1$ then $Q_{\theta}$ is an equivalent local martingale measure, see [42], for example.

We now present the formulation of the general problem.

### 5.3 Problem Formulation

Let $(X_t, Y_t) \in \mathbb{R}^n \times \mathbb{R}^k$ be a given stochastic process. Suppose that $X(t) := X^{\pi(t)}$ is a controlled process where $\pi(t)$ is an $\mathcal{F}_t$-adapted admissible Markov control, such as an investment portfolio. Assume that the process $(X_t, Y_t) \in \mathbb{R}^n \times \mathbb{R}^k$ satisfies the following forward-backward stochastic differential equation (FBSDE, for short)

$$dX_t = b(t, X_t, Y_t, Z_t)dt + \sigma(t, X_t, Y_t, Z_t)dB_t + \int_{\mathbb{R}^l} \gamma(t, X_{t-}, Y_{t-}, e)\tilde{N}(dt, de); \tag{5.2}$$

$$dY_t = g(t, X_t, Y_t, Z_t)dt + Z_tdB_t + \int_{\mathbb{R}^l} U(t, X_{t-}, Y_{t-}, e)\tilde{N}(dt, de); \tag{5.3}$$

$$Y_T = \Upsilon(X_T) \in \mathbb{R}^k; \quad X_0 = x \in \mathbb{R}^n; \tag{5.4}$$

where

$$b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^n$$

$$\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^{n \times d}$$

$$\gamma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^{n \times l}$$

$$g : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^k$$
are given functions, which are not random, satisfying the conditions for the existence and uniqueness of adapted solutions for (5.2)-(5.4).

We also adopt the following assumptions about \( g \)

\begin{enumerate}[(A1)]
    \item There exists a constant \( K > 0 \) such that for any \( t \in [0,T] \), and any \( x_0, x_1 \in \mathbb{R}^n, y_0, y_1 \in \mathbb{R}^k, z_0, z_1 \in \mathbb{R}^{k \times d}, \)

\[ |g_t(x_0, y_0, z_0) - g_t(x_1, y_1, z_1)| \leq K (|x_0 - x_1| + |y_0 - y_1| + ||z_0 - z_1||) \]

\item \( g(\cdot, x, y, z) \in L^2_T(\mathbb{R}; \mathbb{R}) \) for any \( (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \)

\item P-a.s., for any \( (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \) the mapping \( t \to g(t, x, y, z) \) is continuous in \( t \).
\end{enumerate}

In (5.2)-(5.4) \( X_t, Y_t, Z_t \) and \( U_t \) are the unknown processes which are supposed to be \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted.

\textbf{Definition 5.3.1} Let \( \Upsilon \in L^2(\mathcal{F}_T) \). Suppose \( (X_t, Y_t^\Upsilon, Z_t^\Upsilon, U_t^\Upsilon)_{t \in [0,T]} \in \mathbb{R}^n \times L^2(\Omega) \times \mathcal{L}(\mathcal{B}_t) \times \mathcal{L}(\tilde{\mathcal{N}}) \) is the solution of the forward-backward stochastic differential equations (5.2)-(5.4) with terminal and initial conditions \( Y_T = \Upsilon, X_0 = x \in \mathbb{R}^n \), respectively. The g-expectation of \( \Upsilon \), under the local equivalent martingale measure \( Q = Q_\theta \), hereafter denoted by \( \mathcal{E}_{Q,g}(\Upsilon) \), is defined by

\[ \mathcal{E}_{Q,g}(\Upsilon) := Y_0^\Upsilon. \] \hfill (5.5)

\textbf{Problem 5.3.1} The problem is to find the optimal control \( \pi^* \) that minimises \( Y_0^\Upsilon \).

It is well known that under suitable hypotheses on \( g \), the g-expectation induces a class of convex risk measures on the vector space of bounded financial positions, see for example, [20], [25] and [43]. However, to the best of our knowledge, none of the published work, so far, provides a mechanism for determining a solution
\((X_t, Y_t, Z_t, U_t)\) for the FBSDE (5.2)-(5.4), driven by a jump component, which minimises the convex risk measure induced by the \(g\)-expectation.

Before stating the main result we first introduce the five-step scheme for jump-diffusions which enables us to find an explicit solution for equations of the form (5.2)-(5.4). The five-step scheme is an extension of the four step-scheme, presented in [53], for forward backward stochastic differential equations.

### 5.3.2 The five-step scheme for jump diffusions

Suppose that \((X_t, Y_t, Z_t, U_t)\) is an adapted solution to (5.2)-(5.4)). Assume that \(X_t\) and \(Y_t\) are related by

\[ Y_t = (Y^1_t, \ldots, Y^k_t) = \theta(t, X_t) = (\theta^1(t, X_t), \ldots, \theta^k(t, X_t)); \quad \forall t \in [0, T]; \quad P - a.s. \]  

where \(\theta^r \in C^{1,2}([0, T] \times \mathbb{R}^n; \mathbb{R})\) is some function to be determined for \(r = 1, \ldots, k\).

Using Ito’s formula we obtain, for \(1 \leq r \leq k\) and \(1 \leq s \leq l\),

\[
dY^r_t = \left\{ \frac{\partial \theta^r}{\partial t}(t, X_t) + <\theta^r_x(t, X_t), b(t, X_t, \theta(t, X_t), Z_t)> + \frac{1}{2} \text{tr}[\theta^r_{xx}(t, X_t)(\sigma \sigma^T)(t, X_t, \theta(t, X_t), Z_t)] \right. \\
+ \sum_{s=1}^l \int_{\mathbb{R}} \{ \theta^r(t, X^-_t + \gamma^{(s)}(t, X^-_t, Y^-_t, e_s) - \theta^r(t, X^-_t) \\
- <\theta^r_x(t, X_t), \gamma^{(s)}(t, X^-_t, Y^-_t, e_s) > \} \nu_s(de_s) \} dt \\
+ \left. <\theta^r_x(t, X_t), \sigma(t, X_t, \theta(t, X_t), Z_t)dB_t > + \right. \\
+ \sum_{s=1}^l \int_{\mathbb{R}} \{ \theta^r(t, X^-_t + \gamma^{(s)}(t, X^-_t, Y^-_t, e_s) - \theta^r(t, X^-_t) \} \tilde{N}_s(de_s, dt). \]  

\[(5.7)\]

where \(\gamma^{(s)} \in \mathbb{R}^n\) is column number \(s\) of the \(n \times l\) matrix \(\gamma = [\gamma_{is}]\) and \(\gamma^{(s)}_i = \gamma_{is}\) is the coordinate number \(i\) of \(\gamma^{(s)}\). \(M^T\) denotes the transpose of matrix \(M\).
Comparing (5.3) and (5.7) it can be observed that

\begin{align}
& g^r(t, X_t, \theta(t, X_t), Z_t) \\
& = \frac{\partial \theta^r}{\partial t}(t, X_t) + <\theta^r_x(t, X_t), b(t, X_t, \theta(t, X_t), Z_t) > \\
& + \frac{1}{2} \text{tr} [\theta^r_{xx}(t, X_t)(\sigma \sigma^T)(t, X_t, \theta(t, X_t), Z_t)] \\
& + \sum_{s=1}^l \int_{\mathbb{R}} \{ \theta^r(t, X_{t^-} + \gamma(s)(t, X_{t^-}, Y_{t^-}, e_s)) - \theta^r(t, X_t) \\
& - <\theta^r_x(t, X_t), \gamma(s)(t, X_{t^-}, Y_{t^-}, e_s) > \} \nu_s(de_s)
\end{align}

(5.8)

\[
\theta^r(T, X_T) = \Upsilon^r(X_T)
\]

(5.9)

\[
Z(t) = \theta_x(t, X_t) \sigma(t, X_t, \theta(t, X_t), Z_t)
\]

(5.10)

\[
U^r(t, X_{t^-}, \theta(t, X_t), e) = \theta^r(t, X_t + \gamma(s)(t, X_{t^-}, \theta(t, X_t), e_s)) - \theta^r(t, X_t)
\]

(5.11)

From the above discussion we propose the following five-step scheme for solving the forward-backward stochastic differential equations (5.2)-(5.4).

**Step 1** Find \( z(t, x, y, p) \) satisfying the following

\[
z(t, x, y, p) = p\sigma(t, x, y, z(t, x, y, p));
\]

\[
\forall (t, x, y, p) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^{k \times n}
\]

(5.12)

**Step 2** Use the function \( z(t, x, y, p) \) obtained in **Step 1** to solve the following system of integro-differential equations, for \( r = 1, ..., k, \)

\[
\frac{\partial \theta^r}{\partial t}(t, x) + <\theta^r_x(t, x), b(t, x, \theta(t, x), z(t, x, \theta, \theta_x)) > \\
+ \frac{1}{2} \text{tr} [\theta^r_{xx}(t, X_t)(\sigma \sigma^T)(t, X_t, \theta(t, X_t), Z_t)] \\
+ \sum_{s=1}^l \int_{\mathbb{R}} \{ \theta(t, X_{t^-} + \gamma(s)(t, X_{t^-}, Y_{t^-}, e_s)) - \theta^r(t, X_t) \\
- <\theta^r_x(t, X_t), \gamma(s)(t, X_{t^-}, Y_{t^-}, e_s) > \} \nu_s(de_s)
- g(t, X_t, \theta(t, X_t), Z_t) = 0;
\]

108
Step 3 Using the functions $z(t, x, y, p)$ and $\theta(t, x)$, obtained in Steps 1 and 2, we determine $u(t, X_t, Y_t, e)$ using the following equation

\[
  u_t^s(t, x, y, e) = \theta^r(t, x + \gamma^s(t, x), \theta(t, x, e_s)) - \theta^r(t, x)
\]

(5.15)

Step 4 Use the functions $z$, $u$ and $\theta$ obtained in Steps 1-3 to solve the following forward stochastic differential equation

\[
  dX_t = \tilde{b}(t, X_t)dt + \tilde{\sigma}(t, X_t)dB_t \\
  + \int_{\mathbb{R}^l} \tilde{\gamma}(t, X_t, e)\tilde{N}(dt, de); \quad X_0 = x \in \mathbb{R}^n
\]

(5.16)

where

\[
  \tilde{b}(t, x) := b(t, x, \theta(t, x), z(t, x, \theta(t, x), \theta_x(t, x))
\]

(5.17)

\[
  \tilde{\sigma}(t, x) := \sigma(t, x, \theta(t, x), z(t, x, \theta(t, x), \theta_x(t, x)))
\]

(5.18)

\[
  \tilde{\gamma}(t, x) := \gamma(t, x, \theta(t, x), z(t, x, \theta(t, x), \theta_x(t, x)))
\]

(5.19)

Step 5 We now set

\[
  Y_t = \theta(t, X_t)
\]

(5.20)

\[
  Z_t = z(t, X_t, e, \theta(t, X_t), \theta_x(t, X_t))
\]

(5.21)

\[
  U_t = u(t, X_t, \theta(t, X_t), e)
\]

(5.22)
Should this scheme be realizable then the process \((X(.),Y(.),Z(.),U(.))\) would give an adapted solution of (5.2)-(5.4).

We now present our first main result

**Theorem 5.1** Suppose that the equation (5.12) admits a unique solution \(z(t,x,y,u,p)\) that is uniformly Lipschitz continuous in \((x,y,u,p)\) with \(z(t,0,0,0,0)\) being bounded and that (5.13) admits a solution \(\theta(t,x)\) with bounded \(\theta_t, \theta_x, \theta_{xx}\). Additionally, assume that the functions \(b(t,x,y,z); \sigma(t,x,y,z)\) and \(\gamma(t,x,y,e)\) satisfy the conditions for the existence and uniqueness of a solution for (5.2)-(5.4). Then the process \((X,Y,Z,U)\) determined by (5.16) and (5.20)-(5.22) is an adapted solution to the system of equations (5.2)-(5.4). Moreover, if \(g\) is also Lipschitz continuous in \((x,y,z)\) and the Lipschitz constant of \(z \to \sigma(x,y,z)\) denoted by \(L_\sigma \geq 0\) satisfies

\[
\sup_{(t,x) \in [0,T] \times \mathbb{R}^n} |\theta_x^l(t,x)| L_\sigma < 1, \quad 1 \leq l \leq k \quad (5.23)
\]

then the adapted solution is unique and is determined by (5.16) and (5.20)-(5.22).

**Proof**

Under the assumptions adopted in the theorem, it can be verified that the functions \(\tilde{\theta}(t,x), \tilde{\sigma}(t,x)\) and \(\tilde{\gamma}(t,x)\) defined by (5.17)-(5.19) are uniformly Lipschitz in \(x\). Therefore, for any \(x \in \mathbb{R}^n\), (5.16) has a unique strong solution. By defining \((X,Y,Z,U)\) via (5.16) and (5.20)-(5.22), and applying Ito’s formula for jump-diffusions, we can also establish that (5.2)-(5.4) is satisfied.

Thus \((X,Y,Z,U)\) is a solution of (5.2)-(5.4). We now have to prove uniqueness. We claim that any solution \((X,Y,Z,U)\) of (5.2)-(5.4) must be of the form we constructed using the five step scheme. To prove this claim we let \((X,Y,Z,U)\) be a solution of (5.2)-(5.4). Define

\[
\tilde{Y}_t := \theta(t,X_t), \quad \tilde{Z}_t := z(t,X_t,\theta(t,X_t), \frac{\partial \theta^r}{\partial t}(t,X_t)) \quad (5.24)
\]

By assumption, (5.12) admits a unique solution. Thus, it follows from 5.21

\[
\tilde{Z}_t := \theta_x(t,X_t)\sigma(t,X_t,\tilde{Y}_t,\tilde{Z}_t,\tilde{U}_t) \quad \forall t \in [0,T] \quad (5.25)
\]
Applying Ito’s formula to $\theta(t, X_t)$ and noting (5.12) and (5.20)-(5.22) we have the following

\[
d\tilde{Y}_t^r = \left\{ \frac{\partial \theta^r}{\partial t}(t, X_t) + <\theta^r_x(t, X_t), b(t, X_t, \theta(t, X_t), Z_t) > 
+ \frac{1}{2} \text{tr}[\theta^r_{xx}(t, X_t)(\sigma \sigma^T)(t, X_t, \theta(t, X_t), Z_t)] 
+ \sum_{s=1}^l \int_R \{ \theta(t, X_t - + \gamma^s(t, X_{t-}, Y_{t-}, e_s)) - \theta(t, X_{t-}) 
- <\theta^r_x(t, X_t), \gamma^s(t, X_{t-}, Y_{t-}, e_s) > } \nu_s(de_s) \right\} dt 
+ \sum_{s=1}^l \int_R \{ \theta(t, X_t - + \gamma^s(t, X_{t-}, Y_{t-}, e_s)) - \theta(t, X_{t-}) \} \tilde{N}_s(de_s, dt) 
= \left\{ \frac{\partial \theta^r}{\partial t}(t, X_t), b(t, X_t, \tilde{Y}_t, \tilde{Z}_t) > 
+ \frac{1}{2} \text{tr}[\theta^r_{xx}(t, X_t)\left\{ (\sigma \sigma^T)(t, X_t, \tilde{Y}_t, \tilde{Z}_t) - (\sigma \sigma^T)(t, X_t, \tilde{Y}_t, \tilde{Z}_t) \right\} 
+ \sum_{s=1}^l \int_R \{ \theta(t, X_t - + \gamma^s(t, X_{t-}, Y_{t-}, e_s)) - \theta(t, X_{t-} + \gamma^s(t, X_{t-}, \tilde{Y}_{t-}, e_s)) 
- <\theta^r_x(t, X_t), \gamma^s(t, X_{t-}, \tilde{Y}_{t-}, e_s) > - \gamma^s(t, X_{t-}, \tilde{Y}_{t-}, e_s) > } \nu_s(de_s) 
+ \gamma^s(t, X_{t-}, \tilde{Y}_{t-}, \tilde{Z}_t) \right\} dt 
+ \sum_{s=1}^l \int_R \{ \theta(t, X_t - + \gamma^s(t, X_{t-}, Y_{t-}, e_s)) - \theta(t, X_{t-}) \} \tilde{N}_s(de_s, dt). \right\} 
\]  
(5.26)
Combining (5.27) and (5.28) we obtain

\[
dY_t^r = \left\{ \frac{\partial \theta^r}{\partial t}(t, X_t) + < \theta_s(t, X_t), b(t, X_t, \theta(t, X_t), Z_t) > \right. \\
+ \left. \frac{1}{2} \text{tr}\left[ \theta_{xx}^r(t, X_t)(\sigma \sigma^T)(t, X_t, \theta^r(t, X_t), Z_t) \right] \right. \\
+ \left. \sum_{s=1}^l \int_{\mathbb{R}} \{ \theta^r(t, X_{t-} + \gamma^{(s)}(t, X_{t-}, Y_{t-}, e_s)) - \theta^r(t, X_{t-}) \right. \\
- \left. \sum_{i=1}^n \gamma_i^{(s)}(t, X_{t-}, Y_{t-}, e_s) \frac{\partial \theta^r}{\partial x_i}(t, X_t) \} \nu_s(de_s) \} dt \\
+ \left. < \theta_{x}^r(t, X_t), \sigma(t, X_t, \theta(t, X_t), Z_t) dB_t > \right. \\
+ \left. \sum_{s=1}^l \int_{\mathbb{R}} \{ \theta^r(t, X_{t-} + \gamma^{(s)}(t, X_{t-}, Y_{t-}, e_s)) - \theta^r(t, X_{t-}) \} \tilde{N}_s(de_s, dt) \right. \\
\]

(5.28)

Combining (5.27) and (5.28) we obtain

\[
E \mid Y_t - \tilde{Y}_t \mid^2 \\
= E \int_t^\infty \sum_{r=1}^k \sum_{i=1}^n \left\{ 2(Y_t^r - \tilde{Y}_t^r) \left\{ \sum_{i=1}^n [b_i(t, X_t, Y_t, Z_t) - b_i(t, X_t, \tilde{Y}_t, \tilde{Z}_t)] \frac{\partial \theta^r}{\partial x_i}(t, X_t) \right. \\
+ \frac{1}{2} \sum_{i,j=1}^n \left[ (\sigma \sigma^T)_{i,j}(t, X_t, Y_t, Z_t) - (\sigma \sigma^T)_{i,j}(t, X_t, \tilde{Y}_t, \tilde{Z}_t) \right] \frac{\partial^2 \theta^r}{\partial x_i \partial x_j}(t, X_t) \right. \\
+ \left. \sum_{i=1}^n \left( \theta(t, X_{t-} + \gamma^{(s)}(t, X_{t-}, Y_{t-}, e_s)) - \theta(t, X_{t-}) + \gamma^{(s)}(t, X_{t-}, \tilde{Y}_{t-}, e_s) \right) \right. \\
- \left. \sum_{i=1}^n \left[ \gamma_i^{(s)}(t, X_{t-}, Y_{t-}, e_s) - \gamma_i^{(s)}(t, X_{t-}, \tilde{Y}_{t-}, e_s) \right] \frac{\partial \theta^r}{\partial x_i}(t, X_t) \} \nu_s(de_s) \right. \\
+ \left. g^r(t, X_t, \tilde{Y}_t, \tilde{Z}_t) - g^r(t, X_t, Y_t, Z_t) \} \} dt \\
+ \left. \text{tr}\left[ \left( \sigma(t, X_t, Y_t, Z_t) - \sigma(t, X_t, \tilde{Y}_t, \tilde{Z}_t) \right)^T M + \tilde{Z} - Z \right] \right. \\
+ \left. \left( \left( \sigma(t, X_t, Y_t, Z_t) - \sigma(t, X_t, \tilde{Y}_t, \tilde{Z}_t) \right)^T M + \tilde{Z} - Z \right)^T \right. \\
+ \left. \sum_{i=1}^l \int_{\mathbb{R}} \{ \theta^r(t, X_{t-} + \gamma^{(s)}(t, X_{t-}, Y_{t-}, e_s)) - \theta^r(t, X_{t-}) + \gamma^{(s)}(t, X_{t-}, \tilde{Y}_{t-}, e_s) \} \tilde{U}^r - U^r \right. \\
\nu_s(de_s) \}
\]

where

\[
M^r = \left( \frac{\partial \theta^r}{\partial x_i}; \ldots; \frac{\partial^2 \theta^r}{\partial x_i \partial x_j} \right)^T.
\]

Hence, by the boundedness of \( \frac{\partial \theta^r}{\partial x_i} \), \( \frac{\partial^2 \theta^r}{\partial x_i \partial x_j} \) for \( i, j = 1, \ldots, n \), and the uniform

112
Lipschitz continuity of \( b, \sigma, g \) and \( \gamma \), we have
\[
E | Y_t - \tilde{Y}_t |^2 + \int_t^T E | Z_\tau - \tilde{Z}_\tau |^2 \, d\tau + \int_t^T \int_\mathbb{R} E | U_\tau - \tilde{U}_\tau |^2 \, \nu(d\tau) \, d\tau \\
\leq \int_t^T | Y_\tau - \tilde{Y}_\tau | \left( K_1 | Y_\tau - \tilde{Y}_\tau | + K_1 | Z_\tau - \tilde{Z}_\tau | + K_2 \| U_\tau - \tilde{U}_\tau \| \right) \, d\tau \\
\leq (K_1 + K_1 \alpha^2 + K_2 \beta^2) E \int_t^T | Y_\tau - \tilde{Y}_\tau |^2 \, d\tau + \frac{K_1}{\alpha^2} E \int_t^T | Z_\tau - \tilde{Z}_\tau |^2 \, d\tau \\
+ \frac{K_1}{\beta^2} E \int_t^T | Z_\tau - \tilde{Z}_\tau |^2 \, d\tau
\]
where \( \alpha \) and \( \beta \) are constants. Choosing
\[
\frac{K_1}{\alpha^2} = \frac{3}{4} = \frac{K_1}{\beta^2}
\]
we obtain
\[
E | Y_t - \tilde{Y}_t |^2 + \frac{1}{4} \int_t^T E | Z_\tau - \tilde{Z}_\tau |^2 \, d\tau + \frac{1}{4} \int_t^T \int_\mathbb{R} E | U_\tau - \tilde{U}_\tau |^2 \, \nu(d\tau) \, d\tau \\
\leq (K_1 + \frac{3}{4} K_1^2 + \frac{3}{4} K_2^2) \int_t^T | Y_\tau - \tilde{Y}_\tau |^2 \, d\tau
\]
(5.29)
Using this estimate and Gronwall’s lemma we get
\[
Y_t = \tilde{Y}_t, \quad Z_t = \tilde{Z}_t \quad \text{and} \quad U_t = \tilde{U}_t. \tag{5.30}
\]
This proves that the solution of (5.2)-(5.4) must have the form that is constructed through the five-step scheme. To establish uniqueness of solution we let \((X_t, Y_t, Z_t, U_t)\) and \((\tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t, \tilde{U}_t)\) be two solutions of (5.2)-(5.4). Using previous arguments we have
\[
Y_t = \theta(t, X_t), \quad Z_t = z(t, X_t, \theta(t, X_t), \theta_x(t, X_t)), \quad U_t = u(t, x, \theta(t, x), e) \tag{5.31}
\]
It follows that \(X_t\) and \(\tilde{X}_t\) satisfy the same forward stochastic differential equation (16) with the same initial state \(x\). By Theorem 19 in [42] we must have
\[
X_t = \tilde{X}_t \quad \text{for all} \quad t \in [0, T], \quad P \text{ a.s.} \tag{5.32}
\]
which in turn shows that
\[
Y_t = \tilde{Y}_t, \quad Z_t = \tilde{Z}_t, \quad U_t = \tilde{U}_t \quad \text{for all} \quad t \in [0, T], \quad P \text{ a.s.} \tag{5.33}
\]
This completes the proof.
5.3.3 Application: Pricing of European call option in the presence of jumps

Consider the following Lévy version of the Black-Scholes market

(a) A safe asset whose unit price $X_0(t)$, at time $t$, evolves according to the following equation:

$$dX_0(t) = rX_0(t)dt; \quad X_0(0) = x_0 > 0 \quad t \in [0,T]; \quad (5.34)$$

(b) A risky asset whose unit price $X(t)$, at time $t$, is governed by the following stochastic differential equation:

$$dX(t) = X(t^-)[\eta dt + \sigma dB(t) + \int_\mathbb{R} \gamma e^{\tilde{N}(dt, de)}]; \quad X(0) = x > 0 \quad t \in [0,T]; \quad (5.35)$$

where $r$, $\eta$, $\sigma$ and $\gamma$ are constants. Suppose that at time $t$ an agent has a total amount of money denoted by $Y(t)$ available for investment. Of this total sum of money the agent invests an amount denoted by $\pi(t)$ in the risky asset, so that the remainder, $Y(t) - \pi(t)$, is invested in the safe asset. Suppose that by splitting his money in that way, the agent manages to buy $N_0(t)$ units from the safe asset and $N(t)$ units from the risky asset. As before, let $X_0(t)$ be the price of one unit of the safe asset and $X(t)$ be the price of one unit of the risky asset. The total amount of money, $Y(t)$, is then given by

$$Y(t) = N_0(t)X_0(t) + N(t)X(t). \quad (5.36)$$

If there are no interventions then $Y(t)$ satisfies

$$dY(t) = N_0(t)dX_0(t) + N(t)dX(t) = rN_0(t)X_0(t)dt + N(t)X(t^-)[\eta dt + \sigma dB(t) + \int_\mathbb{R} \gamma e^{\tilde{N}(dt, de)}]; \quad X(0) = x > 0; \quad t \in [0,T]$$

$$= [rY(t) + (\eta - r)\pi(t)]dt + \sigma \pi(t)dB(t) + \pi(t^-)\int_\mathbb{R} \gamma e^{\tilde{N}(dt, de)}]; \quad X(0) = x > 0; \quad t \in [0,T]$$

114
By setting $Z(t) = \sigma \pi(t)$ we obtain the following FBSDE

$$dX(t) = X(t^-)[\eta dt + \sigma dB(t) + \int_{\mathbb{R}} \gamma e^{\tilde{N}(dt,de)}]; \quad (5.38)$$

$$dY(t) = \left[ rY(t) + (\eta - r)\frac{Z(t)}{\sigma} \right]dt + Z(t)dB(t) + \frac{Z(t^-)}{\sigma} \int_{\mathbb{R}} \gamma e^{\tilde{N}(dt,de)}; \quad (5.39)$$

$$X(0) = x; \quad Y(T) = (X(T) - K)^+. \quad (5.40)$$

where $(x - q)^+ := \max\{x - q, 0\}$. The above is a decoupled FBSDE. We obtain the result.

**Theorem 5.2** There exists a unique adapted solution $(X(\cdot), Y(\cdot), Z(\cdot), U(\cdot))$ to $(5.38)-(5.40)$. The option price is given by

$$\mathcal{E}_{Q,\gamma}(Y(T)) = \mathcal{E}_{Q,\gamma}((X(T) - K)^+), \quad (5.41)$$

and the portfolio $\pi(t)$ is given by

$$\pi(t) = \frac{Z(t)}{\sigma} \text{ for all } t \in [0, T]. \quad (5.42)$$

We now apply the five-step scheme to achieve the derive an explicit solve the FBSDE.

**Step 1** Set

$$z(t, x, y, u, p) = px\sigma; \quad \forall (t, x, y, u) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}. \quad (5.43)$$
Step 2 Using the function \( z(t,x,y,u,p) \) obtained in Step 1 we solve the following parabolic integro-differential equation
\[
\theta_t + \eta x \theta_x + \frac{1}{2} \sigma^2 x^2 \theta_{xx} + \int_{\mathbb{R}} \left\{ \theta(t,x + \gamma e x) - \theta(t,x) - \gamma xe \theta_x \right\} \nu(de) \\
- \left[ r \theta + (\eta - r) \frac{\sigma x \theta_x}{\sigma} \right] = 0, \quad (t,x) \in (0,T) \times \mathbb{R};
\]
\[(5.44)\]
\[
\theta(T,x) = \Upsilon(x), \ x \in \mathbb{R},
\]
\[(5.45)\]
or
\[
\theta_t + \frac{1}{2} \sigma^2 x^2 \theta_{xx} + rx \theta_x - r \theta \\
+ \int_{\mathbb{R}} \left\{ \theta(t,x + \gamma e x) - \theta(t,x) - \gamma xe \theta_x \right\} \nu(de) = 0,
\]
\[(5.46)\]
\[
\theta(T,x) = \Upsilon(x), \ x \in \mathbb{R}
\]
\[(5.47)\]

Step 3 Use the functions \( z(t,x,y,u,p) \) and \( \theta(t,x) \) obtained in Steps 1 and 2 to solve the following equation
\[
\theta(t,x + \gamma e x) - \theta(t,x) - u(t,e,x) = 0
\]
\[(5.48)\]

Step 4 Now, solving the equation
\[
dX(t) = X(t^-)[\eta dt + \sigma dB(t) + \int_{\mathbb{R}} \gamma e \tilde{N}(dt,de)]; \ X(0) = x;
\]
\[(5.49)\]
we get
\[
X(t) = x \exp\left\{ \left( \eta - \frac{1}{2} \sigma^2 \right) t + \eta B(t) + \int_0^t \int_{\mathbb{R}-\{0\}} \ln(1 + \gamma e) - \gamma e \} \nu(dz) ds \\
+ \int_0^t \int_{\mathbb{R}} \ln(1 + \gamma e) \tilde{N}(ds,ds) \right\}
\]
\[(5.50)\]

Step 5 We now set
\[
Y(t) = \theta(t,X(t)) \quad (5.51)
\]
\[
Z(t) = z(t,X(t),\theta(t,X(t)),\theta_x(t,X(t)),u(t,X(t))) \quad (5.52)
\]
\[
U(t) = u(t,e,X(t)) \quad (5.53)
\]
Chapter 6

Conclusion

This thesis makes several important contributions to the development of stochastic control for jump diffusions with applications to finance and insurance. The starting point is to assume that in the absence of interventions the financial reserves of a firm evolve according to a stochastic differential equation with a jump component. Control variables, namely, dividend policy, reinsurance and investment portfolios, are subsequently introduced at appropriate stages in the development of the thesis. The introduction of these control variables generates complex and interesting stochastic control problems. In order to investigate these control problems the thesis develops new theoretical frameworks in the areas of combined stochastic control theory, stochastic differential game theory and forward-backward stochastic differential equations.

Chapter 1 is a review of existing results in risk and dividend control theory. It is observed that the problem of combined impulse and singular control for jump diffusions has not been investigated before. The general risk/dividend research problem for Lévy markets is presented in this first chapter. As the motivating market we present a generalised model for the evolution of an insurance company’s reserves that are controlled through reinsurance and dividend payout where the dividends are subjected to both fixed and proportional transaction costs. The model extends the one used in [27] and [51] for optimal combined risk and dividend control. Our model is a more realistic description of the dynamics of an insurance company since it caters for significant sudden changes in the value of the liquid reserves and it also considers transaction costs.
In Chapter 2 the focus is on the simplified version of the risk dividend research problem. This simplified case considers that the company does not apply reinsurance but pays out dividends in the presence of fixed and proportional transaction costs. A verification theorem for combined singular-impulse control is derived, proved and applied to solve the insurance problem. Using this verification theorem and a "chattering" strategy we obtain new results presented in form of Theorem 2.2.

Proportional reinsurance is introduced into the problem in Chapter 3. This version of the problem considers that the insurance company applies proportional reinsurance and also pays out dividends that are subjected to proportional transaction costs only. A verification theorem for combined regular-singular control is formulated and proved. As in Chapter 2 the verification theorem is also applied to solve the insurance problem. In this case the value function is similar to the one obtained in Chapter 2. The optimal proportional reinsurance retention level is given as an implicit solution of an integro-differential equation.

Chapter 4 focuses on the game theory with singular control in incomplete markets. The market model describes the evolution of the consolidated financial position of an insurance company that are also invests on the generalised Black-Scholes market. The risk minimization problem is first presented and examined as a two-player zero-sum stochastic differential game, where the first player is the market and the second player is the insurance company. The equivalent probability measure $Q$ is the control for player number 1 (the market/nature) and the portfolio $\pi(t)$ is the control for player number 2 (the agent). The major contribution of this chapter is that for the first time in the literature, we consider singular control in the theory of stochastic differential games for jump diffusions. Our next interesting result in this area is the Hamilton-Jacobi-Bellman-Isaacs variational inequalities (HJBIVI, for short). This class of generalised variational inequalities for the singular control zero-sum stochastic differential game problem is used to examine an example. The HJBIVI is later on extended to the non-zero-sum case, to yield the next important result, that is, the HJBIVI for Nash equilibrium. The HJBIVI for Nash equilibrium is also applied to a practical problem.

The question of minimising convex risk of terminal wealth by means of $g$-expectation in Lévy markets is examined in Chapter 5. In this chapter the main contribution is the five-step-scheme for solving forward backward differential equations with
a jump component. The five-step-scheme culminates in Theorem 5.1. The five-step-scheme is applied to European call option in a jump-diffusion model. In this example, the choice of $g$ enables the investor to replicate the option and thereby obtain a risk-neutral price. This leads to an interesting and surprising result that call options can be replicated in incomplete markets.
References

   **URL**: http://www.math.uio.no/eprint/puremath/2001/02-01.pdf


[28] Hipp, C. (n.d); Stochastic Control with Applications in Insurance.


