

Average Distance and Edge-Connectivity I

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Abstract

The average distance $\mu(G)$ of a connected graph G of order n is the average of the distances between all pairs of vertices of G . We prove that if G is a λ -edge-connected graph of order n , then the bounds $\mu(G) \leq 2n/15 + 9$, if $\lambda = 5, 6$, $\mu(G) \leq n/9 + 10$, if $\lambda = 7$ and $\mu(G) \leq n/(\lambda + 1) + 5$, if $\lambda \geq 8$ hold. Our bounds are shown to be best possible and our results solve a problem of Plesník [12].

1 Introduction

Let $G = (V, E)$ be a connected graph of order n . The *average distance* $\mu(G)$ of G is defined as $\mu(G) = \binom{n}{2}^{-1} \sum_{\{x,y\} \subset V} d_G(x,y)$, where $d_G(x,y)$ denotes the distance between the vertices x and y in G . The average distance has been investigated by several authors and under various names; for instance, Kouider and Winkler [8], Favaron, Kouider, Mahéo [6], Doyle and Graver [4] depict average distance as the *mean distance*; Wuchty and Stadler [16] use the term *average path length*; Bermond, Liu and Syska [1] use the term *mean eccentricity*; whilst Plesník [12], Soltés [14], work with the total distance of a graph and call it the *transmission*. The transmission differs from the average distance by only a factor of $n(n-1)$. The study of average distance seems to have been started by the chemist Wiener [15] in 1947, when he introduced the Wiener index $W(G)$ of a graph G which differs from the average distance by a factor of $\binom{n}{2}$, i.e., $W(G) = \binom{n}{2} \mu(G)$. The applications of the Wiener index in physical chemistry are legion (see for

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example [7, 10, 13] for references); it has been used in the characterization of many different types of chemical species, including alkanes, alkenes and arenes. The Wiener index is also a tool for designing quantitative structure-property relations in organic chemistry.

Several upper and lower bounds on the average distance in terms of other graph parameters are known. The *edge-connectivity* $\lambda(G)$ of G is the minimum number of edges whose deletion from G results in a disconnected or trivial graph. We say G is *k-edge-connected* if $\lambda(G) \geq k$. It has been proved (see for example, [9, 4, 5]) that the average distance of a connected graph with n vertices is at most $(n+1)/3$ and that this bound is attained only by the path of order n . Plesník [12] showed that this bound can be improved for a 2-edge-connected (2-vertex-connected) graph G of order n to $\mu(G) \leq \lfloor \frac{n^2}{4} \rfloor / (n-1)$ and that this bound is achieved if and only if G is a cycle. He posed the problem of finding upper bounds on the average distance of a graph of given order and edge-connectivity. The goal of this paper is to find asymptotically sharp upper bounds on $\mu(G)$ for a graph G of given order and edge-connectivity λ . For $\lambda \geq 8$, we obtain bounds as an immediate consequence of known bounds on the average distance in terms of order and minimum degree. For small values of λ , $\lambda \in \{3, \dots, 7\}$, the problem requires a more intricate analysis. In this paper we give asymptotically sharp bounds on μ for $\lambda \in \{5, 6, 7\}$. The cases $\lambda \in \{3, 4\}$ will be considered in a subsequent paper [3].

We use the following notation. Let $G = (V, E)$ be a connected simple graph. For a subset $S \subseteq V$, $G[S]$ denotes the subgraph induced by S in G and $\text{diam}(S)$ is the maximum value of $d_G(x, y)$, $x, y \in S$. Let $v \in V$. The *eccentricity* $ex_G(v)$ of v is the maximum value of $d_G(v, x)$, $x \in V$. The *distance* $\sigma_G(v)$ of v is the sum $\sum_{x \in V} d_G(v, x)$ whereas the *distance* $\sigma(G)$ of G is $\sum_{u \in V} \sigma_G(u) = \sum_{(x, y) \in V \times V} d_G(x, y)$. Thus $\mu(G) = \sigma(G)/n(n-1)$, where n is the order of G . For subsets $M_1, M_2 \subseteq V$, the distance $\sigma_{M_1}(v)$ of v with respect to M_1 is the sum $\sum_{x \in M_1} d_G(v, x)$ and the distance $\sigma_{M_1}(M_2)$ of M_2 with respect to M_1 is $\sum_{x \in M_2} \sigma_{M_1}(x)$. $E(M_1, M_2)$ denotes the set of edges $\{ab \in E \mid a \in M_1, b \in M_2\}$. $N_G(v) = N(v)$ denotes the set of all vertices adjacent to v in G and its cardinality is the degree of v in G and is denoted by $\text{deg}_G(v)$. We denote the closed neighbourhood of v by $N[v]$, i.e., $N[v] = N(v) \cup \{v\}$. For a positive integer i , the *i-th distance layer* $N_i(v)$ of v is the set of vertices at distance i from v , i.e., $N_i(v) = \{x \in V \mid d_G(v, x) = i\}$. We denote the cardinality of $N_i(v)$ by $k_i(v)$ and if the vertex v is understood we omit the argument v and simply write N_i, k_i . We denote the set $\cup_{0 \leq j \leq i} N_j(v)$ by $N_{\leq i}(v)$ and the set $\cup_{j \geq i} N_j(v)$ by $N_{\geq i}(v)$.

A *disconnecting (separating) set* of G is a set of edges (vertices) of G whose removal increases the number of components of G . A *vertex cutset* of

S is a separating set, S say, such that no proper subset of S is a separating set of G . A vertex v of G is a *cutvertex* if $\{v\}$ is a separating set of G . A maximal subgraph of G with no cutvertex is called a *block* and an *end block* of G is a block of G that contains exactly one cutvertex of G . For vertex disjoint graphs G_1, G_2, \dots, G_k , the *sequential join* $G_1 + G_2 + \dots + G_k$ is the graph obtained from the union of G_1, \dots, G_k by joining every vertex of G_i to every vertex of G_{i+1} for $i = 1, 2, \dots, k-1$.

2 Results on 8-Edge-Connected Graphs

Dankelmann and Entringer [2] proved that if G is a connected graph of order n and minimum degree δ , then $\mu(G) \leq n/(\delta+1) + 5$. An application of the well known inequality $\lambda \leq \delta$ to this bound yields part (a) of the following proposition.

Proposition 1 *Let G be a λ -edge-connected graph of order n .*

- (a) *Then $\mu(G) \leq \frac{n}{\lambda+1} + 5$.*
- (b) *The bound in (a), apart from the additive constant, is best possible for $\lambda \geq 8$.*

Proof: (b) Assume that $\lambda \geq 8$. We show that the bound is sharp apart from the exact value of the additive constant. Given integers n, λ and k with $n = \frac{1}{3}(\lambda+1)(k+2)$, $k \equiv 1 \pmod{3}$, $k \geq 7$ let

$$G_{n,\lambda} = G_1 + G_2 + G_3 + \dots + G_k,$$

where $G_1 = K_1 = G_k$, $G_2 = K_\lambda = G_{k-1}$ and for $3 \leq i \leq k-2$, $G_i =$

$$\begin{cases} K_{\frac{\lambda+1}{3}} & \text{if } \lambda \equiv 2 \pmod{3}, \\ K_{\frac{\lambda}{3}} \text{ for } i \equiv 0, 2 \pmod{3} \text{ and } K_{\frac{\lambda}{3}+1} \text{ for } i \equiv 1 \pmod{3} & \text{if } \lambda \equiv 0 \pmod{3}, \\ K_{\frac{\lambda+2}{3}} \text{ for } i \equiv 0, 2 \pmod{3} \text{ and } K_{\frac{\lambda-1}{3}} \text{ for } i \equiv 1 \pmod{3} & \text{if } \lambda \equiv 1 \pmod{3}. \end{cases}$$

Clearly, since $\lambda \geq 8$, $G_{n,\lambda}$ is λ -edge-connected. It is easy to show that, for constant λ and large n , $\mu(G_{n,\lambda}) \geq \frac{n}{\lambda+1} + O(1)$. \square

3 Results on 5-Edge-Connected Graphs

Lemma 1 *Let G be a 5-edge-connected graph with n vertices and let v be any vertex of G . Then $\sigma_G(v) \leq \frac{1}{5}(n^2 - 4n + 14)$. Moreover, $\sigma(G) \leq \frac{n}{5}(n^2 - 4n + 14)$.*

Proof: Let e be the eccentricity of v and note that

$$\sigma_G(v) = 1k_1 + 2k_2 + \cdots + ek_e. \quad (1)$$

The lemma can easily be verified for $e \leq 4$. Assume that $e \geq 5$. Since G is 5-edge-connected, $k_i k_{i+1} \geq |E(N_i, N_{i+1})| \geq \lambda(G) \geq 5$. From $k_i k_{i+1} \geq 5$ we deduce $k_i + k_{i+1} \geq 5$. Hence it is easy to see that

$$k_i k_{i+1} \geq 5, k_i + k_{i+1} \geq 5 \text{ for } i = 0, 1, \dots, e-1, k_{e-1} + k_e \geq 6, \sum_{i=0}^e k_i = n. \quad (*)$$

Let (a_0, a_1, \dots, a_e) be a sequence that maximizes (1) subject to (*). We first show that we can assume that $a_i > 1$ for $i = 2, \dots, e-1$. If $a_2 = 1$, then since $e \geq 5$ and $a_3 \geq 5$, the sequence (b_0, b_1, \dots, b_e) given by $b_2 = 2$, $b_3 = a_3 - 2$, $b_4 = a_4 + 1$, and $b_i = a_i$ otherwise, also maximizes (1) subject to (*). Hence we can assume that $a_2 > 1$. Now we show that $a_i > 1$ for $i = 3, \dots, e-1$. If this were false, let s , $3 \leq s \leq e-1$, be the smallest integer for which $a_s = 1$. Let (b_0, b_1, \dots, b_e) be the sequence given by (i) $b_i = 3$ if $i = s-1$, (ii) $b_i = 2$ if $i = s$, (iii) $b_i = a_{s-1} + a_{s+1} - 4$ if $i = s+1$, and (vi) $b_i = a_i$ otherwise. By (*), $a_{s-1}, a_{s+1} \geq 5$. This, in conjunction with $b_{s-2} = a_{s-2} \geq 2$, yields that the sequence (b_i) satisfies (*), and, moreover $\sum_{i=0}^e i a_i = \sum_{i=0}^e i b_i + 7 - 2a_{s-1} < \sum_{i=0}^e i b_i$, contradicting our choice of the sequence (a_i) . Therefore, we can assume that $a_i \geq 2$ for all $i = 2, 3, \dots, e-1$.

Hence we maximize (1) subject to the conditions:

$$k_1 \geq 5, k_i \geq 2, k_i + k_{i+1} \geq 5, \text{ for } i = 1, 2, \dots, e-1, k_{e-1} + k_e \geq 6, \sum_{i=0}^e k_i = n.$$

Clearly for fixed e , (1) is maximized, subject to these conditions, for

$$(k_0, k_1, \dots, k_{e-1}) = \begin{cases} (1, 5, 2, 3, 2, 3, \dots, 2, 3) & \text{if } e \text{ is even,} \\ (1, 5, 2, 3, 2, 3, \dots, 2, 3, 2) & \text{if } e \text{ is odd,} \end{cases}$$

and $k_e = n - \sum_{i=0}^{e-1} k_i$. Then

$$\begin{aligned} \sigma_G(v) &= 1k_1 + 2k_2 + 3k_3 + 4k_4 + 5k_5 + \cdots + ek_e \\ &\leq \begin{cases} 1 \cdot 5 + [2 \cdot 2 + 3 \cdot 3] + \cdots + [(e-2) \cdot 2 + (e-1) \cdot 3] + e(n - \frac{5}{2}e - 1) & \text{if } e \text{ is even,} \\ 1 \cdot 5 + [2 \cdot 2 + 3 \cdot 3] + \cdots + [(e-3) \cdot 2 + (e-2) \cdot 3] + (e-1) \cdot 2 + e(n - \frac{5}{2}e - \frac{1}{2}) & \text{if } e \text{ is odd} \end{cases} \\ &\leq en - \frac{5}{4}e^2 - 2e + \frac{9}{4}. \end{aligned} \quad (2)$$

By (*),

$$n = \sum_{i=0}^e k_i \geq \begin{cases} \frac{5e+8}{2} & \text{if } e \text{ is even,} \\ \frac{5e+9}{2} & \text{if } e \text{ is odd.} \end{cases}$$

(Note that for even e we use $k_2 + k_3 + k_4 \geq 7$.) Hence $e \leq \frac{2n-8}{5}$. A simple differentiation shows that (2) is maximized, subject to the constraint $e \leq \frac{2n-8}{5}$, for $e = \frac{2n-8}{5}$ and we obtain $\sigma_G(v) \leq \left(\frac{2n-8}{5}\right)n - \frac{5}{4}\left(\frac{2n-8}{5}\right)^2 - 2\left(\frac{2n-8}{5}\right) + \frac{9}{4} \leq \frac{1}{5}(n^2 - 4n + 14)$, as desired.

Summing $\sigma_G(v)$ for all v , the bound in the second part of the lemma follows from the first part. \square

Proposition 2 *Let G be a 5-edge-connected block and v a vertex of G . If $k_l + k_{l+1} + k_{l+2} \leq 11$ for some l , $1 \leq l \leq ex_G(v) - 3$, then $diam(N_{l+1}) \leq 2$ and $2 \leq k_{l+1} \leq 7$.*

Proof: Suppose to the contrary that for some $a, b \in N_{l+1}$, $d(a, b) > 2$. Then $N[a] \cap N[b] = \emptyset$. Thus, since $N[a], N[b] \subseteq N_l \cup N_{l+1} \cup N_{l+2}$ and $\delta(G) = \delta \geq 5$, we have $k_l + k_{l+1} + k_{l+2} \geq |N[a]| + |N[b]| \geq (\delta + 1) + (\delta + 1) \geq 12$, a contradiction. Therefore, for all $a, b \in N_{l+1}$, $d(a, b) \leq 2$, as desired.

Since G is a block, $k_l, k_{l+1}, k_{l+2} \geq 2$. Therefore, $2 \leq k_{l+1} \leq 11 - (k_l + k_{l+2}) \leq 11 - 4 = 7$. \square

Definition 1 Let G be a 5-edge-connected graph and v a vertex of G . We say that (N_l, N_{l+1}, N_{l+2}) , where $22 \leq l \leq ex_G(v) - 24$, is a *forbidden separating triple* of v if $k_l + k_{l+1} + k_{l+2} \leq 11$.

Clearly, if a vertex v with $ex_G(v) \geq 46$, has no forbidden separating triple, then for all $i \in \{22, \dots, ex_G(v) - 24\}$ we have that $k_i + k_{i+1} + k_{i+2} \geq 12$. This will allow us to show that the distance of a vertex v with no forbidden separating triple is small and bounded by a term of the form $\frac{n^2}{8} + O(n)$.

Lemma 2 *Let G be a 5-edge-connected graph with n vertices and let v be a vertex of G with $ex_G(v) \geq 46$. If v has no forbidden separating triple, then*

$$\sigma_G(v) \leq \frac{1}{8}n^2 + \frac{33}{4}n + \frac{1361}{8}.$$

Proof: Denote the eccentricity of v by e , where e is fixed. We first show that

$$\sigma_G(v) \leq en - 2e^2 + 33e + 34. \quad (3)$$

As above

$$\sigma_G(v) = 1k_1 + 2k_2 + 3k_3 + \dots + ek_e. \quad (4)$$

Since G is a 5-edge-connected graph and v has no forbidden separating triple, the sequence (k_i) satisfies the conditions: $k_1 \geq 5$; $k_{e-1} + k_e \geq 6$; $k_i \geq 1$ for $i = 0, 1, \dots, e$; $k_i + k_{i+1} \geq 5$ for $i = 0, 1, \dots, 20$, $i = e - 21, \dots, e - 1$; $k_i + k_{i+1} + k_{i+2} \geq 12$ for $i = 22, 23, \dots, e - 24$ and $\sum_{i=0}^e k_i = n$. If $e \equiv 0 \pmod{3}$, then clearly (4) is maximized, subject to these

conditions, for $(k_0, k_1, \dots, k_{21}) = (1, 5, 1, 4, \dots, 1, 4)$, $(k_{22}, k_{23}, \dots, k_{e-22}) = (1, 1, 10, 1, 1, 10, \dots, 1, 1, 10, 1, 1)$, $(k_{e-21}, k_{e-20}, \dots, k_{e-1}) = (1, 4, 1, 4, \dots, 1, 4, 1)$ and k_e as large as possible. Then

$$\begin{aligned}
\sigma_G(v) &= 1k_1 + 2k_2 + 3k_3 + \dots + ek_e \\
&\leq 5 + [2 \cdot 1 + 3 \cdot 4] + \dots + [20 \cdot 1 + 21 \cdot 4] + [22 \cdot 1 + 23 \cdot 1 + 24 \cdot 10] \\
&\quad + \dots + [(e-26) \cdot 1 + (e-25) \cdot 1 + (e-24) \cdot 10] + (e-23) \cdot 1 \\
&\quad + (e-22) \cdot 1 + [(e-21) \cdot 1 + (e-20) \cdot 4] + \dots + \\
&\quad [(e-3) \cdot 1 + (e-2) \cdot 4] + (e-1) \cdot 1 + e(n-4e+71) \\
&= en - 2e^2 + 33e + 34.
\end{aligned}$$

By a similar argument, for $e \equiv 1 \pmod 3$ and for $e \equiv 2 \pmod 3$, we have $\sigma_G(v) \leq en - 2e^2 + 33e + 34$.

Note that by the conditions satisfied by the sequence (k_i) , we have

$$n = \sum_{i=0}^e k_i \geq \begin{cases} 4e - 66 & \text{if } e \equiv 0 \pmod 3, \\ 4e - 63 & \text{if } e \equiv 2 \pmod 3, \\ 4e - 60 & \text{if } e \equiv 1 \pmod 3. \end{cases}$$

Hence $e \leq \frac{n+66}{4}$. A simple differentiation shows that (3) is maximized, subject to the constraint $e \leq \frac{n+66}{4}$, for $e = \frac{n+33}{4}$ and we obtain

$$\begin{aligned}
\sigma_G(v) &\leq \left(\frac{n+33}{4}\right)n - 2\left(\frac{n+33}{4}\right)^2 + 33\left(\frac{n+33}{4}\right) + 34 \\
&= \frac{1}{8}n^2 + \frac{33}{4}n + \frac{1361}{8},
\end{aligned}$$

as desired. \square

We first establish an upper bound on distance for blocks.

Theorem 3 *Let G be a 5-edge-connected block of order n . Then*

$$\sigma(G) \leq n(n-1) \left(\frac{2}{15}n + 9\right).$$

Proof: Let v be a vertex of G of largest distance and let e be its eccentricity.

For $e \leq 45$, the result is obtained as follows. If, on one hand, $n \leq 116$, then the second part of Lemma 1 yields the desired bound by a simple calculation. If, on the other hand, $n \geq 117$, then we have

$$\begin{aligned}
\sigma_G(v) &\leq 5 + [2(2) + 3(3)] + \dots + [42(2) + 43(3)] + 44(2) \\
&\quad + 45(n-6-5 \cdot 21-2) \\
&= 45n - 2619.
\end{aligned}$$

Since $45n - 2619 \leq (n - 1)(\frac{2}{15}n + 9)$ for $n \geq 117$, it follows that $\sigma(G) \leq n\sigma_G(v) \leq n(45n - 2619) \leq n(n - 1)(\frac{2}{15}n + 9)$ for all n . Thus the result holds.

So assume that $e \geq 46$ and hence $n \geq 119$. If v has no forbidden separating triple, then summing $\sigma_G(v)$ for all v we obtain the result by Lemma 2.

If v has a forbidden separating triple, we prove the theorem by induction on the order n of G . For $n \leq 143$, the result follows by the second part of Lemma 1. So assume that $n \geq 144$ and that the result holds for any 5-edge-connected block with less than n vertices. Since v has a forbidden separating triple, let N_α , $23 \leq \alpha \leq e - 23$, be chosen in accordance with Proposition 2, that is, $\text{diam}(N_\alpha) \leq 2$ and $2 \leq k_\alpha \leq 7$. Let K_5 be a copy of the complete graph with 5 vertices such that $V(G) \cap V(K_5) = \emptyset$. Denote the vertex set of K_5 by $A = \{a_1, \dots, a_5\}$. We form a new graph H , from $G \cup K_5$ by attaching K_5 to N_α , that is, by joining every vertex in N_α to every vertex in K_5 .

Let $G_1 = H[N_{\leq \alpha} \cup A]$ and $G_2 = H[N_{\geq \alpha} \cup A]$. Since $\text{diam}(N_\alpha) \leq 2$, no distances in G have been reduced, hence $d_G(x, y) = d_H(x, y)$ for all $x, y \in V(G)$. The proof of the following claim is elementary.

Claim 1 Let $X = \sum_{i=1}^5 \sigma_{G_1}(a_i)$ and $Y = \sum_{i=1}^5 \sigma_{G_2}(a_i)$. Then

$$\sigma(G) = \sigma(G_1) + \sigma(G_2) - 2(X + Y) + 2\sigma_A(A) + 2 \sum_{x \in N_{< \alpha}} \sum_{y \in N_{> \alpha}} d_H(x, y) - \sum_{(x, y) \in N_\alpha \times N_\alpha} d_H(x, y).$$

Note that G_1 and G_2 have $N_\alpha \cup A$ in common. Hence, letting $|V(G_i)| = n_i$, $i = 1, 2$, we obtain

$$n_1 + n_2 - k_\alpha - 10 = n. \quad (5)$$

Note also that $n_1 \geq |N_{\leq \alpha}| + |A| \geq (k_0 + k_1) + \dots + (k_{20} + k_{21}) + k_{22} + k_\alpha + 5 \geq 63 + k_\alpha$. Analogously, $n_2 \geq |N_{\geq \alpha}| + |A| \geq 63 + k_\alpha$. It follows from (5), that

$$63 + k_\alpha \leq n_1, \quad n_2 \leq n - 53.$$

We bound $\sigma(G_1)$ and $\sigma(G_2)$ using induction. It is easy to verify that

Claim 2 G_1 and G_2 are 5-edge-connected blocks.

We now find bounds on the terms in Claim 1. First, note that

$$\sigma_A(A) \leq \sigma(K_5) = 20 \quad \text{and} \quad \sum_{(x, y) \in N_\alpha \times N_\alpha} d_H(x, y) \geq k_\alpha(k_\alpha - 1). \quad (6)$$

Now, let a be a fixed vertex in N_α . Then

$$\begin{aligned}
2 \sum_{x \in N_{<\alpha}, y \in N_{>\alpha}} d_H(x, y) &\leq 2 \sum_{x \in N_{<\alpha}} \sum_{y \in N_{>\alpha}} [d_H(x, a) + d_H(a, y)] \\
&= 2 \sum_{x \in N_{<\alpha}} [|N_{>\alpha}| d_H(x, a) + \sigma_{G_2}(a) - \sigma_{N_\alpha \cup A}(a)] \\
&= 2[|N_{>\alpha}|(\sigma_{G_1}(a) - \sigma_{N_\alpha \cup A}(a)) \\
&\quad + |N_{<\alpha}|(\sigma_{G_2}(a) - \sigma_{N_\alpha \cup A}(a))].
\end{aligned}$$

Note that $\sigma_{N_\alpha \cup A}(a) \geq |(N_\alpha \cup A) - \{a\}| = k_\alpha + 4$. Thus,

$$\begin{aligned}
&2 \sum_{x \in N_{<\alpha}, y \in N_{>\alpha}} d_H(x, y) \\
&\leq 2[(n_2 - k_\alpha - 5)(\sigma_{G_1}(a) - k_\alpha - 4) + (n_1 - k_\alpha - 5)(\sigma_{G_2}(a) - k_\alpha - 4)]. \quad (7)
\end{aligned}$$

We find lower bounds on X and Y . Note that for $i = 1, 2, \dots, 5$,

$$\begin{aligned}
\sigma_{G_1}(a) &= \sum_{x \in V(G_1)} d(a, x) \leq \sum_{x \in V(G_1)} [d(a, a_i) + d(a_i, x)] \\
&= \sum_{x \in V(G_1)} [1 + d(a_i, x)] = n_1 + \sigma_{G_1}(a_i).
\end{aligned}$$

Therefore, $\sigma_{G_1}(a_i) \geq \sigma_{G_1}(a) - n_1$ and

$$X \geq 5\sigma_{G_1}(a) - 5n_1. \quad (8)$$

Analogously,

$$Y \geq 5\sigma_{G_2}(a) - 5n_2. \quad (9)$$

Using induction, we have $\sigma(G_i) \leq n_i(n_i - 1) \left(\frac{2}{15}n_i + 9\right)$, whereas by Lemma 1 we have $\sigma_{G_i}(a) \leq \frac{1}{5}(n_i^2 - 4n_i + 14)$ for $i = 1, 2$.

This, together with (5), Claim 1, (6)-(9), yields

$$\begin{aligned}
\sigma(G) &\leq n(n-1)\left(\frac{2}{15}n + 9\right) + \left\{ (n_1 + n_2)\left(\frac{28}{3}k_\alpha + \frac{2414}{15} - \frac{2}{5}k_\alpha^2\right) \right. \\
&\quad \left. - n_1n_2\left(\frac{194}{15} - \frac{4}{5}k_\alpha\right) + \frac{2}{15}k_\alpha^3 - \frac{28}{15}k_\alpha^2 - \frac{1808}{15}k_\alpha - \frac{2506}{3} \right\}.
\end{aligned}$$

Denote the term in curly brackets of the right hand side of the last inequality by $f(n_1, n_2)$. It is sufficient to show that, subject to $63 + k_\alpha \leq n_1, n_2$, we have $f(n_1, n_2) \leq 0$. Clearly, $(n_1 - (63 + k_\alpha))(n_2 - (63 + k_\alpha)) \geq 0$, hence $n_1n_2 \geq (63 + k_\alpha)(n_1 + n_2) - (63 + k_\alpha)^2$. This, in conjunction with (5), yields $f(n_1, n_2) \leq n\left(\frac{2}{5}k_\alpha^2 + \frac{234}{5}k_\alpha - \frac{9808}{15}\right) - \frac{4}{15}k_\alpha^3 - \frac{584}{15}k_\alpha^2 - 1852k_\alpha + \frac{219792}{5}$

and thus from $2 \leq k_\alpha \leq 7$ we obtain $f(n_1, n_2) \leq \frac{601448}{15} - \frac{920}{3}n \leq 0$ for all $n \geq 131$, as desired. \square

We now show that the upper bound on the distance proved above for 5-edge-connected blocks holds for all 5-edge-connected graphs.

Theorem 4 *Let G be a 5-edge-connected graph of order n . Then*

$$\mu(G) \leq \frac{2}{15}n + 9.$$

Apart from the additive constant, this inequality is best possible.

Proof: We prove this result by induction on the number of blocks in G . If G is a block then the result follows by Theorem 3. So assume that G has at least 2 blocks. We prove the equivalent statement

$$\sigma(G) \leq n(n-1) \left(\frac{2}{15}n + 9 \right).$$

Let G_1 be an end block, G_2 be the union of the other blocks, and u be the unique common vertex of G_1 and G_2 . For $i = 1, 2$, let $n_i = |V(G_i)|$. Then the relation $n_1 + n_2 - 1 = n$ holds. By definition,

$$\begin{aligned} \sigma(G) &= \sum_{(x,y) \in V(G) \times V(G)} d(x,y) \\ &= \sigma(G_1) + 2 \sum_{x \in V(G_1) - \{u\}} \sum_{y \in V(G_2) - \{u\}} d(x,y) + \sigma(G_2) \\ &= \sigma(G_1) + 2[(n_2 - 1)\sigma_{G_1}(u) + (n_1 - 1)\sigma_{G_2}(u)] + \sigma(G_2). \end{aligned} \quad (10)$$

Clearly, G_1 and G_2 are 5-edge-connected. Thus, by induction, $\sigma(G_i) \leq n_i(n_i - 1) \left(\frac{2}{15}n_i + 9 \right)$ for $i = 1, 2$. By Lemma 1, we have $\sigma_{G_i}(u) \leq \frac{1}{5}(n_i^2 - 4n_i + 14)$ for $i = 1, 2$. This, in conjunction with (10) and $n_1 + n_2 - 1 = n$, yields $\sigma(G) \leq n(n-1) \left(\frac{2}{15}n + 9 \right) + \left\{ \frac{368}{15}(n_1 + n_2) - \frac{302}{15}n_1n_2 - \frac{434}{15} \right\}$. Denote the term in curly brackets of the right hand side of the last inequality by $f(n_1, n_2)$. We show that $f(n_1, n_2) \leq 0$ for all $n_1, n_2 \geq 6$. Clearly, $(n_1 - 6)(n_2 - 6) \geq 0$; hence $n_1n_2 \geq 6(n_1 + n_2) - 36$. This, in conjunction with $n_1 + n_2 - 1 = n$, yields $f(n_1, n_2) \leq \frac{2998}{5} - \frac{1444}{15}n \leq 0$ for all $n \geq 7$. Note that if $n = 6$, then $G = K_6$, and the result holds.

We now show that, apart from the additive constant, the bound is best possible. Given integers n, k with $n = \frac{1}{2}(5k + 9)$, $k \geq 5$ odd, let

$$G_n = G_1 + G_2 + \cdots + G_k,$$

where $G_1 = K_5 = G_k$ and for $2 \leq i \leq k - 1$,

$$G_i = \begin{cases} K_2 & \text{if } i \text{ is even,} \\ K_3 & \text{if } i \text{ is odd.} \end{cases}$$

Clearly, G_n is 6-edge-connected (and thus 5-edge-connected) of order n . Straightforward calculations show that

$$\mu(G_n) \geq \frac{2}{n(n-1)} \left(\frac{1}{15}(n^3 - 15n^2 + 20n + 57) \right) = \frac{2}{15}n - \frac{28}{15} + \frac{2(6n+57)}{15n(n-1)},$$

as desired. \square

Since every 6-edge-connected graph is also 5-edge-connected and since the graph G_n defined above is 6-edge-connected, the following theorem becomes immediate.

Theorem 5 *Let G be a 6-edge-connected graph of order n . Then*

$$\mu(G) \leq \frac{2}{15}n + 9.$$

Apart from the additive constant, this inequality is best possible. \square

4 Results on 7-Edge-Connected Graphs

We now focus our attention on 7-edge-connected graphs.

Lemma 6 *Let G be a 7-edge-connected graph with n vertices and let v be an arbitrary vertex of G . Then $\sigma_G(v) \leq \frac{1}{6}(n^2 - 6n + 27)$. Moreover, $\sigma(G) \leq \frac{n}{6}(n^2 - 6n + 27)$.*

Proof: Let e be the eccentricity of v . Then

$$\sigma_G(v) = 1k_1 + 2k_2 + \cdots + ek_e. \quad (11)$$

The lemma can easily be verified for $e \leq 4$. Assume that $e \geq 5$. Since G is 7-edge-connected and as in the proof of Lemma 1, we have

$$k_1, k_i k_{i+1} \geq 7, k_i + k_{i+1} \geq 6 \text{ for } i = 1, \dots, e-1, k_{e-1} + k_e \geq 8, n = \sum_{i=0}^e k_i. (*)$$

Let (a_0, a_1, \dots, a_e) be a sequence that maximizes (11) subject to (*). We first show that $a_i > 1$ for $i = 2, \dots, e-1$. If this were not true, let s , $2 \leq s \leq e-1$, be the smallest integer for which $a_s = 1$. First assume that $s \geq 3$. Let (b_0, b_1, \dots, b_e) be the sequence defined by (i) $b_i = 4$ if $i = s-1$, (ii) $b_i = 2$ if $i = s$, (iii) $b_i = a_{s-1} + a_{s+1} - 5$ if $i = s+1$ and (vi) $b_i = a_i$ otherwise. Then from $a_{s-2} \geq 2$, $a_{s-1}, a_{s+1} \geq 7$, we have that the sequence (b_i) satisfies (*) and $\sum_{i=0}^e ia_i = \sum_{i=0}^e ib_i - (2a_{s-1} - 9) < \sum_{i=0}^e ib_i$, contradicting our choice of the sequence (a_i) . If $s = 2$, then let (b_0, b_1, \dots, b_e) be defined by $b_2 = 2$, $b_3 = 4$, $b_4 = a_3 + a_4 - 5$ and $b_i = a_i$ otherwise. Since $a_3 \geq 7$ and $e \geq 5$, the sequence (b_i) satisfies (*) and $\sum_{i=0}^e ia_i = \sum_{i=0}^e ib_i - (a_3 - 6) < \sum_{i=0}^e ib_i$,

again contradicting our choice of the sequence (a_i) . Therefore, $a_i \geq 2$ for all $i = 2, \dots, e-1$. Hence we maximize (11) subject to the conditions:

$$k_1 \geq 7, k_i \geq 2, k_i + k_{i+1} \geq 6 \text{ for } i = 1, 2, \dots, e-1, k_{e-1} + k_e \geq 8, \sum_{i=0}^e k_i = n.$$

Clearly for fixed e , (11) is maximized, subject to these constraints, for

$$(k_0, k_1, \dots, k_{e-1}) = \begin{cases} (1, 7, 2, 4, 2, 4, \dots, 2, 4) & \text{if } e \text{ is even,} \\ (1, 7, 2, 4, 2, 4, \dots, 2, 4, 2) & \text{if } e \text{ is odd.} \end{cases}$$

As in the proof of Lemma 1, we obtain $\sigma_G(v) \leq \frac{1}{6}(n^2 - 6n + 27)$, as desired. The second part also follows similarly. \square

The proposition below is an analogue to Proposition 2. Its proof is similar, so we omit it.

Proposition 3 *Let G be a 7-edge-connected block and v a vertex of G . If for some l , $1 \leq l \leq \text{ex}_G(v) - 3$, we have $k_l + k_{l+1} + k_{l+2} \leq 15$, then $\text{diam}(N_{l+1}) \leq 2$ and $2 \leq k_{l+1} \leq 11$.*

Definition 2 Let G be a 7-edge-connected graph and v a vertex of G . We say that (N_l, N_{l+1}, N_{l+2}) , where $22 \leq l \leq \text{ex}_G(v) - 24$, is a *forbidden separating triple* of v if $k_l + k_{l+1} + k_{l+2} \leq 15$.

The bound on the distance of a vertex of G with no forbidden separating triple is given below. Its proof is analogous to the proof of Lemma 2.

Lemma 7 *Let G be a 7-edge-connected graph of order n and v a vertex of G with $\text{ex}_G \geq 46$. If v has no forbidden separating triple, then*

$$\sigma_G(v) \leq \frac{3}{32}n^2 + \frac{153}{16}n + \frac{9275}{32}.$$

Theorem 8 *Let G be a 7-edge-connected block of order n . Then*

$$\mu(G) \leq \frac{1}{9}n + 10.$$

Proof: Let v be a vertex of G of largest distance and let e be its eccentricity.

If $e \leq 45$, then the result is obtained as follows. If, on one hand, $n \leq 142$, then by the second part of Lemma 6, $\sigma(G) \leq \frac{1}{6}n(n^2 - 6n + 27) \leq n(n-1)(\frac{1}{9}n + 10)$ for all $n \leq 142$ and the theorem holds. If, on the other hand, $n \geq 143$, then as in the proof of Lemma 6, we have

$$\begin{aligned} \sigma_G(v) &\leq 7 + [2(2) + 3(4)] + \dots + [42(2) + 43(4)] + 44(2) + 45(n - 136) \\ &= 45n - 3169. \end{aligned}$$

Since $45n - 3169 \leq (n - 1)(\frac{1}{9}n + 10)$ for $n \geq 143$, it follows that $\sigma(G) \leq n\sigma_G(v) \leq n(n - 1)(\frac{1}{9}n + 10)$ for all n . Thus the theorem holds for $e \leq 45$.

So assume that $e \geq 46$ and hence $n \geq 144$. If v has no forbidden separating triple, then summing $\sigma_G(v)$ for all v we obtain result by Lemma 7. If v has a forbidden separating triple we proceed with our proof as follows. We prove the equivalent statement

$$\sigma(G) \leq n(n - 1) \left(\frac{1}{9}n + 10 \right)$$

by induction on the order n of G . For $n \leq 194$, the result follows by the second part of Lemma 6. So assume that $n \geq 195$ and that the result holds for any 7-edge-connected block with less than n vertices. Since v has a forbidden separating triple, let N_α , $23 \leq \alpha \leq e - 23$, be chosen in accordance with Proposition 3, that is, $\text{diam}(N_\alpha) \leq 2$ and $2 \leq k_\alpha \leq 11$. Let K_7 be a copy of the complete graph with 7 vertices such that $V(G) \cap V(K_7) = \emptyset$. Denote the vertex set of K_7 by $A = \{a_1, \dots, a_7\}$. We form a new graph H , from $G \cup K_7$ by attaching K_7 to N_α , that is, by joining each vertex in N_α to every vertex in K_7 .

Let $G_1 = H[N_{\leq \alpha} \cup A]$ and $G_2 = H[N_{\geq \alpha} \cup A]$. Since $\text{diam}(N_\alpha) \leq 2$, no distances in G have been reduced, hence $d_G(x, y) = d_H(x, y)$ for all $x, y \in V(G)$. The claim below is easy to establish.

Claim 3 Let $X = \sum_{i=1}^7 \sigma_{G_1}(a_i)$ and $Y = \sum_{i=1}^7 \sigma_{G_2}(a_i)$. Then

$$\begin{aligned} \sigma(G) &= \sigma(G_1) + \sigma(G_2) - 2(X + Y) + 2\sigma_A(A) \\ &\quad + 2 \sum_{x \in N_{< \alpha}} \sum_{y \in N_{> \alpha}} d_H(x, y) - \sum_{(x, y) \in N_\alpha \times N_\alpha} d_H(x, y). \end{aligned}$$

By bounding the terms on the right hand side of the equation in Claim 3 in a similar manner to that done in Theorem 3, we obtain $\sigma(G) \leq n(n - 1)(\frac{1}{9}n + 10)$, as desired. \square

Theorem 9 *Let G be a 7-edge-connected graph of order n . Then*

$$\mu(G) \leq \frac{1}{9}n + 10.$$

Apart from the additive constant, this inequality is best possible.

Proof: The proof of this result is treated similarly to the proof of Theorem 4. To show that the bound is best possible apart from the exact value of the additive constant, let n and k be integers satisfying $n = 3k + 4$, $k \geq 6$ and let $G_{n,7} = G_1 + G_2 + \dots + G_k$ where $G_1 = K_1 = G_k$, $G_2 = K_7 = G_{k-1}$ and $G_i = K_3$ for $i = 3, \dots, k - 2$. Clearly, $G_{n,7}$ is 7-edge-connected and $\mu(G_{n,7}) = \frac{1}{9}n + \frac{1}{9} + \frac{2(736 - 97n)}{9n(n-1)}$, as desired. \square

References

- [1] J.C. Bermond, Z. Liu and M. Syska, Mean eccentricities of DeBruijn networks. *Networks* **30** (1997), 187-203.
- [2] P. Dankelmann and R.C. Entringer, Average distance, minimum degree and spanning trees. *J. Graph Theory* **33** (2000), 1-13.
- [3] P. Dankelmann, S. Mukwembi and H.C. Swart, Average Distance and Edge-Connectivity II. *SIAM J. Discrete Math.* (to appear).
- [4] J. K. Doyle and J. E. Graver, Mean distance in a graph. *Discrete Math.* **7** (1977), 147-154.
- [5] R. C. Entringer, D. E. Jackson and D. A. Snyder, Distance in graphs. *Czech Math. J.* **26** (1976), 283-296.
- [6] O. Favaron, M. Kouider and M. Mahéo, Edge-vulnerability and mean distance. *Networks* **19** (1989), 493-504.
- [7] I. Gutman and O. E. Polansky, *Mathematical concepts in organic chemistry*. Springer, Berlin, (1986).
- [8] M. Kouider and P. Winkler, Mean distance and minimum degree. *J. Graph Theory* **25** (1997), 95-99.
- [9] L. Lovász, *Combinatorial problems and exercises*. Akadémiai Kiadó, Budapest (1979).
- [10] Z. Mihalic, D. Veljan, D. Amic, S. Nikolic, D. Plavsic and N. Trinajstic, The distance matrix in chemistry. *J. Math. Chem.* **11** (1992), 233-258.
- [11] S. Mukwembi, *Bounds on distances in graphs*. PhD Thesis, University of KwaZulu-Natal (2006).
- [12] J. Plesník, On the sum of all distances in a graph or digraph. *J. Graph Theory* **8** (1984), 1-24.
- [13] D. H. Rouvray, The modeling of chemical phenomena using topological indexes. *J. Comput. Chem.* **8** (1987), 470-480.
- [14] L. Soltés, Transmission in graphs: a bound and vertex removing. *Math. Slovaca* **41** (1991), 11-16.
- [15] H. Wiener, Structural determination of paraffin boiling points. *J. Amer. Chem. Soc.* **69** (1947), 17-20.
- [16] S. Wuchty and P. F. Stadler, Centers of complex networks. *J. Theoret. Biol.* **223** (2003), 45-53.